Li, F., Y. Wang, **A. Emrouznejad**, and Q. Zhu and G. Kou (2021). Allocating a fixed cost across decision-making units with undesirable outputs: A bargaining game approach. *Journal of the Operational Research Society,* **Accepted**. https://doi.org/10.1080/01605682.2021.1981781.

Allocating a fixed cost across decision-making units with undesirable outputs: A bargaining game approach

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ABSTRACT

Allocating a fixed cost among a set of peer decision-making units (DMUs) represents one of the most important applications of data envelopment analysis (DEA). However, almost all existing studies have addressed the fixed cost allocation (FCA) problem within a traditional framework while ignoring the existence of undesirable outputs. Undesirable outputs are neither scarce in various production activities in the real world nor trivial in efficiency evaluation and subsequent decision making. Motivated by this observation, this paper attempts to explicitly extend the traditional FCA problem to situations in which DMUs are necessarily involved with undesirable outputs. To this end, we first investigate the efficiency evaluation of DMUs considering undesirable outputs based on the joint weak disposability assumption. Then, flexible FCA schemes are considered to revisit the efficiency evaluation process. Results show that some feasible allocation schemes always exist such that all DMUs can be simultaneously efficient. Further, we define the comprehensive satisfaction degree and develop a satisfaction degree bargaining game approach to determine a unique FCA scheme. Finally, the proposed approach is demonstrated with an empirical study of banking activities based on real conditions.

Keywords: Data envelopment analysis (DEA), Fixed cost allocation (FCA), Undesirable outputs, Bargaining game, Comprehensive satisfaction degree

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1. Introduction

Data envelopment analysis (DEA) is a nonparametric mathematical method originally designed to evaluate the relative efficiency of a set of homogenous decision-making units (DMUs) that are responsible for converting multiple inputs to produce multiple outputs (Charnes et al. 1978; Banker et al. 1984). Since its inception, DEA has been applied to various activities in the public and private sectors (Emrouznejad & Yang 2018; Li, Emrouznejad, et al. 2020; Li et al. 2021; Zhu, Li, Wu & Sun 2021), among which the fixed cost allocation (FCA) problem is one of the most important applications (Cook & Kress 1999; Beasley 2003; Li et al. 2009; Lin 2011a; Li et al. 2013; Du et al. 2014; Zhu, Aparicio, et al. 2021). In many real managerial applications, FCA problems arise how to divide the total amount costs among multiple units in a reasonable way. The fixed cost refers to the expense of building a common platform for a group of units (Li et al. 2009; Li, Zhu & Liang 2019), and typical examples of fixed costs include the advertisement expenditures of a manufacturer across its retailers (Cook & Kress 1999), the cost of a common communication cable among users (Beasley 2003) and the maintenance charges of an information and technology service across bank branches (Li, Liang, et al. 2018; Li, Zhu & Chen 2019). The underlying issue of DEA-based FCA problems concern the determination of a unique scheme that fairly assigns cost shares to all peer DMUs based on intrinsic efficiency principles.

DEA-based FCA approaches would investigate DMUs' efficiency scores through the inclusion of allocated costs. The existing studies can be mainly divided into two categories based on their focus on efficiency invariance or efficiency maximization (Cook and Kress 1999; Beasley 2003). Cook & Kress (1999) made the first attempt to allocate a fixed cost based on efficiency analysis, and they left efficiencies unchanged to ensure fairness. Efficiency invariance is related to the fact that the allocation scheme is out of the control of individual DMUs, and hence each DMU's efficiency score should not be affected. Jahanshahloo et al. (2004) argued that Cook & Kress (1999) violated the Pareto-minimality principle and then proposed an alternative approach using a simple formula, but their approach was further thought to lack feasibility and acceptability (Lin 2011b). Cook & Zhu (2005) extended the Cook & Kress (1999) approach to an output-oriented version and developed a new feasible method to generate the FCA scheme in multi-input–multi-output situations. However, Lin (2011a) argued that the Cook & Zhu (2005) approach will be infeasible when some special constraints are added, and Lin (2011a) modified the constraints to generate the final allocation plan through minimizing the gaps between the largest and smallest allocated cost. Interestingly, Amirteimoori & Kordrostami (2005) and Jahanshahloo et al. (2017) considered the efficiency invariance principle with a set of common weights, and Lin & Chen (2016) proposed a FCA approach based on super CCR (Charnes-Cooper-Rhodes) efficiency and practical feasibility. Lin et al. (2016) combined efficiency invariance and zero slacks to generate a FCA plan that is unique, partially dependent on DMUs' inputs and unit-invariant.

Another important stream is based on efficiency maximization first proposed by Beasley (2003), who used a set of common weights to maximize the overall efficiency for all DMUs. In fact, the efficiency maximization principle will ensure the full efficiency of one for each DMU (Li et al. 2013; Si et al. 2013), and thus each DMU will deem the corresponding allocation scheme to be fair and beneficial since each DMU will reach the highest efficiency (Beasley 2003). A significant extension of this concept was presented by Li et al. (2013), who proved the existence of efficient allocation schemes that can render all DMUs simultaneously efficient under a set of common weights. The authors also defined a degree of satisfaction concept for determining a unique allocation scheme. Du et al. (2014) used a game-like cross efficiency approach to address the FCA problem. The results showed that all DMUs will be efficient, but multiple optimal allocation schemes will be obtained (Li, Li, et al. 2019). Chen et al. (2020) proposed another cross-efficiency DEA-based iterative approach, which took the overall goal of the whole organization and individual preferences into account in a centralized environment. Further, Sharafi et al. (2020) developed an alternative cross-efficiency based on the Pareto concept, and found that the allocated results would be Pareto cross-efficient. Li, Zhu & Liang (2018) and Li, Li, et al. (2019) developed game-based approaches for selecting an optimal scheme from efficient allocation schemes using Shapley and nucleolus solutions. Recently, Meng et al. (2020) proposed an approach incorporating the perspectives of coalition efficiency and the Shapley value, which considers the relationships among DMUs across their formed coalitions to determine their interaction types and then generates an allocation scheme that represents the Shapley value.

Noting that most studies consider DMUs as *black boxes* without considering internal production structures, some researchers have extended the traditional FCA problem to network environments. Yu et al. (2016) extended the game-like cross efficiency approach developed by Du et al. (2014) to two-stage network structures where the intermediate outputs produced in the first stage are used as inputs in the second stage. Ding et al. (2019) and Zhu et al. (2019) then extended the satisfaction degree method developed by Li et al. (2013), and some variants of two-stage structures and satisfaction degree concepts were also studied in reference to the two-stage FCA problem. Li, Zhu & Chen (2019) studied the same problem and it can obtain a unique allocation scheme through an iterative procedure. Recently, Chu et al. (2020), An, Wang, et al. (2020) and An, Wang & Shi (2020) studied the same two-stage FCA problem by considering game relationships.

Other FCA studies address some special cases. For instance, Lotfi et al. (2007) extended the work of Jahanshahloo et al. (2004) by proposing a means of targeting fuzzy inputs, fuzzy outputs and fuzzy fixed costs. Pendharkar (2018) developed a DEA-based hybrid genetic algorithm to generate a cost allocation scheme minimizing the correlation between inefficient DMUs' efficiency levels and allocated resources. Dai et al. (2016) considered returns to scale properties, and Khodabakhshi and Aryavash (2014) took the relationship between existing measures and fixed costs into account. Mostafaee (2013) left returns to scale properties and efficiencies unchanged and specified an allocation scheme by minimizing the gap in the allocated costs among all DMUs. Ding et al. (2018) studied a centralized FCA problem while considering heterogeneous technology across DMUs. Li, Zhu & Liang (2019) presented a non-egoistic principle requiring that the allocation proposed by each DMU itself will result in cost sharing of no less than those of other DMUs. Most recently, Li, Yan, et al. (2020) proposed a FCA approach based on the efficiency ranking concept, which addressed the performance and efficiency ranking interval by considering all relative weights. Li et al. (2009), Dai et al. (2016) and Lin & Chen (2020) studied cases in which allocated costs are considered as complements of other cost measures while all other studies in the existing literature consider allocated costs as additional inputs.

In surveying articles from the existing literature, we find that almost all previous articles have developed approaches for FCA problems under a traditional framework while ignoring the existence of undesirable outputs. However, most FCA problems in the real world are involved in sectors where undesirable outputs are inevitable. To provide a simple view of these observations, we summarize the existing studies in Table 1, including the application sectors and possible undesirable outputs. It can be demonstrated from Table 1 that most real applications inevitably provide undesirable outputs, while Li, Zhu & Chen (2019) is the only study that addresses the FCA problem in a case where the bad debt in banking activities was explicitly considered as a kind of undesirable output. However, Li, Zhu & Chen (2019) developed their approach considering only traditional inputs and desirable outputs, while the bad debt was only considered in an empirical study to illustrate the usefulness and efficacy of their proposed approach. Therefore, Li, Zhu & Chen (2019) simply adopted the data transformation method developed by Seiford and Zhu (2002) to change undesirable outputs into desirable outputs. It is rather remarkable that undesirable outputs are neither scarce nor trivial in both academic research and real applications since undesirable outputs are inevitable and frequently appear in many production activities. It is well known that undesirable outputs have intrinsic features and that significantly affect evaluation results, and therefore we should focus on such outputs. Motivated by this idea, in this paper, we will explicitly extend the traditional FCA problem to situations in which DMUs convert inputs to produce desirable outputs and undesirable outputs simultaneously.

Table 1 DEA-based fixed cost allocation studies

Study	Application	Typical undesirable outputs	Are undesirable outputs considered
Cook & Kress (1999)	Numerical	--	NO
Beasley (2003)	Numerical	--	NO
Jahanshahloo et al. (2004)	Numerical	--	NO
Amirteimoori & Kordrostami (2005)	Numerical	--	NO
Cook & Zhu (2005)	Numerical	--	NO
Lin (2011a)	Numerical	--	NO
Lin (2011b)	Numerical	--	NO
Li et al. (2013)	Numerical	--	NO
Si et al. (2013)	Numerical	--	NO
Mostafaee (2013)	Numerical	--	NO
Du et al. (2014)	Numerical	--	NO
Lin et al. (2016)	Numerical	--	NO
Pendharkar (2018)	Numerical	--	NO
Zhu et al. (2019)	Numerical	--	NO
Li et al. (2009)	Car manufacturer	Pollution	NO
Khodabakhshi & Aryavash(2014)	Gas company	Pollution	NO
Dai et al. (2016)	Multinational firm	Not clear	NO
Lin & Chen (2016)	School	Not clear	NO
Yu et al. (2016)	Bank	Bad-debt	NO
Jahanshahloo et al. (2017)	Bank	Bad-debt	NO
Ding et al. (2018)	Gas company	Pollution	NO
Li, Zhu & Liang (2018)	Bank	Bad-debt	NO
Chu & Jiang (2019)	Car manufacturer	Pollution	NO
Ding et al. (2019)	Bank	Bad-debt	NO
Li, Zhu & Chen (2019)	Bank	Bad-debt	YES
Li, Zhu, Liang (2019)	Transportation	Pollution	NO

Li, Li, et al. (2019)	Steel and iron enterprise	Pollution	NO
An, Wang et al. (2020)	Bank	Bad-debt	NO
An, Wang & Shi (2020)	Bank	Bad-debt	NO
Chu et al. (2020)	Bank	Bad-debt	NO
Li, Yan, et al. (2020b)	Bank	Bad-debt	NO
Lin & Chen (2020)	Car manufacturer	Pollution	NO
Meng et al. (2020)	Bank	Bad-debt	NO

Note: *"--"means that the numerical example has nothing to do with undesirable outputs as it is a virtual dataset.*

In allocating a fixed cost across DMUs with undesirable outputs, we first examine the initial relative efficiency level by assuming the free disposability of inputs and the joint weak disposability of desirable outputs and undesirable outputs. We then take flexible FCA schemes into account to reexamine the potential efficiency scores, and results show that some feasible FCA schemes can always render all DMUs simultaneously efficient under a set of common weights. Further, we propose a new approach for determining a unique allocation scheme. The approach is based on a comprehensive satisfaction degree concept and a satisfaction degree bargaining game model. Finally, we present an empirical study to illustrate the proposed approach. In summation, this article makes a number of contributions to the existing literature. First, this study explicitly explores the FCA problem in an environment with undesirable outputs, while all previous studies ignored undesirable outputs. An inception is Li, Zhu & Chen (2019), which simply transformed undesirable outputs into their desirable version to follow a conventional framework. Since the transformation method ignores the weak disposability of undesirable outputs and different results and implications would be obtained, it indeed makes sense to explicitly consider undesirable outputs for such FCA problems. Second, this study develops a feasible and practical approach for determining a unique allocation scheme, rectifying the satisfaction degree concept and resulting in the development of a Nash bargaining game model. Third, the newly proposed FCA approach is used in an empirical study of real conditions to illustrate its applicability and usefulness.

The remainder of this paper is organized as follows. First, section 2 develops the mathematical methodology. Second, section 3 applies the proposed approach to an empirical case involving twenty-seven commercial bank branches. Finally, section 4 concludes the paper and provides additional notes.

2. Methodology

2.1. Preliminary

Consider a common situation involving n DMUs with each DMU being responsible for converting m inputs into s outputs. The input and output vectors of DMU_j $(j = 1, \dots, n)$ are written as $\boldsymbol{X}_j = (x_{1j}, \dots, x_{ij}, \dots, x_{mj})$ and $\boldsymbol{Y}_j = (y_{1j}, \dots, y_{rj}, \dots, y_{sj})$, respectively. For a certain $DMU_d (d = 1, \dots, n)$, Banker et al. (1984) proposed model (1) for calculating the maximal proportional input reduction while maintaining existing output production under the variable returns to scale (VRS) assumption:

$$
\begin{aligned}
& e_d^* = Min\,\theta \\
& s.t. \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{id}, i = 1, \dots, m \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rd}, r = 1, \dots, s \\
& \quad \sum_{j=1}^{n} \lambda_j = 1 \\
& \quad \lambda_j \ge 0, j = 1, \dots, n
\end{aligned}
\tag{1}
$$

where λ_j is the intensity variable corresponding with DMU_j $(j = 1, \dots, n)$. Intensity variables are used to construct a production frontier based on the current input consumption and output production of all observed DMUs. Then, each DMU can be projected onto that frontier and be evaluated by comparing itself to its ideal projection. The optimal objective function of model (1) e_d^* estimates the maximal reduction proportion of inputs while maintaining current levels of output production. It is easy to show that the optimal objective function of model (1) ranges from zero to unity. Furthermore, the evaluated DMU is identified as efficient when it cannot proportionally reduce any inputs, that is, $e_d^* = 1$. By contrast, the evaluated DMU is assumed to be inefficient when $e_d^* < 1$, which implies that it is possible to proportionally reduce current inputs while producing unreduced outputs.

Note in addition that when we drop the third constraint $\sum_{j=1}^{n} \lambda_j = 1$ from model (1), it reduces to the classic CCR model (Charnes et al. 1978) under the constant returns to scale (CRS) assumption. The CRS model will contain both pure technique efficiency and scale efficiency, while the VRS model computes only the pure technique efficiency that is free from scale effects. It is clear that the VRS model is more general in its mathematical formulation than the CRS model, and the VRS can be easily changed into the CRS model by removing one constraint or adding an additional constrained condition. However, Dyson et al. (2001) argued that adopting the VRS model might lead to a pitfall if it is used in situations in which there are no inherent scale effects. Therefore, we will develop mathematical models in the more general VRS case throughout this paper, but in the real applications, it is necessary to first test the data separately for scale effects and adopt the VRS assumption only when scale effects can be demonstrated. Scale effects can be a priori information in real applications, but if they are not a priori information, some procedures have also been developed in the literature for the purpose of testing for scale effects and returns to scale properties (Zhu & Shen 1995; Banker 1996; Simar & Wilson 2002; Wei & Yan, 2004; Banker & Natarajan 2011).

Model (1) considers only traditional intended outputs that are maximized as possible, while undesirable outputs such as pollution and bad debt are also generated frequently in various production processes (Färe et al. 1989; Seiford & Zhu 2002). As Murty et al. (2012) pointed out that the generation of undesirable outputs proceeds hand-in-hand with the processes of consumption and production, and it exhibits the inevitability of a certain minimal amount of the undesirable outputs.

With the consideration of undesirable outputs $\boldsymbol{B}_j = \left(b_{1j}, \dots, b_{pj}, \dots, b_{qj}\right)$ for DMU_j $(j = 1, \dots, n)$, a core task is to identify means of addressing undesirable outputs with desirable outputs, as the results would be very different depending on the used approach. The DEA literature contains many strands dealing with undesirable outputs, see, for example, treating undesirable outputs as inputs (Seiford & Zhu 2002), the directional distance function model (Chung et al. 1997), strong and weak disposability modelling (Färe & Grosskopf 2003; Kuosmanen 2005; Färe & Grosskopf 2009; Kuosmanen & Podinovski 2009), managerial and natural disposability modelling (Sueyoshi & Goto 2012; Song et al. 2012), by-production technology (Førsund 2009; Murty et al. 2012; Dakpo 2016; Førsund 2018; Boussemart et al. 2020; Fukuyama et al. 2020, 2021), eco-inefficiency (Chen & Delas 2012). We note that the strong and weak disposability assumptions of undesirable outputs and desirable outputs are the two most common used methods for dealing with the by-production mode in the literature (Zhou et al. 2012; Zhu et al. 2020; Zhu, Li, Li, et al. 2021; Li, Emrouznejad, et al. 2020; Li, Li, et al. 2020), although both have some weaknesses (Førsund 2009; Murty et al. 2012). A significant feature between the strong and weak disposability assumptions is whether undesirable outputs can be produced without damage or a subsequent cost for the desirable outputs (Kuosmanen 2005; Chen & Delmas 2012). If undesirable outputs can be freely generated without damage or a subsequent cost, implying that both inputs and outputs can change unilaterally without compromising each other, then undesirable outputs are assumed to be strongly disposable. In contrast, if the production of undesirable outputs indeed causes some damage or a subsequent cost for inputs or desirable outputs, implying that a reduction in undesirable outputs would also result in a reduction of desirable outputs simultaneously (Kuosmanen 2005; Kuosmanen & Podinovski 2009), then undesirable outputs are assumed to be weakly disposable.

Since undesirable outputs are usually not free from desirable outputs in many real managerial applications, such as the empirical application in a later section in which the undesirable bad-debts cannot be freely reduced without affecting incomes and changing banking operations, we develop our models assuming joint weak disposability, and the weakly disposable DEA model is indeed a useful empirical tool in the analysis of undesirable outputs (Färe & Grosskopf 2003; Kuosmanen 2009). It should be also noted that nowadays there are several approaches for dealing with the by-production mode, but it has not reached a consensus. Different approaches can be used for developing FCA approaches depending on the problem environment, and results would be hence different. According to Kuosmanen (2005) and Kuosmanen & Podinovski (2009), model (2) can be developed to calculate the maximal proportional input reduction of DMUs with desirable outputs and undesirable outputs:

$$\begin{aligned} \hat{e}_d^* &= Min\,\theta \\ s.t. &\sum_{j=1}^{n}\left(\lambda_j+\eta_j\right)x_{ij} \le \theta x_{id}, i=1,\dots,m \\ &\sum_{j=1}^{n}\lambda_j y_{rj} \ge y_{rd}, r=1,\dots,s \\ &\sum_{j=1}^{n}\lambda_j b_{pj} = b_{pd}, p=1,\dots,q \\ &\sum_{j=1}^{n}\left(\lambda_j+\eta_j\right)=1 \\ &\lambda_j,\eta_j \ge 0, j=1,\dots,n. \end{aligned} \tag{2}$$

Denote the production technology with inputs $\boldsymbol{X}$, desirable outputs $\boldsymbol{Y}$ and undesirable outputs $\boldsymbol{B}$ as $P(\boldsymbol{X})=\{(\boldsymbol{Y},\boldsymbol{B})|\boldsymbol{X}\text{ can produce }(\boldsymbol{Y},\boldsymbol{B})\}$. Model (2) assumes the free disposability of inputs and the joint weak disposability of desirable outputs and undesirable outputs. As indicated in Kuosmanen (2005) that, outputs are weakly disposable if $(\boldsymbol{Y},\boldsymbol{B})\in P(\boldsymbol{X})$ and $0\le\rho\le 1$ implies $(\rho\boldsymbol{Y},\rho\boldsymbol{B})\in P(\boldsymbol{X})$. The joint weak disposability of desirable and undesirable outputs allows, by definition, a proportional abatement of all of them. The VRS mathematical formulation of the corresponding technology requires including such non-uniform abatement factors. This can be done as indicated in Kuosmanen (2005) by modifying the production possibility set $\{(x,y,b) | \sum_{j=1}^{n} \beta_j x_{ij} \le x, \sum_{j=1}^{n} \beta_j \rho_j y_{rj} \ge y, \sum_{j=1}^{n} \beta_j \rho_j b_{pj} = b, \sum_{j=1}^{n} \beta_j = 1, \beta_j \ge 0\}$, where $0 \le \rho_j \le 1$ is the non-uniform proportional abatement factor of outputs that scales down both desirable and undesirable outputs by the same proportion, being consistent with Shephard (1970)'s definition. Through the simple algebraic operations of $\beta_j \rho_j = \lambda_j$ and $\beta_j(1-\rho_j) = \eta_j$, the modified production possibility set $\{(x,y,b) | \sum_{j=1}^{n} (\lambda_j + \eta_j) x_{ij} \le x, \sum_{j=1}^{n} \lambda_j y_{rj} \ge y, \sum_{j=1}^{n} \lambda_j b_{pj} = b, \sum_{j=1}^{n} (\lambda_j + \eta_j) = 1, \lambda_j, \eta_j \ge 0\}$ is used in the above model (2). The free disposability of inputs and desirable outputs is modeled by inequality constraints, the jointly weakly disposability of desirable outputs and undesirable outputs is modeled through the inclusion of abatement factors, and the undesirable outputs are not necessarily freely disposable due to the strict equality constraint. It is clear that model (2) always holds for Kuosmanen's (2005) production specification, as discussed and debated in Kuosmanen (2005), Färe & Grosskopf (2009) and Kuosmanen & Podinovski (2009).

Compared to the traditional BCC (Banker-Charnes-Cooper) formulation in model (1), here, the intensity variables are divided into the nondisposed intensity variable λ_j and the disposed intensity variable η_j. The inputs of DMU_j $(j = 1, \dots, n)$ are weighted by the sum of both the disposed intensity variable η_j and nondisposed intensity variable λ_j while the desirable and undesirable outputs of DMU_j $(j = 1, \dots, n)$ are only weighted by the nondisposed intensity variable λ_j. Model (2) would reduce to the CRS model by dropping the fourth constraint, and in that case, η_j is redundant and the weak disposability is satisfied by default. Note in addition that the weak disposability of desirable outputs and undesirable outputs can also be modeled in other formulations. One of important features in model (2) is that it allows for nonuniform abatement factors ρ_j across different units. Readers can refer to Kuosmanen (2005), Färe & Grosskopf (2009) and Kuosmanen & Podinovski (2009) for further discussions.

The following model (3) is a dual of model (2) in the multiplier formulation:

$$\begin{aligned} &\bar{e}_d^* = Max \sum_{r=1}^{s} u_r y_{rd} + \sum_{p=1}^{q} w_p b_{pd} + u_0 \\ &\quad s.t. \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{p=1}^{q} w_p b_{pj} + u_0 \le 0, j = 1, \dots, n \\ &\quad -\sum_{i=1}^{m} v_i x_{ij} + u_0 \le 0, \quad j = 1, \dots, n \\ &\quad \sum_{i=1}^{m} v_i x_{id} = 1 \\ &\quad u_r, v_i \ge 0, r = 1, \dots, s; i = 1, \dots, m; w_p \text{ and } u_0 \text{ are free}, p = 1, \dots, q. \end{aligned} \tag{3}$$

Here, u_r $(r = 1, \dots, s)$, v_i $(i = 1, \dots, m)$, w_p $(p = 1, \dots, q)$ and u_0 are dual variables derived from model (2) and are also unknown decision variables in model (3). Each dual variable measures the level of efficiency increments attributable to a unit increase/reduction in the corresponding input-output bundle. In particular, the first constraint corresponds to a nondisposed intensity variable λ_j and the second constraint corresponds to a disposed intensity variable η_j. According to the dual theory, it holds that $\hat{e}_d^* = \bar{e}_d^*$ for any DMU_d $(d = 1, \ldots, n)$.

2.2. Performance with flexible fixed cost allocation schemes

Now consider a problem environment in which a total fixed cost R is allocated across all n DMUs. Without any loss of generality, each DMU_j $(j = 1, \ldots, n)$ is supposed to provide a proportion of the total fixed cost R_j such that the following holds:

$$\sum\nolimits_{j=1}^{n} R_j = R, R_j \geq 0\,. \tag{4}$$

Equation (4) above represents a full cover condition guaranteeing that the individual allocated cost R_j $(j = 1, \ldots, n)$ takes the same value as the total fixed cost R. Since any allocation scheme satisfying the full cover condition is a feasible allocation scheme in the real world, considerable flexibility is available in the determination of a unique FCA scheme.

By affording a non-negative allocated cost R_j $(j = 1, \ldots, n)$ for each DMU_j $(j = 1, \ldots, n)$, the input-output bundle is changed for the DMUs. Therefore, the performance of all DMUs is affected regardless of whether the allocated cost is considered as a new independent input or a complement to existing inputs. In this work, we consider the allocated cost as an additional input, and readers can refer to Li et al. (2009), Dai et al. (2016) and Lin and Chen (2020) for another case in which the allocated cost is considered as a complement to existing inputs. To this end, model (5) is developed below to investigate the potential sources of post-allocation efficiency with the flexible allocation scheme $(R_1, \ldots, R_n)$:

$$\begin{aligned}
E_d^* = & Max \sum\nolimits_{r=1}^{s} u_r y_{rd} + \sum\nolimits_{p=1}^{q} w_p b_{pd} + u_0 \\
& s.t. \sum\nolimits_{r=1}^{s} u_r y_{rj} - \sum\nolimits_{i=1}^{m} v_i x_{ij} - v_{m+1} R_j + \sum\nolimits_{p=1}^{q} w_p b_{pj} + u_0 \leq 0, j = 1, \ldots, n \\
& \quad -\sum\nolimits_{i=1}^{m} v_i x_{ij} - v_{m+1} R_j + u_0 \leq 0, \quad j = 1, \ldots, n \\
& \quad \sum\nolimits_{i=1}^{m} v_i x_{id} + v_{m+1} R_d = 1 \\
& \quad \sum\nolimits_{j=1}^{n} R_j = R \\
& \quad u_r, v_i, R_j \geq 0, r = 1, \ldots, s; i = 1, \ldots, m; j = 1, \ldots, n \\
& \quad v_{m+1} > 0; w_p \text{ and } u_0 \text{ are free}, p = 1, \ldots, q.
\end{aligned} \tag{5}$$

It is well known that the DEA-based FCA approaches determine the allocation scheme by investigating the impact of allocated costs on efficiency scores (Cook & Kress 1999; Beasley 2003; Li et al. 2009; Lin 2011a; Du et al. 2014; An, Wang, et al. 2020). Therefore, the allocation of other DMUs' costs is needed for the computation of the efficiency of DMU$_d$ in above model (5). Model (5) is obtained from model (3) with two differences. On the one hand, the equation (4) of a total cost covered by n DMUs is added to model (3) to develop model (5). On the other hand, the impact of possible allocated costs is considered in the evaluated efficiency scores in model (5), which is implemented by replacing the inputs $\sum_{i=1}^{m} v_i x_{ij}$ by $\left(\sum_{i=1}^{m} v_i x_{ij} + v_{m+1} R_j\right)$. Here, an additional relative weight $v_{m+1} > 0$ is attached to the allocated costs. Further, model (5) can be linearized to model (6) below by substituting $r_j = v_{m+1} R_j (j = 1, \dots, n)$:

$$
\begin{aligned}
E_d^* = & Max \sum_{r=1}^{s} u_r y_{rd} + \sum_{p=1}^{q} w_p b_{pd} + u_0 \\
& s.t. \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - r_j + \sum_{p=1}^{q} w_p b_{pj} + u_0 \le 0, \ j = 1, \dots, n \\
& -\sum_{i=1}^{m} v_i x_{ij} - r_j + u_0 \le 0, \quad j = 1, \dots, n \\
& \sum_{i=1}^{m} v_i x_{id} + r_d = 1 \\
& \sum_{j=1}^{n} r_j = v_{m+1} R \\
& u_r, v_i, r_j \ge 0, r = 1, \dots, s; i = 1, \dots, m; j = 1, \dots, n \\
& v_{m+1} > 0; w_p \ and \ u_0 \ are \ free, p = 1, \dots, q.
\end{aligned}
\tag{6}
$$

Assume that $\left(u_r^*, v_i^*, w_p^*, r_j^*, v_{m+1}^*, u_0^*, \forall r, i, p, j\right)$ is the optimal solution of model (6) when it is solved for DMU_d $(d = 1, \dots, n)$. Then, a DMU_d's relative efficiency is calculated as $E_d^* = \sum_{r=1}^{s} u_r^* y_{rd} + \sum_{p=1}^{q} w_p^* b_{pd} + u_0^*$. It is clear that the optimal objective function E_d^* is smaller than one, and the optimum is reached if $\sum_{r=1}^{s} u_r\ y_{rd} - \sum_{i=1}^{m} v_i\ x_{id} - r_d + \sum_{p=1}^{q} w_p\ b_{pd} + u_0\ = 0$. Otherwise, the objective function can be further improved according to the first and third constraints of model (6). Following a similar practice used by Li et al. (2013), Li, Zhu & Chen (2019) and Ding et al. (2020), we easily find a set of efficient allocation schemes. For further details, we provide the following theorems.

Theorem 1. *The optimal objective function of model (6) is always one, and thus* $E_d^* = 1$ *for any* $d = 1, \dots, n$.
Proof: See Appendix A.

From Theorem 1, we find that any DMU can be efficient by determining a feasible FCA scheme and by selecting a set of optimal relative weights, which means that any proportional input reduction is impossible without proportionally reducing desirable outputs and undesirable outputs. This theorem also demonstrates that DMUs' efficiency levels can be affected by affording the fixed cost, and hence all DMUs have enough incentives to respond to certain FCA schemes according to the effects on their efficiency scores. Therefore, a certain FCA scheme is considered to be fair and acceptable when it can determine a maximal efficiency score for all DMUs.

Theorem 2. *There is always at least one feasible FCA scheme that can make all DMUs simultaneously efficient under a set of common weights.*
Proof: See Appendix B.

Theorem 2 extends Theorem 1 to a broader platform where all DMUs rather than an individual DMU can be efficient by taking allocated costs into account in an environment with undesirable outputs. Therefore, these efficient allocation schemes are considered to be fair and acceptable. According to the above theorems and conclusions, the efficient allocation set (EAS), which contains all allocation schemes that can make all DMUs simultaneously efficient under a set of common weights, can be expressed as shown in formula (7):

$$EAS = \left\{ R_j \left| \begin{array}{l} R_j = \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{p=1}^{q} w_p b_{pj} + u_0 \,,\ j = 1,\dots,n \\ R = \sum_{j=1}^{n} R_j ,\ R_j \geq 0,\ j = 1,\dots,n \\ -\sum_{i=1}^{m} v_i x_{ij} - R_j + u_0 \leq 0, \quad j = 1,\dots,n \\ u_r, v_i \geq 0, r = 1,\dots,s; i = 1,\dots,m; w_p \text{ and } u_0 \text{ are free}, p = 1,\dots,q \end{array} \right. \right\} \tag{7}$$

The first constraint of formula (7) can ensure that all DMUs are efficient simultaneously. By rethinking model (5) or equivalently model (6), we can find that the efficient status would be simultaneously achieved if $\sum_{r=1}^{s} u_r\ y_{rj} - \sum_{i=1}^{m} v_i\ x_{ij} - v_{m+1}R_j + \sum_{p=1}^{q} w_p\ b_{pj} + u_0 = 0\ (v_{m+1} > 0)$ is held for any $DMU_j\ (j = 1, \dots, n)$. Then its equivalent version $\sum_{r=1}^{s} u_r\ y_{rj} - \sum_{i=1}^{m} v_i\ x_{ij} - R_j + \sum_{p=1}^{q} w_p\ b_{pj} + u_0 = 0$ would be obtained by assuming $u_r = u_r/v_{m+1}$, $v_i = v_i/v_{m+1}$, $w_p = w_p/v_{m+1}$ and $u_0 = u_0/v_{m+1}$.

By comparing formula (7) with that developed by Li et al. (2013), two features are significant except for scalar u_0, which means that this work considers a VRS case while Li et al. (2013) studied a CRS case. First, undesirable outputs are involved and their relative weights $w_p(p = 1, \dots, q)$ are free of a sign, diverging from other measures whose relative weights are nonnegative. This applies because this article adopts the joint weak disposability of desirable outputs and undesirable outputs, and undesirable outputs are constrained by equalities, implying a free dual variable. Second, the third constraint, which corresponds with the disposed intensity variable η_j in the envelopment formulation, represents DMUs' abated outputs through the scaling down of activity levels. This constraint can ensure nonnegative aggregated outputs by linking to the first constraint as $\sum_{r=1}^{s} u_r y_{rj} + \sum_{p=1}^{q} w_p b_{pj} = \sum_{i=1}^{m} v_i x_{ij} + R_j - u_0 \geq 0$. It is remarkable that both of the above two features emerge due to concerns related to undesirable outputs. It thus indeed makes sense to explicitly address undesirable outputs in an FCA problem.

2.3. FCA using a satisfaction degree bargaining game approach

It is notable that the efficient allocation set given in formula (7) includes more than one feasible allocation scheme, and many concepts and methods have been developed to address the FCA problem on the basis of efficient allocation schemes (Li et al., 2013; Si et al., 2013; Li, Zhu & Chen 2019; Li, Li, et al. 2019; Chu et al., 2020). In this section, we adopt a satisfaction degree bargaining game approach to determine a unique FCA scheme. To this end, we first calculate the interval of allocated costs for all DMU_j $(j = 1, \dots, n)$. When DMU_d $(d = 1, \dots, n)$ is under consideration, then model (8) can be used to calculate the maximal/minimal allocated cost while all DMUs remain efficient with undesirable outputs:

$$
\begin{aligned}
& R_d^{\max} / R_d^{\max} = Max / Min\, R_d \\
& \quad R_j = \sum\nolimits_{r=1}^{s} u_r y_{rj} - \sum\nolimits_{i=1}^{m} v_i x_{ij} + \sum\nolimits_{p=1}^{q} w_p b_{pj} + u_0,\ j = 1, \dots, n \\
& \quad R = \sum\nolimits_{j=1}^{n} R_j,\ R_j \geq 0,\ j = 1, \dots, n \\
& \quad -\sum\nolimits_{i=1}^{m} v_i x_{ij} - R_j + u_0 \leq 0, \quad j = 1, \dots, n \\
& \quad u_r, v_i \geq 0, r = 1, \dots, s; i = 1, \dots, m; w_p \text{ and } u_0 \text{ are free}, p = 1, \dots, q.
\end{aligned}
\tag{8}
$$

The constraint of model (8) is just the efficient allocation set included in Formula (7). Therefore, the optimal objective function of model (8) determines an allocation interval for DMU_d. Namely, the final allocated cost is no more than the maximum R_d^{max} and no less than the minimum R_d^{min} since otherwise the fully efficient status of some DMUs would be sacrificed.

In practical and rational applications, each DMU is intrinsically willing to reduce its allocated cost. Therefore, each DMU is more willing to afford a cost share closer to the minimum value of R_d^{min} or is more willing to afford a cost share farther from the maximum R_d^{max}. Therefore, we can follow Li et al. (2013) and Li, Yan, et al. (2020) in defining two different satisfaction degree concepts from the given allocation scheme $(R_1, \dots, R_n)$.

Definition 1. *The optimistic satisfaction degree of* DMU_d $(d = 1, \dots, n)$ *for the FCA scheme is* $\rho_d^o = \frac{R_d^{max} - R_d}{R_d^{max} - R_d^{min}}$.

Definition 2. *The pessimistic satisfaction degree of* DMU_d $(d = 1, \dots, n)$ *for the FCA* *scheme is* $\rho_d^p = \frac{R_d^{min} - R_d}{R_d^{max} - R_d^{min}}$.

As a comparison, the first definition focuses mainly on the maximal allocated cost R_d^{max} while the second definition focuses mainly on the minimal allocated cost R_d^{min}. The first definition of satisfaction degree is optimistic since it measures how different the allocated cost is from the worst result; and the greater the distance is, the higher the degree of satisfaction. The optimistic degree of satisfaction ranges from zero to unity, and each $DMU_d (d = 1, \dots, n)$ will prefer a cost share that is related to a greater optimistic degree of satisfaction. By contrast, the second satisfaction degree is pessimistic since it measures how different the allocated cost is from the best result. The pessimistic degree of satisfaction ranges from -1 to 0, and similarly an FCA scheme with a greater degree of pessimistic satisfaction is preferred to another one with a smaller degree of pessimistic satisfaction.

In fact, previous studies have only considered either optimistic or pessimistic degrees of satisfaction (Li et al. 2013; Ding et al. 2019; Zhu et al. 2019; Chu et al. 2020). In this work, we believe that neither the optimistic degree of satisfaction nor the pessimistic degree of satisfaction can provide enough information on satisfaction with allocation schemes. Based on this observation, we further define a comprehensive degree of satisfaction that is a convex combination of the optimistic satisfaction degree and the pessimistic satisfaction degree.

Definition 3. *The comprehensive satisfaction degree of* DMU_d $(d = 1, \dots, n)$ *for the FCA scheme is* $\rho_d^c = k\rho_d^o + (1-k)\rho_d^p, k \in [0,1]$.

To simplify this, we express the comprehensive satisfaction degree as k-optimistic. Decision makers can incorporate information on optimistic and pessimistic attitudes into the comprehensive satisfaction degree. Therefore, considering the comprehensive satisfaction degree can generate more sophisticated results. It is easy to verify that $\rho_d^c = \frac{kR_d^{max} + (1-k)R_d^{min} - R_d}{R_d^{max} - R_d^{min}}$. Hence, a larger k implies a more optimistic attitude. When $k = 0$, then DMU_d is fully pessimistic since the minimal allocated cost R_d^{min} has a significant impact on the comprehensive satisfaction degree. By contrast, when $k = 1$, DMU_d is fully optimistic.

In determining a unique FCA scheme, we believe that the most preferred allocation scheme is associated with the maximal comprehensive satisfaction degree for all DMUs. To this end, we develop a satisfaction degree bargaining game model as shown below. It is notable that there are many solution concepts for game-based cost allocation problems, such as Shapley value, nucleolus, kernel, core, τ-value, cost gap method, etc. (Tijs and Driessen 1986; Nakabayashi & Tone 2006; Lozano 2012; Li, Emrouznejad, et al. 2020). It is possible for decision makers to select the game solution depending on the context in which the problem is located, and the results might be different depending on the selected solution. Here in this article, we put our main focus on developing a FCA approach with undesirable outputs, but readers should recognize the possibility of using other solutions.

Within the bargaining game process, each DMU is considered to be a player. The relative weights and allocation schemes of the EAS are players' strategies and the comprehensive satisfaction degree is considered as a payoff. In addition, the breakdown point is linked to the maximal allocated cost $R_j^{max}(j = 1, \dots, n)$, which represents the minimal comprehensive satisfaction degree $\hat{\rho}_j^c = \frac{kR_j^{max}+(1-k)R_j^{min}-R_j^{max}}{R_j^{max}-R_j^{min}} = k - 1$. In addition, the breakdown point increases along with k, which implies that different allocation results should be determined from different levels of optimistic attitudes.

$$
\begin{aligned}
& Max\prod_{j=1}^{n}\left(\rho_j^c - \hat{\rho}_j^c\right) \\
& \quad R_j = \sum\nolimits_{r=1}^{s} u_r y_{rj} - \sum\nolimits_{i=1}^{m} v_i x_{ij} + \sum\nolimits_{p=1}^{q} w_p b_{pj} + u_0,\ j = 1,\dots,n \\
& \quad R = \sum\nolimits_{j=1}^{n} R_j,\ R_j \ge 0,\ j = 1,\dots,n \\
& \quad -\sum\nolimits_{i=1}^{m} v_i x_{ij} - R_j + u_0 \le 0,\quad j = 1,\dots,n \\
& \quad \rho_j^c = \frac{kR_j^{max} + \left(1-k\right)R_j^{min} - R_j}{R_j^{max} - R_j^{min}},\ k \in \left[0,1\right],\ j = 1,\dots,n \\
& \quad u_r, v_i \ge 0, r = 1,\dots,s; i = 1,\dots,m; w_p\ and\ u_0\ are\ free,\ p = 1,\dots,q.
\end{aligned}
\tag{9}
$$

Following a practice similar to that adopted by Wu et al. (2009) and Wu et al. (2013), we can easily demonstrate that the feasible region of model (9) is a convex set. To this end, we have the following Theorem 3.

Theorem 3. *Denote S as all the restrictions in above model* (9)*, i.e., S is the feasible regions of* $(u_1, \dots, u_s, v_1, \dots, v_m, w_1, \dots, w_q, u_0)$. *Then, S is a convex set.*

Proof. See Appendix C.

Further, for a game problem like the considered satisfaction degree Nash bargaining game, Nash (1950) presented a Nash bargaining solution characterized by four properties: Pareto efficiency, invariance with respect to affine transformation, independence of irrelevant alternatives, and symmetry. According to the Nash bargaining theorem, if the feasible region S is convex, then there would be only one solution satisfying the four properties. Therefore, we have the important lemma below, which can be easily verified by referring to Nash (1950, 1953) and Wu et al. (2009) and we omit the proof here.

Lemma: *There exists only one solution satisfying the satisfaction degree Nash bargaining game.*

For the traditional bargaining problem, Nash (1950, 1953) has shown that there exists a unique solution called the Nash bargaining solution, and the solution can be obtained by solving the above maximization problem in model (9), which can maximize the whole comprehensive satisfaction degree of all DMUs. The above model is a nonlinear programming model, but it can be immediately solved using standard modules with tools such as MATLAB. Then, the unique Nash bargaining solution of model (9) determines a unique FCA scheme $(R_1^*, \dots, R_n^*)$.

3. Illustration

In this section, we apply the proposed approach to a real case of commercial bank activities originally described by Li, Zhu & Chen (2019) and studied by Chu et al. (2020). In more detail, there is a commercial bank with twenty-seven branches in China. The bank's headquarters decide to allocate 8000 units (unit: 10 thousand Chinese Yuan (CNY)) of maintenance charges for information and technology services across these branches. While Li, Zhu & Chen (2019) presented a two-staged production system in their work and dealt with undesirable outputs via a data transformation method, we ignore the internal structure while maintaining our main interest in undesirable outputs. The approach proposed in this paper is the only one that can explicitly deal with the FCA problem with undesirable outputs.

In this case, each branch is regarded as a DMU. In addition, there are three inputs and three outputs, as given in Table 2. The three inputs are labor (x_1), fixed assets (x_2) and operation costs except for labor costs (x_3); and the three outputs are interest income (y_1), noninterest income (y_2) and bad debt (z_1). The input-output data are given in Table 3. Note in particular that the bad debt is a jointly produced undesirable output of two other desirable outputs: interest income and noninterest income. A problem arises in the allocation of the total maintenance charges for information and technology services (8000 units) across the 27 bank branches. Keep in mind that we also consider the allocated cost as an additional input that differs from existing inputs for simplification. In this section we will conduct the analysis under the VRS case since these twenty-seven bank branches have relatively different input usage and output production, while the proposed allocation model can be theoretically used under the CRS assumption, and for readers' interest we also provide the CRS results in Appendix D. In fact, by comparing efficiency scores under the CRS and VRS cases shown in later tables we find that almost one half of all bank branches have scale inefficiencies, implying that it is more appropriate to conduct the empirical analysis under the VRS case.

Table 2 Notations of input-output measures

Input/output	Variable	Unit	Notation
Inputs	Labor	Person	x_1
	Fixed asset	10 thousand CNY	x_2
	Operations costs other than labor costs	10 thousand CNY	x_3
Desirable outputs	Interest income	10 thousand CNY	y_1
	Non-interest income	10 thousand CNY	y_2
Undesirable outputs	Bad debt	10 thousand CNY	z_1

Table 3 Input-output data for the twenty-seven bank branches

DMUs	x_1	x_2	x_3	y_1	y_2	z_1
1	25	619	538	2947	913	224
2	27	419	489	3138	478	516
3	40	1670	1459	5494	1242	877
4	42	2931	1497	3144	870	1138
5	52	2587	797	6705	854	618
6	45	2181	697	8487	1023	2096
7	33	989	1217	4996	767	713
8	107	6277	2189	21265	6282	6287
9	88	3197	949	8574	1537	1739
10	146	6222	1824	21937	5008	3261
11	57	1532	2248	8351	1530	2011
12	42	1194	1604	5594	858	1203
13	132	5608	1731	15271	4442	2743
14	77	2136	906	10070	2445	1487
15	43	1534	438	4842	1172	1355
16	43	1711	1069	6505	1469	1217
17	59	3686	820	6552	1209	1082
18	33	1479	2347	8624	894	2228
19	38	1822	1577	9422	967	1367
20	162	5922	2330	18700	4249	6545
21	60	2158	1153	10573	1611	2210
22	56	2666	2683	10678	1589	1834
23	71	2969	1521	8563	905	1316
24	117	5527	2369	15545	2359	2717
25	78	3219	2738	14681	3477	3134
26	51	2431	741	7964	1318	1158

27	48	2924	1561	11756	2779	1398

We first investigate the efficiency levels of these bank branches. When we do not recognize the undesirable feature of bad debt and consider it as a kind of desirable output, there would be 14 efficient branches and the lowest efficiency score is found for branch 23 at 0.6364 as shown in the second column of Table 4. Further, different results are generated when we immediately deal with the bad debts by assuming the free disposability of inputs and the joint weak disposability of desirable outputs and undesirable outputs. As seen from the third and fourth columns of Table 4, all DMUs have an efficiency score that is no less than the previous one. This result demonstrates the need to manage undesirable outputs and it indeed makes sense and difference to explicitly deal with undesirable outputs. Further, all DMUs are fully efficient when the flexible FCA scheme is considered as illustrated by the last column of Table 4 and by Theorem 1.

Table 4 Preliminary efficiency results

DMUs	Model (1)	Model (2)	Model (3)	Model (6)
1	1.0000	1.0000	1.0000	1.0000
2	1.0000	1.0000	1.0000	1.0000
3	0.7719	0.7719	0.7719	1.0000
4	0.7297	0.7297	0.7297	1.0000
5	0.7716	1.0000	1.0000	1.0000
6	1.0000	1.0000	1.0000	1.0000
7	0.8761	0.9108	0.9108	1.0000
8	1.0000	1.0000	1.0000	1.0000
9	0.7616	0.7814	0.7814	1.0000
10	1.0000	1.0000	1.0000	1.0000
11	1.0000	1.0000	1.0000	1.0000
12	0.7941	0.7941	0.7941	1.0000
13	0.9003	1.0000	1.0000	1.0000
14	1.0000	1.0000	1.0000	1.0000
15	1.0000	1.0000	1.0000	1.0000
16	0.8700	0.8700	0.8700	1.0000
17	0.7343	0.7822	0.7822	1.0000
18	1.0000	1.0000	1.0000	1.0000
19	1.0000	1.0000	1.0000	1.0000
20	1.0000	1.0000	1.0000	1.0000
21	1.0000	1.0000	1.0000	1.0000
22	0.8255	0.8300	0.8300	1.0000
23	0.6364	0.6620	0.6620	1.0000
24	0.7236	0.7312	0.7312	1.0000
25	1.0000	1.0000	1.0000	1.0000

26	0.9278	1.0000	1.0000	1.0000
27	1.0000	1.0000	1.0000	1.0000

Since there are multiple efficient allocation schemes as given in Theorem 2 and Formula (7), we further calculate the interval of possible allocated costs while keeping all DMUs efficient. If the minimal allocated cost is equal to the maximal allocated cost, then a unique allocation amount would be determined. We can learn from the second and third columns of Table 5 that all DMUs present a relatively large allocation interval ranging from zero for most DMUs to a relatively large value. This result implies that all bank branches have flexibility in paying for the maintenance charge for information and technology services. We then adopt the satisfaction degree concept and use the satisfaction degree bargaining game approach to determine the final allocation scheme.

Since the Nash bargaining gain model proposed in this paper is based on the comprehensive satisfaction degree, which is a convex combination of optimistic and pessimistic degrees of satisfaction, the value of k should affect the allocation results. Decision makers can incorporate preferred information on optimistic and pessimistic attitudes into the comprehensive satisfaction degree, and hence change the value of k and generate different allocation results. In this section, we consider a special case where k=0.5 as an illustration, which means that the comprehensive satisfaction degree is the arithmetic mean of the optimistic satisfaction degree and the pessimistic satisfaction degree. As the breakdown point is only related to the value of k, it is always defined as -0.5 for all DMUs in our empirical illustration. Therefore, the objective function of the Nash bargaining game model is $\prod_{j=1}^{n}(\rho_j^c + 0.5)$.

Table 5 Calculation results and allocation results

DMUs	R_j^{min}	R_j^{max}	ρ_j^c	R_j^*	Li, Zhu & Chen (2019)	Chu et al. (2020)
1	0.0000	577.1865	0.0459	262.1154	134.4306	84.1034
2	0.0000	594.4732	0.0823	248.2889	61.7595	94.3866
3	0.0000	405.9803	-0.0102	207.1441	150.8768	164.7918
4	0.0000	375.2481	0.2174	106.0278	47.8869	105.3818
5	0.0000	486.0476	0.0175	234.5133	125.6671	193.2387
6	0.0000	730.6505	-0.0102	372.8013	195.6915	267.4747
7	0.0000	477.1937	0.0482	215.6030	85.3113	148.3827
8	0.0000	1657.0593	0.0439	755.7614	1118.2562	688.2128
9	0.0000	421.3070	-0.0102	214.9643	214.2724	263.6254
10	0.0000	1118.9466	0.0366	518.5040	877.1561	653.8042
11	0.0000	508.1724	0.1282	188.9192	170.1160	262.2932

12	0.0000	457.8628	0.1168	175.4365	73.7728	173.1905
13	0.0000	942.2770	0.1278	350.6987	708.5666	463.3684
14	63.0262	651.3131	0.0011	356.5208	410.7174	299.9504
15	76.1476	575.6960	0.0441	303.8858	188.8462	155.3719
16	156.6240	421.1089	-0.0102	291.5727	224.9445	198.2271
17	0.0000	451.5719	-0.0102	230.4065	174.7134	197.1552
18	0.0000	666.3405	0.1501	233.1743	88.6087	273.5019
19	39.4108	568.1979	0.0030	302.2379	153.7766	280.2254
20	0.0000	1495.5025	0.2558	365.2215	646.7800	622.8994
21	162.6180	564.2589	-0.0102	367.5480	279.5816	326.2304
22	0.0000	478.0152	0.0894	196.2524	187.6059	322.5401
23	0.0000	341.5584	0.0360	158.4847	90.9078	255.9596
24	0.0000	564.8941	0.1351	206.1117	329.7382	470.3725
25	0.0000	733.8698	-0.0102	374.4439	540.7023	454.1207
26	129.2969	528.2120	0.0124	323.8271	229.0715	236.9063
27	74.0810	851.5954	0.0300	439.5348	490.2422	344.2850

When assessing the satisfaction degree Nash bargaining game model with MATLAB 2018a, we have an optimal function and a series of optimal solutions, which are associated with satisfaction degrees and allocated costs across all bank branches, as shown in Table 5. It can be seen from Table 5 that the allocated costs range from a minimum of 106.0278 for branch 4 to a maximum of 755.7614 for branch 8. This result is similar with the results of Li, Zhu & Chen (2019) and Chu et al. (2020), according to whom the highest costs of 1118.2562 and 668.2128 are found for branch 8 while the lowest costs of 47.8869 and 84.1034 are found for branches 4 and 1, respectively. Further, it shows a little similar tendency across all branches, however, the allocated cost in this paper is largely different from that of Li, Zhu & Chen (2019) and Chu et al. (2020), as shown in Table 5 and Figure 1. This result may be attributed to the fact that the examined problem involves undesirable outputs while only the current work explicitly considers undesirable outputs and their intrinsic features with the proposed allocation approach. Further, the gap between the maximal and minimal allocated costs across all DMUs is 649.7336, which is smaller than the value 1070.3693 given by Li, Zhu & Chen (2019) and a little larger than that of Chu et al. (2020) by 604.1094. Since a smaller gap implies the presence of more fairness, this result suggests that the explicit consideration of undesirable outputs may generate fairer allocation results with less implementation difficulty and organizational resistance.

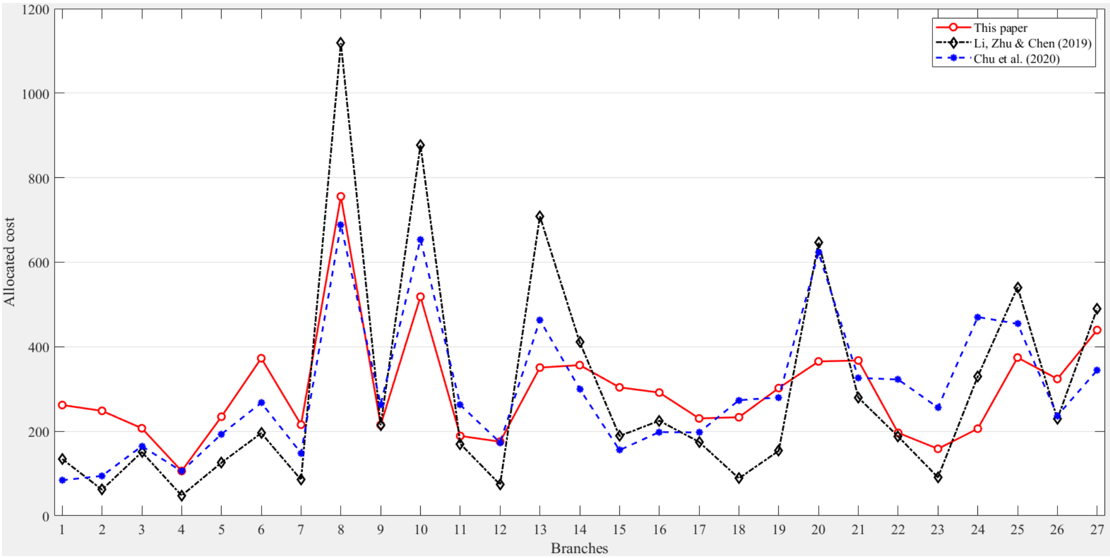

Figure 1 Allocation results with different approaches

Further, the allocation results reveal a number of other features. First, the simultaneously efficient assumption is indeed upheld. In fact, considering the allocated cost as the fourth input and applying model (2) or model (3) determines the full efficiency of one for each DMU. Second, the proposed approach can ensure positive cost sharing for all DMUs. Third, all DMUs will have benefits relative to the worst and maximal allocated cost $R_j^{max}(j = 1, \ldots, n)$, that is, the final allocated cost R_j^* is strictly smaller than R_j^{max}; otherwise, the objective function of model (9) would be zero if $R_j^* = R_j^{max}$ for some DMUs and in turn it would not be an optimum. In addition, the final allocation scheme is always unique due to the bargaining game features, and this explanation is theoretically supported by game theory. This feature is very important for the studied problem since the deep logic behind the FCA problem shows that multiple possibilities can occur. In addition, our analysis shows that the allocation results are significantly different from the initial efficiencies, meaning that the proposed approach does not necessarily compromise DMUs with higher efficiencies. Li, Zhu & Chen (2019) consider this phenomenon to be a very important advantage of DEA-based FCA approaches. In addition, since the proposed bargaining game model is based on the comprehensive satisfaction degree, which is a combination of the optimistic satisfaction degree and the pessimistic satisfaction degree, the final allocation results are determined from the perspective of both the maximal and minimal allocated costs. Therefore, a DMU's comprehensive satisfaction degree is positive when it is allocated a cost share similar to the minimal cost relative to the maximal cost. This result is intuitive. DMUs trade off their baselines (*i.e.*, maximal costs) and ideal goals (*i.e.*, minimal costs) and are negatively satisfied with an allocation scheme when it must offer a cost share similar to the maximal cost, meaning that they have made more concessions. More importantly, the proposed approach considers the bargaining game played by all DMUs, and the derived allocation scheme serves as a Nash bargaining equilibrium for maximizing the comprehensive satisfaction degree across all DMUs, rendering the allocation scheme fair and acceptable.

Figure 1 shows that the different approaches for allocating cost are providing different results. Further, we investigate the difference from a statistical point of view. Since the empirical study is the only FCA problem with undesirable outputs in the literature, we can compare our results with similar methods only on this example. Given the total fixed cost, the average allocated cost would be exactly R/n regardless of which approach is used. Therefore, we will investigate the difference by comparing variance instead of mean value. The null hypothesis for the F-test is that there isn't any statistically significant difference between this paper and similar approaches, that is, the allocation in this paper is not statistically significant different from those by Li, Zhu & Chen (2019) and Chu et al. (2020). As Table 6 shows that this paper provides different allocations with respect to Li, Zhu & Chen (2019) under 1% significance level, while there is no statistically significant difference between this paper and Chu et al. (2020). Note in addition that the allocation scheme is generated by mathematical optimization models, even allocation schemes that are not statistically significant different are indeed different if these allocation schemes are not identical.

Table 6 Two-sample F-test between this paper and similar approaches

	Mean	Variance	F-statistics	P-value
This paper	296.2963	17169.72		
Li, Zhu & Chen (2019)	296.2963	72731.61	0.2361	0.0002
Chu et al. (2020)	296.2963	27330.28	0.6282	0.1213

Based on the above observations and discussion, we can conclude that it indeed makes sense to explicitly address undesirable outputs and their intrinsic characteristics in the FCA problem. It is of vital importance to immediately develop FCA approaches for cases involving undesirable outputs, and for this purpose, the approach proposed in this paper is useful and of practical value.

4. Conclusions

This paper extends the traditional FCA problem to situations in which DMUs generate both desirable outputs and undesirable outputs simultaneously. To this end, it immediately deals with undesirable outputs by assuming the joint weak disposability of desirable outputs and undesirable outputs, and flexible FCA schemes are taken into account to investigate the efficiency evaluation processes. Our analysis demonstrates the existence of multiple efficient allocation schemes. To determine a unique and fair allocation scheme, this paper rectifies the satisfaction degree concept and further develops a Nash bargaining game approach based on the comprehensive satisfaction degree. Finally, an empirical study of twenty-seven bank branches is provided to illustrate the usefulness and efficacy of the proposed approach.

This paper can be further extended in several respects. First, it would be of vital significance to develop resource allocation approaches considering undesirable outputs given an ever-increasing awareness of environmental concerns. Second, although this paper takes undesirable outputs into account, the degree of undesirable outputs or desirable outputs can be further incorporated. For example, neutral outputs such as customers being less satisfied with banking services are less desirable than desirable outputs (e.g., satisfied customers) but are more desirable than undesirable outputs (e.g., unsatisfied customers). Thus, future work can attempt to incorporate relations and preferred information among desirable, neutral and undesirable outputs into the development of fixed cost and resource allocation approaches. Third, noting that the existing FCA studies are based on efficiency scores, future work can develop approaches based on efficiency rankings. Fourth, the Nash bargaining solution is just one alternative solution, so the approach to determine the final FCA plan could be extended based on other game solutions considering the context in which the application problem is located. Lastly, there exist several approaches dealing with undesirable outputs, and it is clear that the results might be very different depending on the selected approach. Future research can be developed for different approaches such as by-production pollution-generating technologies of Murty et al. (2012) and Dakpo (2016).

Acknowledgement

The authors would like to thank editors and two anonymous reviewers for their kind work and insightful comments.

Disclosure Statement

No potential conflict of interest was reported by the authors.

Funding

This research was financially supported by the National Natural Science Foundation of China (Nos. 71901178, 71904084, 71725001 and 71910107002), the Natural Science Foundation for Jiangsu Province (No. BK20190427), the Social Science Foundation of Jiangsu Province (No. 19GLC017), and the Fundamental Research Funds for the Central Universities (No. NR2019003).

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Appendix

Appendix A

Theorem 1. *The optimal objective function of model (6) is always one, and thus* $E_d^* = 1$ *for any* $d = 1, \dots, n$.

Proof: The proof is simple adaptation of the existing ones in Li et al. (2013), Li, Zhu & Chen (2019) and Ding et al. (2020).

First, it is clear that the optimal objective function of model (6) is no more than one since $\sum_{r=1}^{s} u_r y_{rd} + \sum_{p=1}^{q} w_p b_{pd} + u_0 \leq \sum_{i=1}^{m} v_i x_{id} + r_d = 1$.

Second, we show that the optimal objective function of model (6) can reach one. To this end, consider $\bar{v}_i = 0 (i = 1, \dots, m)$, $\bar{v}_{m+1} = \sum_{j=1}^{n} y_{sj}/2R y_{sd} + \sum_{j=1}^{n} b_{qj}/2R b_{qd}$, $\bar{u}_s = 1/2y_{sd}$, $\bar{u}_r = 0 (r = 1, \dots, s; r \neq s)$, $\bar{u}_0 = 0$, $\bar{w}_q = 1/2b_{qd}$, $\bar{w}_p = 0 (p = 1, \dots, q; p \neq q)$ and $\bar{r}_j = y_{sj}/2y_{sd} + b_{qj}/2b_{qd} \, (j = 1, \dots, n)$. Then, it is easy to verify that $\bar{\alpha} = (\bar{v}_i, \bar{v}_{m+1}, \bar{u}_r, \bar{u}_0, \bar{w}_p, \bar{r}_j, \forall i, r, p)$ is a feasible solution of model (6) since it satisfies all constraints in model (6). Therefore, the optimal objective function of model (6) is no less than that with $\bar{\alpha} = (\bar{v}_i, \bar{v}_{m+1}, \bar{u}_r, \bar{u}_0, \bar{w}_p, \bar{r}_j, \forall i, r, p)$, namely, $E_d^* \geq E_d^*(\bar{\alpha}) = 1$.

To sum up, the optimal objective function of model (6) is exactly one. Furthermore, the outputs labeled as *s* and q are randomly selected, that is, there exists at least one feasible solution that can achieve the optimal objective function model (6) of one. Therefore, the optimal objective function model (6) is always one. This completes the proof of Theorem 1.

Appendix B

Theorem 2. *There is always at least one feasible FCA scheme that can make all DMUs simultaneously efficient under a set of common weights.*

Proof: The proof follow Li et al. (2013). Let us consider again the feasible solution $\bar{\alpha} = (\bar{v}_i, \bar{v}_{m+1}, \bar{u}_r, \bar{u}_0, \bar{w}_p, \bar{r}_j, \forall i, r, p)$ in the proof process of Theorem 1. In addition, we can learn from Theorem 1 that the efficiency score of DMU_d is one, and hence DMU_d is efficient.

In addition, we can easily find the following results that can be used to characterize other DMUs' efficiency scores.

$$E_j^* \geq E_j(\bar{\alpha}) = \frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{p=1}^{q} w_p b_{pj} + u_0}{\sum_{i=1}^{m} v_i x_{ij} + r_j} = \frac{y_{sj}/2y_{sd} + b_{qj}/2b_{qd} + 0}{0 + y_{sj}/2y_{sd} + b_{qj}/2b_{qd}} = 1, \forall \mathrm{j} \neq \mathrm{d} \qquad \text{(B1)}$$

Therefore, we have $E_j^* \geq 1$ for all $DMU_j (j = 1, \dots, n)$.

In addition, it always holds that $-\sum_{i=1}^{m} v_i x_{ij} - r_j + u_0 = -y_{sj}/2y_{sd} - b_{qj}/2b_{qd} \leq 0 (j = 1, \dots, n)$, and the efficiency of any DMU is no more than one since $\sum_{r=1}^{s} u_{rj} + \sum_{p=1}^{q} w_p b_{pj} + u_0 - \sum_{i=1}^{m} v_i x_{ij} + r_j \leq 0$, namely, $E_j^* \leq 1$ for all DMU_j $(j = 1, \dots, n)$.

To sum up, we have $E_j^* = 1$ for all DMU_j $(j = 1, \dots, n)$, which means that all DMUs can be simultaneously efficient under a set of common weights $(\bar{v}_i, \bar{v}_{m+1}, \bar{u}_r, \bar{u}_0, \bar{w}_p, \forall i, r, p)$ and the feasible allocation scheme $\bar{r}_j = y_{sj}/2y_{sd} + b_{qj}/2b_{qd}$ $(j = 1, \dots, n)$. This completes the proof of Theorem 2.

Appendix C

Theorem 3. *Denote S as all the restrictions in above model* (9)*, i.e., S is the feasible regions of* $(u_1, \dots, u_s, v_1, \dots, v_m, w_1, \dots, w_q, u_0)$*. Then, S is a convex set.*

Proof. Suppose that both $\pi_1 = (\rho_{1j}^c, R_{11}, \dots, R_{1n}, u_{11}, \dots, u_{1s}, v_{11}, \dots, v_{1m}, w_{11}, \dots, w_{1q}, u_{10})$ and $\pi_2 = (\rho_{2j}^c, R_{21}, \dots, R_{2n}, u_{21}, \dots, u_{2s}, v_{21}, \dots, v_{2m}, w_{21}, \dots, w_{2q}, u_{20})$ belong to *S*. For any $\lambda \in [0,1]$, we have the following:

$$\begin{aligned}(1-\lambda)R_{1j} + \lambda R_{2j} &= (1-\lambda)\left(\sum_{r=1}^{s} u_{1r} y_{rj} - \sum_{i=1}^{m} v_{1i} x_{ij} + \sum_{p=1}^{q} w_{1p} b_{pj} + u_{10}\right) \\ &\quad + \lambda\left(\sum_{r=1}^{s} u_{2r} y_{rj} - \sum_{i=1}^{m} v_{2i} x_{ij} + \sum_{p=1}^{q} w_{2p} b_{pj} + u_{20}\right) \\ &= \sum_{r=1}^{s}\left[(1-\lambda)u_{1r} + \lambda u_{2r}\right] y_{rj} - \sum_{i=1}^{m}\left[(1-\lambda)v_{1i} + \lambda v_{2i}\right] x_{ij} \\ &\quad + \sum_{p=1}^{q}\left[(1-\lambda)w_{1p} + \lambda w_{2p}\right] b_{pj} + \left[(1-\lambda)u_{10} + \lambda u_{20}\right]\end{aligned} \tag{C1}$$

$$R = (1-\lambda)R + \lambda R = (1-\lambda)\sum_{i=1}^{m} R_{1j} + \lambda \sum_{i=1}^{m} R_{2j} = \sum_{i=1}^{m}\left[(1-\lambda)R_{1j} + \lambda R_{2j}\right] \tag{C2}$$

$$\begin{aligned}&-\sum_{i=1}^{m}\left[(1-\lambda)v_{1i} + \lambda v_{2i}\right] x_{ij} - \left[(1-\lambda)R_{1j} + \lambda R_{2j}\right] R_j + \left[(1-\lambda)u_{10} + \lambda u_{20}\right] \\ &= (1-\lambda)\left(-\sum_{i=1}^{m} v_{1i} x_{ij} - R_{1j} + u_{10}\right) - \lambda\left(\sum_{i=1}^{m} v_{2i} x_{ij} - R_{2j} + u_{20}\right) \le 0 + 0 = 0\end{aligned} \tag{C3}$$

$$\begin{aligned}(1-\lambda)\rho_{1j}^c + \lambda \rho_{2j}^c &= (1-\lambda)\frac{kR_j^{max} + (1-k)R_j^{min} - R_{1j}}{R_j^{max} - R_j^{min}} + \lambda \frac{kR_j^{max} + (1-k)R_j^{min} - R_{2j}}{R_j^{max} - R_j^{min}} \\ &= \frac{kR_j^{max} + (1-k)R_j^{min} - \left[(1-\lambda)R_{1j} + \lambda R_{2j}\right]}{R_j^{max} - R_j^{min}}.\end{aligned} \tag{C4}$$

Together with (C1)-(C4), we have $(1-\lambda)\pi_1 + \lambda\pi_2 \in S$. Therefore, *S* is a convex set. This completes the proof of Theorem 3.

Appendix D

Table A Results under the CRS assumption

DMUs	Model (1)	Model (2)	Model (3)	Model (6)	R_j^{min}	R_j^{max}	ρ_j^c	R_j^*
1	1.0000	1.0000	1.0000	1.0000	0.0000	250.1171	111.2690	0.0551
2	1.0000	1.0000	1.0000	1.0000	15.0149	121.0986	76.0201	-0.0751
3	0.7117	0.7196	0.7196	1.0000	0.0000	249.9209	142.1954	-0.0690

4	0.4414	0.4414	0.4414	1.0000	0.0000	173.4954	2.8839	0.4834
5	0.6926	1.0000	1.0000	1.0000	0.0000	346.4381	121.5028	0.1493
6	1.0000	1.0000	1.0000	1.0000	11.9937	450.4135	249.6891	-0.0422
7	0.8684	0.9099	0.9099	1.0000	0.0000	209.7754	108.8794	-0.0190
8	1.0000	1.0000	1.0000	1.0000	270.0481	1613.5451	1042.6490	-0.0751
9	0.7513	0.7513	0.7513	1.0000	0.0000	333.4550	191.7590	-0.0751
10	1.0000	1.0000	1.0000	1.0000	400.8738	1104.4489	805.4767	-0.0751
11	1.0000	1.0000	1.0000	1.0000	89.1326	400.4718	214.9571	0.0959
12	0.7930	0.7930	0.7930	1.0000	0.0000	224.1958	106.6352	0.0244
13	0.8963	0.9803	0.9803	1.0000	197.3671	894.1907	574.2157	-0.0408
14	1.0000	1.0000	1.0000	1.0000	163.7416	555.2580	379.5280	-0.0512
15	1.0000	1.0000	1.0000	1.0000	76.1476	307.0211	173.2597	0.0794
16	0.8511	0.8511	0.8511	1.0000	165.4697	278.6759	218.5524	0.0311
17	0.6671	0.6907	0.6907	1.0000	0.0000	247.3520	130.0207	-0.0257
18	1.0000	1.0000	1.0000	1.0000	0.0000	545.1867	194.5305	0.1432
19	1.0000	1.0000	1.0000	1.0000	42.2992	426.7259	232.3876	0.0055
20	1.0000	1.0000	1.0000	1.0000	0.0000	1416.4555	638.6197	0.0491
21	1.0000	1.0000	1.0000	1.0000	175.5411	446.8738	331.5756	-0.0751
22	0.7974	0.8062	0.8062	1.0000	7.7165	418.5955	243.9995	-0.0751
23	0.6309	0.6517	0.6517	1.0000	0.0000	341.5584	117.5065	0.1560
24	0.6749	0.6903	0.6903	1.0000	184.2779	557.2423	340.4073	0.0814
25	1.0000	1.0000	1.0000	1.0000	387.3550	727.4923	550.9908	0.0189
26	0.8973	0.9751	0.9751	1.0000	134.2021	330.2947	230.8085	0.0073
27	1.0000	1.0000	1.0000	1.0000	143.8141	818.1647	469.6809	0.0168