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Quantum filtering for multiple measurements driven by fields in single-photon states*

Zhiyuan Dong[1] Guofeng Zhang[1] and Nina H. Amini[2]

Abstract— **In this paper, we derive the stochastic master equations for quantum systems driven by a single-photon input state which is contaminated by quantum vacuum noise. To improve estimation performance, quantum filters based on multiple-channel measurements are designed. Two cases, namely diffusive plus Poissonian measurements and two diffusive measurements, are considered.**

Key words: single-photon state, quantum filtering, homodyne detection, photon counting, quantum trajectories.

1. INTRODUCTION

When light interacts with a quantum system, e.g., a two-level atom or an optical cavity, partial system information can be transferred to the output light. The output light may be measured, say via homodyne detection, to produce photocurrent upon which the state of quantum system can be conditioned. The stochastic evolution of the conditional system state is usually called quantum trajectory. Quantum filter can be designed to estimate these trajectories [3], [4], [11], [14]-[17], [21], [23].

In quantum optics, the quantum filtering problem is known under the names of stochastic master equation and quantum trajectory theory [5], [11], [23], and it was first developed by Belavkin [3], [4]. The formalism of quantum filtering for Gaussian input fields, including the vacuum state, coherent state, squeezed state and thermal state, have been considered and well studied [8], [11], [19], [23]. With lots of experimental results, such as cavity quantum electrodynamics (QED) [18], circuit QED [9] and quantum dots in semiconductors [24], nonclassical states of light have also been discussed in connection with quantum networks. A range of nonclassical states, single-photon states and coherent states have been considered in [16]. Particularly, the master equations and stochastic master equations are presented for an arbitrary quantum system probed by a continuous-mode single-photon input field. As an application, the conditional dynamics for the cross phase modulation in a doubly resonant cavity are considered in [6], where both homodyne detection and photon-counting measurements are simulated for a cavity driven by a single-photon input field. The interaction of a two-level atom with a propagating mode single-photon in free space has been discussed in the literature, see e.g., [22]. The dependence of the atomic excitation probability on the temporal and spectral features of single-photon pulse shapes and coherent states pulse shapes are also considered in [22].

In real physical experiments, there may be limitations for the case of single measurements due to the existence of noise. To circumvent this imperfection, quantum filtering problem with multiple output fields has been developed using quantum trajectory theory with multi-input-multi-output (MIMO) quantum feedback [7]. A finite dimensional discrete-time Markov model in the cases of perfect and imperfect measurements are described in [2]. For the state estimations used in the feedback scheme, the quantum filters are discussed and a general robustness property for perfect and imperfect measurements are proved. An experimental implementation has been conducted by using the photon box and closed-loop simulations are also presented [20]. The observed system in [1] is assumed to be governed by a continuous-time stochastic master equation driven by Wiener and Poisson processes. Particularly, the incompleteness and errors in measurements have been taken into account and the measurement imperfections are modeled by a stochastic matrix.

In this paper, we extend the single-photon filtering framework proposed in [6], [16] by including imperfect measurements. More specifically, we study the case when the output light field is corrupted by a vacuum noise. We show how to design filters based on multiple measurements to achieve desired estimation performance.

2. OPEN QUANTUM SYSTEMS

The system model we discuss is an arbitrary quantum system G driven by a single-photon input field. Here, we will describe the system by using the triple language (S,L,H) [13], [26]. The scattering operator S is unitary, which satisfies $S^{\dagger}S = SS^{\dagger} = I$. The coupling between system and field is described by the operator L and the self-adjoint operator H is the initial Hamiltonian of the system.

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[1]Zhiyuan Dong and Guofeng Zhang are with the Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom Hong Kong, China zhiyuan.dong@connect.polyu.hk, guofeng.zhang@polyu.edu.hk

[2]Nina H. Amini is with CNRS, Laboratoire des signaux et systèmes (L2S), CentraleSupélec, 3 rue Joliot Curie, 91192 Gif-Sur-Yvette, France nina.amini@lss.supelec.fr

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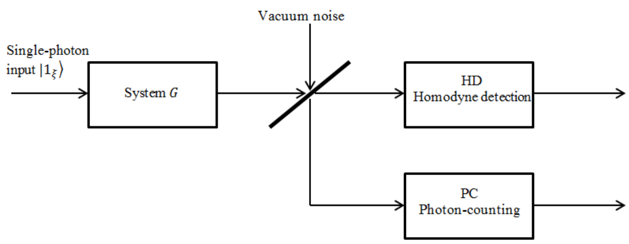

Fig. 1. Simultaneous homodyne detection and photon-counting at the outputs of a beam splitter in quantum system.

The input field is represented by annihilation operator $b(t)$ and creation operator $b^\dagger(t)$ on the Fock space H_F, which satisfies $[b(t), b^\dagger(s)] = \delta(t-s)$.

The dynamical evolution can be described by a unitary operator $U(t)$ on the tensor product Hilbert space $\mathsf{H}_S \otimes \mathsf{H}_F$ which is given by the following quantum stochastic differential equations (QSDE)

$$dU(t) = \Big\{(S-I)d\Lambda(t) + LdB^\dagger(t) - L^\dagger SdB(t) - \left(\frac{1}{2}L^\dagger L + iH\right)dt\Big\}U(t)$$

where $U(0) = I$.

By Itō calculus, we can find the following evolution

$$\begin{aligned} dB_{\text{out}}(t) =& S(t)dB(t) + L(t)dt, \\ d\Lambda_{\text{out}}(t) =& S^*(t)d\Lambda(t)S^T(t) + S^*(t)dB^*(t)L^T(t) \\ &+ L^*(t)dB^T(t)S^T(t) + L^*(t)L^T(t)dt. \end{aligned} \tag{2.1}$$

3. QUANTUM FILTER FOR MULTIPLE MEASUREMENTS

A. Continuous-mode Single-photon State

The creation operator for a photon with wave packet [12] $\xi(t)$ in time domain is defined as

$$B^*(\xi) = \int_0^\infty \xi(t)b^*(t)dt,$$

with the normalization condition $\int_0^\infty |\xi(t)|^2 dt = 1$. Then the single-photon state is given by

$$|1_\xi\rangle = B^*(\xi)|0\rangle.$$

B. Quantum Filter for Multiple Measurements Driven by Vacuum Input

To derive the quantum filter for system driven by a single-photon input state, we firstly introduce the result of multiple measurements with vacuum input.

Lemma 3.1: ([10, Theorem 3.2]) Let $\{Y_{i,t}, i = 1,2,\ldots,N\}$ be a set of N compatible measurement outputs for a quantum system G. With vacuum initial state, the corresponding joint measurement quantum filter is given by

$$d\hat{X} = \pi_t[\mathscr{L}_G(X_t)]dt + \sum_{i=1}^{N} \beta_{i,t} dW_{i,t},$$

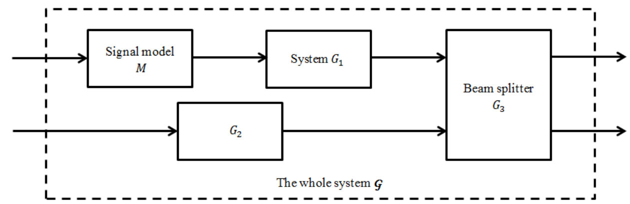

Fig. 2. Quantum system depiction of Figure 1.

where $dW_{i,t} = dY_{i,t} - \pi_t(dY_{i,t})$ is a martingale process for each measurement output and $\beta_{i,t}$ is the corresponding gain given by

$$\begin{aligned} &\zeta^T = \pi_t(X_t dY_t^T) - \pi_t(X_t)\pi_t(dY_t^T) + \pi_t\left([L_t^\dagger, X_t]S_t dB dY_t^T\right), \\ &\Sigma = \pi_t(dY_t dY_t^T), \quad \beta = \Sigma^{-1}\zeta, \end{aligned} \tag{3.2}$$

where Σ is assumed to be non-singular.

Remark 3.1: A general measurement equation, which is a function of annihilation, creation and conservation processes in the output field, is defined as [10]

$$dY(t) = F^* dB_{\text{out}}^*(t) + F dB_{\text{out}}(t) + G\text{diag}(d\Lambda_{\text{out}}(t)). \tag{3.3}$$

Particularly, a combination of homodyne detection and photon-counting measurement is given by

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

C. Quantum Filter for Joint Homodyne and Photon-counting Detections Driven by Single-photon State

Assume the system is in an initial state $\rho_0 = |\eta\rangle\langle\eta|$ and the single-photon input state is $|1_\xi\rangle$. The quantum filter for the conditional expectation for the system G driven by a single-photon field is given by

$$\pi_t^{11}(X) = \mathbb{E}_{\eta\xi}[X(t)|Y(s), 0 \le s \le t].$$

The whole system $\mathcal{G}$ with the measurements in Fig. 1 can be depicted as shown in Fig. 2. $G_1 = (S, L, H)$ is the original system G, which has been connected with a signal model (ancilla) $M = (I, \lambda(t)\sigma_-, 0)$. By introducing a second open quantum system $G_2 = (1, 0, 0)$, we concatenate the vacuum noise into our system. The last open quantum system is a beam splitter $G_3 = (S_b, 0, 0)$, where

$$S_b = \begin{bmatrix} \sqrt{1-r^2}e^{i\theta} & re^{i(\theta+\frac{\pi}{2})} \\ re^{i(\theta+\frac{\pi}{2})} & \sqrt{1-r^2}e^{i\theta} \end{bmatrix}, \quad 0 \le r \le 1.$$

The quantum signal generating filter is given by $M = (S_M, L_M, H_M)$, where

$$(S_M, L_M, H_M) = (I, \lambda(t)\sigma_-, 0),$$

Here, σ_- is the lowering operator and the coupling strength $\lambda(t) = \frac{\xi(t)}{\sqrt{w(t)}}$, where $w(t) = \int_t^\infty |\xi(s)|^2 ds$.

By the concatenation and series product, the whole system $\mathscr{G}$ is given by

$$\mathscr{G} = G_3 \lhd [(G_1 \lhd M) \boxplus G_2] = (S_t, L_t, H_t),$$

where $S_t = S_b \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix}$, $L_t = \begin{bmatrix} L + \lambda(t) S\sigma_- \\ 0 \end{bmatrix}$, $H_t = H + \lambda(t)\mathrm{Im}\{L^\dagger S \sigma_-\}$.

In what follows, we denote by $B_{i,t}$, which is a vacuum state, the input of signal model M. $B_{v,t}$ is the vacuum noise for system G_2, by the evolution of output fields (2.1), the measurements stochastic equations are given by

$$\begin{aligned} dY_{1,t} =& \sqrt{1-r^2}\Big\{ \Big[e^{i\theta}(L+SL_M) + e^{-i\theta}(L^\dagger + L_M^\dagger S^\dagger)\Big] dt \\ &+ e^{i\theta} S dB_{i,t} + e^{-i\theta} S^\dagger dB_{i,t}^\dagger \Big\} \\ &+ ir\left(e^{i\theta} dB_{v,t} - e^{-i\theta} dB_{v,t}^\dagger\right), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} dY_{2,t} =& r^2\Big[S d\Lambda_{i,t} S^\dagger + (L+SL_M) S^\dagger dB_{i,t}^\dagger + S(L^\dagger + L_M^\dagger S^\dagger) dB_{i,t} \\ &+ (L^\dagger + L_M^\dagger S^\dagger)(L+SL_M) dt\Big] + (1-r^2) d\Lambda_{v,t} \\ &+ ir\sqrt{1-r^2}\Big[S d\Lambda_{vi,t} - S^\dagger d\Lambda_{iv,t} + (L+SL_M) dB_{v,t}^\dagger \\ &- (L^\dagger + L_M^\dagger S^\dagger) dB_{v,t}\Big], \end{aligned} \tag{3.5}$$

where $dY_{1,t}$ is the first channel with homodyne detection and $dY_{2,t}$ is the second channel with photon-counting measurement.

Thus, the corresponding gain $\beta = [\beta_1 \ \beta_2]$ can be calculated by (3.2)

$$\begin{aligned} \beta_1 =& \sqrt{1-r^2} e^{i\theta} \tilde{\pi}_t\left(A \otimes XL + AL_M \otimes XS\right) \\ &+ \sqrt{1-r^2} e^{-i\theta} \tilde{\pi}_t\left(A \otimes L^\dagger X + L_M^\dagger A \otimes S^\dagger X\right) \\ &- \sqrt{1-r^2}\tilde{\pi}_t\left(A \otimes X\right) \\ &\times \tilde{\pi}_t\left[e^{i\theta}(L+SL_M) + e^{-i\theta}(L^\dagger + L_M^\dagger S^\dagger)\right], \end{aligned} \tag{3.6}$$

$$\begin{aligned} \beta_2 =& \left[\tilde{\pi}_t(L^\dagger L + L_M^\dagger S^\dagger L + L^\dagger S L_M + L_M^\dagger L_M)\right]^{-1} \\ &\times \tilde{\pi}_t\big(A \otimes L^\dagger X L + L_M^\dagger A \otimes S^\dagger X L + AL_M \otimes L^\dagger X S \\ &+ L_M^\dagger A L_M \otimes S^\dagger X S\big) - \tilde{\pi}_t\left(A \otimes X\right), \end{aligned} \tag{3.7}$$

where A is any ancilla operator and X is the system operator.

If we define [16]

$$\pi_t^{jk}(X) = \frac{\tilde{\pi}_t(Q_{jk} \otimes X)}{w_{jk}}, \quad j,k = 0,1,$$

where Q_{jk} and w_{jk} are given by

$$Q_{jk} = \begin{bmatrix} Q_{00} & Q_{01} \\ Q_{10} & Q_{11} \end{bmatrix} = \begin{bmatrix} \sigma_+\sigma_- & \sigma_+ \\ \sigma_- & I \end{bmatrix},$$

$$w_{jk} = \begin{bmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{bmatrix} = \begin{bmatrix} w(t) & \sqrt{w(t)} \\ \sqrt{w(t)} & 1 \end{bmatrix},$$

the theorem which presents the quantum filter can be obtained as follows.

Theorem 3.1: Let $\{Y_{i,t}, i = 1,2\}$ be a combination of homodyne detection and photon-counting measurement for a quantum system G. With single-photon input state, the quantum filter for the conditional expectation in the Heisenberg picture is given by (3.8).

Here,

$$\begin{aligned} K_t :=& e^{i\theta}\pi_t^{11}(L) + e^{-i\theta}\pi_t^{11}(L^\dagger) \\ &+ e^{-i\theta}\pi_t^{01}(S^\dagger)\xi^*(t) + e^{i\theta}\pi_t^{10}(S)\xi(t), \\ \nu_t :=& \pi_t^{11}(L^\dagger L) + \pi_t^{01}(S^\dagger L)\xi^*(t) \\ &+ \pi_t^{10}(L^\dagger S)\xi(t) + \pi_t^{00}(I)|\xi(t)|^2, \end{aligned}$$

the Wiener process $W(t)$ and compensated Poisson process $N(t)$ are given by

$$dW(t) = dY_{1,t} - \sqrt{1-r^2} K_t dt, \quad dN(t) = dY_{2,t} - r^2 \nu_t dt,$$

respectively. We have $\pi_t^{10}(X) = \pi_t^{01}(X^\dagger)^\dagger$, the initial conditions are $\pi_0^{11}(X) = \pi_0^{00}(X) = \langle \eta, X\eta\rangle$, $\pi_0^{10}(X) = \pi_0^{01}(X) = 0$.

Remark 3.2: If we let $r = 0$, $\theta = 0$, the filter equations reduce to an estimation problem with a single homodyne detection. On the other hand, if we let $r = 1$, $\theta = -\frac{\pi}{2}$, the filter equations reduce to an estimation problem with a single photon-counting measurement [16].

Corollary 3.1: With a combination of homodyne detection and photon-counting measurement, the quantum filter for the system G driven by single-photon input state in the Schrödinger picture is given by (3.9).

Here,

$$\begin{aligned} K_t =& e^{-i\theta}\mathrm{Tr}[L^\dagger \rho^{11}(t)] + e^{i\theta}\mathrm{Tr}[L\rho^{11}(t)] \\ &+ e^{i\theta}\mathrm{Tr}[S\rho^{01}(t)]\xi(t) + e^{-i\theta}\mathrm{Tr}[S^\dagger \rho^{10}(t)]\xi^*(t), \\ \nu_t =& \mathrm{Tr}[L^\dagger L \rho^{11}(t)] + \mathrm{Tr}[L^\dagger S \rho^{01}(t)]\xi(t) \\ &+ \mathrm{Tr}[S^\dagger L \rho^{10}(t)]\xi^*(t) + \mathrm{Tr}[\rho^{00}(t)]|\xi(t)|^2, \end{aligned}$$

and the initial conditions are

$$\rho^{11}(0) = \rho^{00}(0) = |\eta\rangle\langle\eta|, \quad \rho^{10}(0) = \rho^{01}(0) = 0.$$

D. Quantum Filter for Both Homodyne Detection Measurements

In this subsection, we will derive the filter equations for the case of joint homodyne-homodyne measurements, see Fig. 3. Here, by the general measurement equation (3.3), we choose $F = I$, $G = 0$. Then, the measurements stochastic equations are given by (3.4) and

$$\begin{aligned} dY_{2,t} =& \sqrt{1-r^2}\left(e^{i\theta} dB_{v,t} + e^{-i\theta} dB_{v,t}^\dagger\right) \\ &+ ir\big\{\left[e^{i\theta}(L+SL_M) - e^{-i\theta}(L^\dagger + L_M^\dagger S^\dagger)\right] dt \\ &+ e^{i\theta} S dB_{i,t} - e^{-i\theta} S^\dagger dB_{i,t}^\dagger\big\}, \end{aligned} \tag{3.10}$$

where $dY_{2,t}$ is the second channel with homodyne detection measurement. Thus, the corresponding gain β can also be

$$
\begin{aligned}
d\pi_t^{11}(X) =&\{\pi_t^{11}(\mathscr{L}_G X)+\pi_t^{01}(S^\dagger[X,L])\xi^*(t)+\pi_t^{10}([L^\dagger,X]S)\xi(t)+\pi_t^{00}(S^\dagger XS-X)|\xi(t)|^2\}dt\\
&+\sqrt{1-r^2}\left[e^{i\theta}\pi_t^{11}(XL)+e^{-i\theta}\pi_t^{11}(L^\dagger X)+e^{-i\theta}\pi_t^{01}(S^\dagger X)\xi^*(t)+e^{i\theta}\pi_t^{10}(XS)\xi(t)-\pi_t^{11}(X)K_t\right]dW(t)\\
&+\{\nu_t^{-1}\left[\pi_t^{11}(L^\dagger XL)+\pi_t^{01}(S^\dagger XL)\xi^*(t)+\pi_t^{10}(L^\dagger XS)\xi(t)+\pi_t^{00}(S^\dagger XS)|\xi(t)|^2\right]-\pi_t^{11}(X)\}dN(t),\\
d\pi_t^{10}(X) =&\{\pi_t^{10}(\mathscr{L}_G X)+\pi_t^{00}(S^\dagger[X,L])\xi^*(t)\}dt\\
&+\sqrt{1-r^2}\left[e^{i\theta}\pi_t^{10}(XL)+e^{-i\theta}\pi_t^{10}(L^\dagger X)+e^{-i\theta}\pi_t^{00}(S^\dagger X)\xi^*(t)-\pi_t^{10}(X)K_t\right]dW(t)\\
&+\{\nu_t^{-1}\left[\pi_t^{10}(L^\dagger XL)+\pi_t^{00}(S^\dagger XL)\xi^*(t)\right]-\pi_t^{10}(X)\}dN(t),\\
d\pi_t^{01}(X) =&\{\pi_t^{01}(\mathscr{L}_G X)+\pi_t^{00}([L^\dagger,X]S)\xi(t)\}dt\\
&+\sqrt{1-r^2}\left[e^{i\theta}\pi_t^{01}(XL)+e^{-i\theta}\pi_t^{01}(L^\dagger X)+e^{i\theta}\pi_t^{00}(XS)\xi(t)-\pi_t^{01}(X)K_t\right]dW(t)\\
&+\{\nu_t^{-1}\left[\pi_t^{01}(L^\dagger XL)+\pi_t^{00}(L^\dagger XS)\xi(t)\right]-\pi_t^{01}(X)\}dN(t),\\
d\pi_t^{00}(X) =&\pi_t^{00}(\mathscr{L}_G X)dt+\sqrt{1-r^2}\left[e^{i\theta}\pi_t^{00}(XL)+e^{-i\theta}\pi_t^{00}(L^\dagger X)-\pi_t^{00}(X)K_t\right]dW(t)\\
&+\{\nu_t^{-1}\left[\pi_t^{00}(L^\dagger XL)\right]-\pi_t^{00}(X)\}dN(t).
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
d\rho^{11}(t) =&\{\mathscr{L}_G^\star\rho^{11}(t)+[S\rho^{01}(t),L^\dagger]\xi(t)+[L,\rho^{10}(t)S^\dagger]\xi^*(t)+[S\rho^{00}(t)S^\dagger-\rho^{00}(t)]|\xi(t)|^2\}dt\\
&+\sqrt{1-r^2}\left[e^{-i\theta}\rho^{11}(t)L^\dagger+e^{i\theta}L\rho^{11}(t)+e^{i\theta}S\rho^{01}(t)\xi(t)+e^{-i\theta}\rho^{10}(t)S^\dagger\xi^*(t)-K_t\rho^{11}(t)\right]dW(t)\\
&+\{\nu_t^{-1}\left[L\rho^{11}(t)L^\dagger+S\rho^{01}(t)L^\dagger\xi(t)+L\rho^{10}(t)S^\dagger\xi^*(t)+S\rho^{00}(t)S^\dagger|\xi(t)|^2\right]-\rho^{11}(t)\}dN(t),\\
d\rho^{10}(t) =&\{\mathscr{L}_G^\star\rho^{10}(t)+[S\rho^{00}(t),L^\dagger]\xi(t)\}dt\\
&+\sqrt{1-r^2}\left[e^{-i\theta}\rho^{10}(t)L^\dagger+e^{i\theta}L\rho^{10}(t)+e^{i\theta}S\rho^{00}(t)\xi(t)-K_t\rho^{10}(t)\right]dW(t)\\
&+\{\nu_t^{-1}[L\rho^{10}(t)L^\dagger+S\rho^{00}(t)L^\dagger\xi(t)]-\rho^{10}(t)\}dN(t),\\
d\rho^{01}(t) =&\{\mathscr{L}_G^\star\rho^{01}(t)+[L,\rho^{00}(t)S^\dagger]\xi^*(t)\}dt\\
&+\sqrt{1-r^2}\left[e^{-i\theta}\rho^{01}(t)L^\dagger+e^{i\theta}L\rho^{01}(t)+e^{-i\theta}\rho^{00}(t)S^\dagger\xi^*(t)-K_t\rho^{01}(t)\right]dW(t)\\
&+\{\nu_t^{-1}[L\rho^{01}(t)L^\dagger+L\rho^{00}(t)S^\dagger\xi^*(t)]-\rho^{01}(t)\}dN(t),\\
d\rho^{00}(t) =&\mathscr{L}_G^\star\rho^{00}(t)dt+\sqrt{1-r^2}\left[e^{-i\theta}\rho^{00}(t)L^\dagger+e^{i\theta}L\rho^{00}(t)-K_t\rho^{00}(t)\right]dW(t)\\
&+\{\nu_t^{-1}[L\rho^{00}(t)L^\dagger]-\rho^{00}(t)\}dN(t).
\end{aligned}
\tag{3.9}
$$

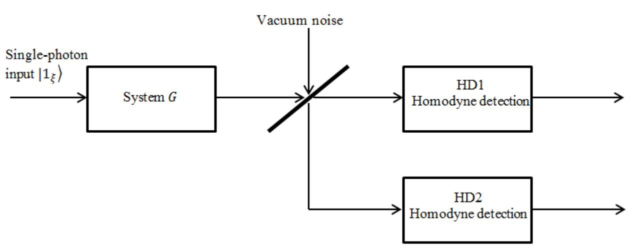

Fig. 3. Both homodyne detection measurements at the outputs of a beam splitter in quantum system.

calculated by (3.2), where β_1 is given by (3.6) and β_2 is given by

$$
\begin{aligned}
\beta_2 =& ire^{i\theta}\tilde{\pi}_t(A\otimes XL+AL_M\otimes XS)\\
&-ire^{-i\theta}\tilde{\pi}_t(A\otimes L^\dagger X+L_M^\dagger A\otimes S^\dagger X)\\
&-ir\tilde{\pi}_t(A\otimes X)\tilde{\pi}_t[e^{i\theta}(L+SL_M)-e^{-i\theta}(L^\dagger+L_M^\dagger S^\dagger)].
\end{aligned}
\tag{3.11}
$$

Theorem 3.2: Let $\{Y_{i,t}, i=1,2\}$ be the two homodyne detection measurements for a quantum system G. With single-photon input state, the quantum filter for the conditional expectation in the Heisenberg picture is given by (3.12). Here,

$$
\begin{aligned}
K_{1,t} =& e^{i\theta}\pi_t^{11}(L)+e^{-i\theta}\pi_t^{11}(L^\dagger)\\
&+e^{-i\theta}\pi_t^{01}(S^\dagger)\xi^*(t)+e^{i\theta}\pi_t^{10}(S)\xi(t),\\
K_{2,t} =& e^{i\theta}\pi_t^{11}(L)-e^{-i\theta}\pi_t^{11}(L^\dagger)\\
&-e^{-i\theta}\pi_t^{01}(S^\dagger)\xi^*(t)+e^{i\theta}\pi_t^{10}(S)\xi(t),
\end{aligned}
$$

the Wiener processes $W_1(t)$ and $W_2(t)$ are given by

$$
dW_1(t)=dY_{1,t}-\sqrt{1-r^2}K_{1,t}dt,\quad dW_2(t)=dY_{2,t}-irK_{2,t}dt,
$$

respectively. We have $\pi_t^{10}(X)=\pi_t^{01}(X^\dagger)^\dagger$, the initial conditions are $\pi_0^{11}(X)=\pi_0^{00}(X)=\langle\eta,X\eta\rangle$, $\pi_0^{10}(X)=\pi_0^{01}(X)=0$.

Corollary 3.2: With the two homodyne detection measurements, the quantum filter for the system G driven by

$$
\begin{aligned}
d\pi_t^{11}(X) = &\left\{\pi_t^{11}(\mathscr{L}_G X) + \pi_t^{01}(S^\dagger[X,L])\xi^*(t) + \pi_t^{10}([L^\dagger,X]S)\xi(t) + \pi_t^{00}(S^\dagger XS - X)|\xi(t)|^2\right\} dt \\
&+ \sqrt{1-r^2}\left[e^{i\theta}\pi_t^{11}(XL) + e^{-i\theta}\pi_t^{11}(L^\dagger X) + e^{-i\theta}\pi_t^{01}(S^\dagger X)\xi^*(t) + e^{i\theta}\pi_t^{10}(XS)\xi(t) - \pi_t^{11}(X)K_{1,t}\right] dW_1(t) \\
&+ ir\left[e^{i\theta}\pi_t^{11}(XL) - e^{-i\theta}\pi_t^{11}(L^\dagger X) - e^{-i\theta}\pi_t^{01}(S^\dagger X)\xi^*(t) + e^{i\theta}\pi_t^{10}(XS)\xi(t) - \pi_t^{11}(X)K_{2,t}\right] dW_2(t), \\
d\pi_t^{10}(X) = &\left\{\pi_t^{10}(\mathscr{L}_G X) + \pi_t^{00}(S^\dagger[X,L])\xi^*(t)\right\} dt \\
&+ \sqrt{1-r^2}\left[e^{i\theta}\pi_t^{10}(XL) + e^{-i\theta}\pi_t^{10}(L^\dagger X) + e^{-i\theta}\pi_t^{00}(S^\dagger X)\xi^*(t) - \pi_t^{10}(X)K_{1,t}\right] dW_1(t) \\
&+ ir\left[e^{i\theta}\pi_t^{10}(XL) - e^{-i\theta}\pi_t^{10}(L^\dagger X) - e^{-i\theta}\pi_t^{00}(S^\dagger X)\xi^*(t) - \pi_t^{10}(X)K_{2,t}\right] dW_2(t), \\
d\pi_t^{01}(X) = &\left\{\pi_t^{01}(\mathscr{L}_G X) + \pi_t^{00}([L^\dagger,X]S)\xi(t)\right\} dt \\
&+ \sqrt{1-r^2}\left[e^{i\theta}\pi_t^{01}(XL) + e^{-i\theta}\pi_t^{01}(L^\dagger X) + e^{i\theta}\pi_t^{00}(XS)\xi(t) - \pi_t^{01}(X)K_{1,t}\right] dW_1(t) \\
&+ ir\left[e^{i\theta}\pi_t^{01}(XL) - e^{-i\theta}\pi_t^{01}(L^\dagger X) + e^{i\theta}\pi_t^{00}(XS)\xi(t) - \pi_t^{01}(X)K_{2,t}\right] dW_2(t), \\
d\pi_t^{00}(X) = &\pi_t^{00}(\mathscr{L}_G X)dt + \sqrt{1-r^2}\left[e^{i\theta}\pi_t^{00}(XL) + e^{-i\theta}\pi_t^{00}(L^\dagger X) - \pi_t^{00}(X)K_{1,t}\right] dW_1(t) \\
&+ ir\left[e^{i\theta}\pi_t^{00}(XL) - e^{-i\theta}\pi_t^{00}(L^\dagger X) - \pi_t^{00}(X)K_{2,t}\right] dW_2(t).
\end{aligned} \tag{3.12}
$$

single-photon input state in the Schrödinger picture is given by (3.13). Here,

$$
\begin{aligned}
K_{1,t} =& e^{-i\theta}\mathrm{Tr}[L^\dagger\rho^{11}(t)] + e^{i\theta}\mathrm{Tr}[L\rho^{11}(t)] \\
&+ e^{i\theta}\mathrm{Tr}[S\rho^{01}(t)]\xi(t) + e^{-i\theta}\mathrm{Tr}[S^\dagger\rho^{10}(t)]\xi^*(t), \\
K_{2,t} =& e^{-i\theta}\mathrm{Tr}[L^\dagger\rho^{11}(t)] - e^{i\theta}\mathrm{Tr}[L\rho^{11}(t)] \\
&- e^{i\theta}\mathrm{Tr}[S\rho^{01}(t)]\xi(t) + e^{-i\theta}\mathrm{Tr}[S^\dagger\rho^{10}(t)]\xi^*(t),
\end{aligned}
$$

and the initial conditions are

$$\rho^{11}(0) = \rho^{00}(0) = |\eta\rangle\langle\eta|, \quad \rho^{10}(0) = \rho^{01}(0) = 0.$$

E. Simulation Results

The atom is in the ground state initially $|g\rangle\langle g|$ with the Hamiltonian $H = 0$. The wave packet $\xi(t)$ for the single-photon is given by

$$\xi(t) = \left(\frac{\Omega^2}{2\pi}\right)^{1/4} \exp\left[-\frac{\Omega^2}{4}(t-t_0)^2\right].$$

Now we choose the bandwidth $\Omega = 1.46\kappa$ and the exciting probability for quantum filtering equations is given by

$$P_e^c(t) = \mathrm{Tr}[\rho^{11}(t)|e\rangle\langle e|], \tag{3.14}$$

where $\rho^{11}(t)$ is the solution to (3.13) and $|e\rangle$ means the excited state.

In Fig. 4, 72 different stochastic trajectories are simulated as colorful lines in each case given by (3.14). Fig. 4(a) ($r = 0$) denotes the ideal case which is equivalent to the single measurement (HD1) without any noise, [16]. For $r = 1$, the case will be similar to Fig. 4(a) since the single measurement becomes HD2. We can see that many of the stochastic trajectories begin to decay at $t = 4$. Meanwhile, some trajectories continue to rise towards $P_e^c(t) = 1$, it means that the atom may be fully excited. In Fig. 4(b), $r = \sqrt{0.5}$, that is the output field is contaminated by vacuum noise. Nevertheless, it can be seen that by means of joint measurement the estimation performance is close to those for the ideal case. The exciting probabilities become bad if we only use single measurement, see Fig. 4(c) and (d). By comparing Fig. 4(b), (c) and (d), it is clear that multiple measurements is much better.

4. CONCLUSIONS

In this paper, we have derived the quantum filter for a quantum system driven by single-photon input state with multiple compatible measurements. Particularly, the explicit form of stochastic master equations with two homodyne detection measurements and a combination of homodyne detection and photon-counting are given. A numerical study of a two-level system driven by a single-photon state demonstrated the advantage of filtering design based on multiple measurement when the output filed is contaminated by quantum vacuum noise.

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$$
\begin{aligned}
d\rho^{11}(t) = & \left\{ \mathscr{L}_G^{\star}\rho^{11}(t) + [S\rho^{01}(t), L^{\dagger}]\xi(t) + [L, \rho^{10}(t)S^{\dagger}]\xi^*(t) + [S\rho^{00}(t)S^{\dagger} - \rho^{00}(t)]|\xi(t)|^2 \right\} dt \\
& + \sqrt{1-r^2}\left[e^{-i\theta}\rho^{11}(t)L^{\dagger} + e^{i\theta}L\rho^{11}(t) + e^{i\theta}S\rho^{01}(t)\xi(t) + e^{-i\theta}\rho^{10}(t)S^{\dagger}\xi^*(t) - K_{1,t}\rho^{11}(t) \right] dW_1(t) \\
& - ir\left[e^{-i\theta}\rho^{11}(t)L^{\dagger} - e^{i\theta}L\rho^{11}(t) - e^{i\theta}S\rho^{01}(t)\xi(t) + e^{-i\theta}\rho^{10}(t)S^{\dagger}\xi^*(t) + K_{2,t}\rho^{11}(t) \right] dW_2(t), \\
d\rho^{10}(t) = & \left\{ \mathscr{L}_G^{\star}\rho^{10}(t) + [S\rho^{00}(t), L^{\dagger}]\xi(t) \right\} dt \\
& + \sqrt{1-r^2}\left[e^{-i\theta}\rho^{10}(t)L^{\dagger} + e^{i\theta}L\rho^{10}(t) + e^{i\theta}S\rho^{00}(t)\xi(t) - K_{1,t}\rho^{10}(t) \right] dW_1(t) \\
& - ir\left[e^{-i\theta}\rho^{10}(t)L^{\dagger} - e^{i\theta}L\rho^{10}(t) - e^{i\theta}S\rho^{00}(t)\xi(t) + K_{2,t}\rho^{10}(t) \right] dW_2(t), \\
d\rho^{01}(t) = & \left\{ \mathscr{L}_G^{\star}\rho^{01}(t) + [L, \rho^{00}(t)S^{\dagger}]\xi^*(t) \right\} dt \\
& + \sqrt{1-r^2}\left[e^{-i\theta}\rho^{01}(t)L^{\dagger} + e^{i\theta}L\rho^{01}(t) + e^{-i\theta}\rho^{00}(t)S^{\dagger}\xi^*(t) - K_{1,t}\rho^{01}(t) \right] dW_1(t) \\
& - ir\left[e^{-i\theta}\rho^{01}(t)L^{\dagger} - e^{i\theta}L\rho^{01}(t) + e^{-i\theta}\rho^{00}(t)S^{\dagger}\xi^*(t) + K_{2,t}\rho^{01}(t) \right] dW_2(t), \\
d\rho^{00}(t) = & \mathscr{L}_G^{\star}\rho^{00}(t)dt + \sqrt{1-r^2}\left[e^{-i\theta}\rho^{00}(t)L^{\dagger} + e^{i\theta}L\rho^{00}(t) - K_{1,t}\rho^{00}(t) \right] dW_1(t) \\
& - ir\left[e^{-i\theta}\rho^{00}(t)L^{\dagger} - e^{i\theta}L\rho^{00}(t) + K_{2,t}\rho^{00}(t) \right] dW_2(t).
\end{aligned}
\tag{3.13}
$$

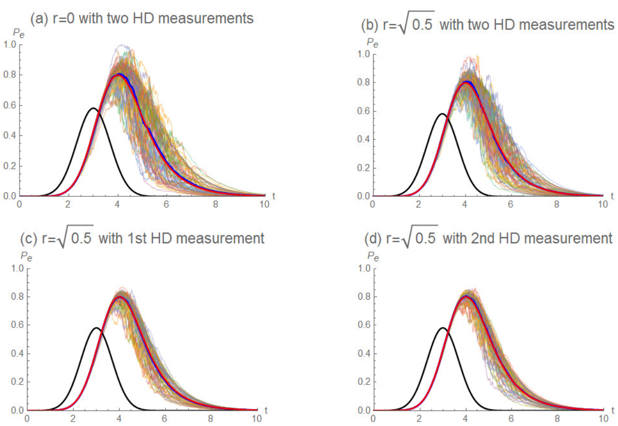

Fig. 4. (Color online) The exciting probability for a two-level system interacting with one photon in a Gaussian pulse shape with different beam splitter parameters. The black line is the wave packet $|\xi(t)|^2$, the red line is $P_e(t)$ given by the master equation, the colorful lines are the trajectories $P_e^c(t)$ and the blue line denotes the average of these trajectories.

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