Mean Square Capacity of Power Constrained Fading Channels with Causal Encoders and Decoders*

Liang Xu[1], Lihua Xie[1] and Nan Xiao[2]

Abstract— **This paper is concerned with the mean square stabilization problem of discrete-time LTI systems over a power constrained fading channel. Different from existing research works, the channel considered in this paper suffers from both fading and additive noises. We allow any form of causal channel encoders/decoders, unlike linear encoders/decoders commonly studied in the literature. Sufficient conditions and necessary conditions for the mean square stabilizability are given in terms of channel parameters such as transmission power and fading and additive noise statistics in relation to the unstable eigenvalues of the open-loop system matrix. The corresponding mean square capacity of the power constrained fading channel under causal encoders/decoders is given. It is proved that this mean square capacity is smaller than the corresponding Shannon channel capacity. In the end, numerical examples are presented, which demonstrate that the causal encoders/decoders render less restrictive stabilizability conditions than those under linear encoders/decoders studied in the existing works.**

I. Introduction

Control over communication networks has been a hot research topic in the past decade [1]. This is mainly motivated by the rapid development of wireless communication technology that enables the connection of geographically distributed systems and devices. However, the insertion of wireless communication networks also poses challenges in analysis and design of control systems due to constraints and uncertainties in communications. One must take the communication networks into consideration and analyze how they affect the stability and performance of the closed-loop control systems.

Until now, there have been plentiful results that reveal requirements on communication channels to ensure the stabilizability. For noiseless digital channels, the celebrated data rate theorem is given in [2]. For noisy channels, the problem is complicated by the fact that different channel capacities are required under different stability definitions. For almost sure stability, [3] shows that the Shannon capacity in relation to unstable dynamics of a system constitutes the critical condition for its stabilizability. While for moment stability, [4] shows that the Shannon capacity is too optimistic while the zero-error capacity is too pessimistic, and the anytime capacity introduced in this paper characterizes the stabilizability conditions. Essentially, to keep the η-moment of the state of an unstable scalar plant bounded, it is necessary and sufficient for the feedback channel's anytime capacity corresponding to anytime-reliability $\alpha = \eta \log_2|\lambda|$ to be greater than $\log_2|\lambda|$, where λ is the unstable eigenvalue of the plant. The anytime capacity has a more stringent reliability requirement than the Shannon capacity. However, it is worthy noting that there exist no systematic method to calculate the anytime capacities of channels.

In control community, the anytime capacity is usually studied under the mean square stability requirement, for which the anytime capacity is commonly named as the mean square capacity. For example, [5] characterizes the mean square capacity of a fading channel. [6] studies the mean square stabilization problem over a power constrained AWGN channel and characterizes the critical capacity to ensure mean square stabilizability. They further show that the extension from linear encoders/decoders to more general causal encoders/decoders cannot provide additional benefits of increasing the channel capacity [7]. Specifically, the results stated above deal with fading channels or AWGN channels separately. While in wireless communications, it is practical to consider them as a whole. In this paper, we are interested in a power constrained fading channel which is corrupted by both fading and AWGN. We aim to find the critical condition on the channel to ensure the mean square stabilizability of the system. Note that [8] has derived the necessary and sufficient condition for such kind of channel to ensure mean square stabilizability under a linear encoder/decoder. It is still unknown whether we can achieve a higher channel capacity with more general causal strategies. This paper provides a positive answer to this question.

This paper is organized as follows. Problem formulation and some preliminaries are given in Section 2. Section 3 provides the results for scalar systems. Section 4 discusses the extension to vector systems. Section 5 provides numerical illustrations and this paper ends with some concluding remarks in Section 6.

II. Problem Formulation and Preliminaries

This paper studies the following single-input discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state and $u \in \mathbb{R}$ is the control input. Without loss of generality, we assume that all the

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[1]Liang Xu and Lihua Xie are with EXQUISITUS, Centre for E-City, School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798, Singapore lxu006@e.ntu.edu.sg, elhxie@ntu.edu.sg

[2]Nan Xiao is with the Singapore-MIT Alliance for Research and Technology Centre, Singapore 138602, Singapore xiaonan@smart.mit.edu

eigenvalues of A are unstable, i.e., $|\lambda_i(A)| \geq 1$ for all $i = 1, 2, \ldots, n$ [7]. The initial value x_0 is randomly generated from a Gaussian distribution with zero mean and bounded covariance Σ_{x_0}. The system state x_t is observed by a sensor and then encoded and transmitted to the controller through a power constrained fading channel. The communication channel is modeled as

$$r_t = g_t s_t + n_t \tag{2}$$

in which s_t denotes the channel input; r_t represents the channel output; $\{g_t\}$ is an i.i.d. stochastic process modeling the fading effects and $\{n_t\}$ is the additive white Gaussian noise with zero-mean and known variance σ_n^2. The channel input s_t must satisfy an average power constraint, i.e., $\mathbb{E}\{s_t^2\} \leq P$. We also assume that $x_0, g_0, n_0, g_1, n_1, \ldots$ are independent. In the paper, it is assumed that after each transmission, the instantaneous value of the fading factor g_t is known to the decoder, which is a reasonable assumption for slowly varying channels with channel estimation [9]. The instantaneous Shannon channel capacity is $c_t = \frac{1}{2}\ln\left(1 + \frac{g_t^2 P}{\sigma_n^2}\right)$ with c_t being measured in nats/transmission. The feedback configuration among the plant, the sensor and the controller, and the channel encoder/decoder structure are depicted in Fig. 1. In this paper, we try to find requirements

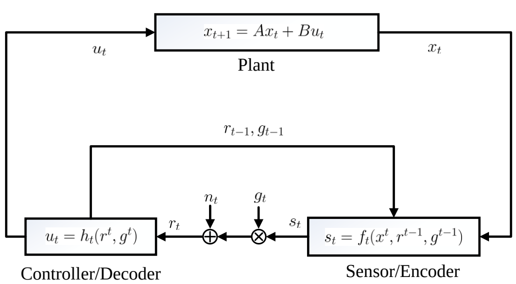

Fig. 1. Network control structure over power constraint fading channel

on the power constrained fading channel such that there exists a pair of causal encoder/decoder $\{f_t\}, \{h_t\}$ that can mean square stabilize the LTI dynamics (1), i.e., to render $\lim_{t\to\infty}\mathbb{E}\{x_t x_t'\} = 0$.

To solve this problem, the following preliminaries are needed, which are borrowed from [7]. Throughout the paper, a sequence $\{\chi_i\}_{i=0}^t$ is denoted by χ^t; random variables are denoted by uppercase letters, and their realizations by lower case letters. All random variables are assumed to exist on a common probability space with measure $\mathcal{P}$. The probability density of a random variable X in Euclidean space with respect to Lebesgue measure on the space is denoted by p_X, and the probability density of X conditioned on the σ-field generated by the event $Y = y$ by $p_{X|y}$. Let the expectation operator be denoted by $\mathbb{E}$, and the expectation conditioned on the event $Y = y$ by $\mathbb{E}_y$. We use $\log$ to denote the logarithm to the base two, and $\ln$ to denote the natural logarithm.

The differential entropy of X is defined by $H(X) = -\mathbb{E}\{\ln p_X\}$, provided that the defining integral exists. Denote the conditional entropy of X given the event $Y = y$ by $H_y(X) = H(X|Y = y) = -\mathbb{E}_y\{\ln p_{X|y}\}$, and the random variable associated with $H_y(X)$ by $H_Y(X)$. The average conditional entropy of X given the event $Y = y$ and averaged over Y is defined by $H(X|Y) = \mathbb{E}\{H_Y(X)\}$, and the average conditional entropy of X given the events $Y = y$ and $Z = z$ and averaged only over Y by $H_z(X|Y) = \mathbb{E}_z\{H_{Y,Z}(X)\}$. The conditional mutual information between two random variables X and Y given the event $Z = z$ is defined by $I_z(X;Y) = H_z(X) - H_z(X|Y)$. Given a random variable $X \in \mathbb{R}^n$ with entropy $H(X)$, the entropy power of X is defined by $N(X) = \frac{1}{2\pi e} e^{\frac{2}{n}H(X)}$. Denote the conditional entropy power of X given the event $Y = y$ by $N_y(X) = \frac{1}{2\pi e} e^{\frac{2}{n}H_y(X)}$, and the random variable associated with $N_y(X)$ by $N_Y(X)$. The average conditional entropy power of X given the event $Y = y$ and averaged over Y is defined by $N(X|Y) = \mathbb{E}\{N_Y(X)\}$, and the average conditional entropy power of X given the events $Y = y$ and $Z = z$ and averaged only over Y by $N_z(X|Y) = \mathbb{E}_z\{N_{Y,Z}(X)\}$. The following lemma shows that the entropy power of a random variable provides an estimation of the lower bound for its variance.

Lemma 1 ([7]): Let X be an n-dimensional random variable. Then $N_y(X) \leq \frac{1}{n}\mathbb{E}_y\{\|X\|^2\}$.

Lemma 2: Let X be an n-dimensional random variable, $f(X)$ be a function of X, and $Y = f(X) + N$ with N being a random variable that is independent with X. Then $I(X;Y) = I(f(X);Y)$.

Proof: Since $H(Y|X) = H(Y|X, f(X)) \leq H(Y|f(X))$, we have $H(Y) = I(X;Y) + H(Y|X) \leq I(X;Y) + H(Y|f(X))$. Thus $H(Y) - H(Y|f(X)) = I(Y; f(X)) \leq I(X;Y)$. Besides, noting that $X \to f(X) \to Y$ forms a Markov chain, the data processing inequality [10] implies that $I(X;Y) \leq I(f(X);Y)$. Combining the two facts, we have $I(X;Y) = I(f(X);Y)$. ■

Remark 1: Lemma 2 indicates that for the AWGN channel, the amount of information that the channel output contains about the source is equal to the amount of information that the channel output contains about the channel input.

III. Scalar Systems

To better convey our ideas, we start with scalar systems. Consider the following scalar system

$$x_{t+1} = \lambda x_t + u_t \tag{3}$$

where $|\lambda| \geq 1$ and $\mathbb{E}\{x_0^2\} = \sigma_{x_0}^2$. With the communication channel given in (2), the stabilizability result is stated in the following theorem.

Theorem 1: There exists a causal encoder/decoder pair $\{f_t\}, \{h_t\}$, such that the system (3) can be stabilized over the communication channel (2) in mean square sense if and only if

$$\log|\lambda| < -\frac{1}{2}\log\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2 + g_t^2 P}\} \tag{4}$$

Theorem 1 indicates that the mean square capacity of the power constraint fading channel is $C_{\text{MSC}} = -\frac{1}{2}\log\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\}$. In the following, we will prove the necessity and sufficiency of Theorem 1, respectively. The proof essentially follows the same steps as in [11], [7], [12], however, with some differences due to the channel structure.

A. Proof of Necessity

The proof of necessity follows from the intuition below. In view of Lemma 1, the entropy power provides a lower bound for the mean square value of the system state. We thus can use the average entropy power as a measure of the uncertain region of the system state and analyze its update. At time t, the controller maintains a knowledge of the uncertain region of x_t. When it takes action on the plant, the average uncertain region of x_{t+1} predicated by the controller is expanded to λ^2 times that of x_t. This is the iteration we term as dynamics update, which describes the update of the uncertain region of x maintained by the controller from time t to $t+1$. After receiving information about x_{t+1} from the sensor through the communication channel, the controller can reduce the predication error of the uncertain region of x_{t+1} by a factor of $\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2 P}\}$. This is the iteration we term as communication update, which describes the update of the uncertain region of x maintained by the controller at time $t+1$ after it has received the information about x_{t+1} from the sensor through the communication channel. Thus to ensure mean square stability, the average expanding factor $\lambda^2\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2 P}\}$ of the system state's uncertain region should be smaller than one, which gives the necessary requirement in Theorem 1. The formal proof is stated as follows. Here we use the uppercase letters X, S, R, G to denote the random variables of the system state, the channel input, the channel output and the channel fading coefficient. We use the lowercase letters x, s, r, g to denote their realizations.

1) Communication Update: The average entropy power of X_t conditioned on (R^t, G^t) is $N(X_t|R^t,G^t)=\mathbb{E}\{N_{R^t,G^t}(X_t)\}\overset{(a)}{=}\mathbb{E}\{\mathbb{E}\{N_{R^t,G^t}(X_t)|R^{t-1},G^t\}\}\overset{(b)}{=}\frac{1}{2\pi e}\mathbb{E}\{\mathbb{E}\{e^{2H_{R^t,G^t}(X_t)}|R^{t-1},G^t\}\}$ where (a) follows from the law of total expectation and (b) follows from the definition of entropy power. Since $\mathbb{E}\{e^{2H_{R^t,G^t}(X_t)}|R^{t-1}=r^{t-1},G^t=g^t\}$

$$
\begin{aligned}
&\overset{(c)}{\geq} e^{2\mathbb{E}\{H_{R^t,G^t}(X_t)|R^{t-1}=r^{t-1},G^t=g^t\}}\\
&\overset{(d)}{=} e^{2H(X_t|R_t,R^{t-1}=r^{t-1},G^t=g^t)}\\
&\overset{(e)}{=} e^{2\left(H(X_t|R^{t-1}=r^{t-1},G^t=g^t)-I(X_t,R_t|R^{t-1}=r^{t-1},G^t=g^t)\right)}\\
&\overset{(f)}{=} e^{2\left(H(X_t|R^{t-1}=r^{t-1},G^t=g^t)-I(S_t,R_t|R^{t-1}=r^{t-1},G^t=g^t)\right)}\\
&\overset{(g)}{\geq} e^{2\left(H(X_t|R^{t-1}=r^{t-1},G^t=g^t)-c_t\right)}\\
&\overset{(h)}{=} e^{-2c_t}e^{2H(X_t|R^{t-1}=r^{t-1},G^{t-1}=g^{t-1})}
\end{aligned}
$$

where (c) follows from Jensen's inequality; (d) follows from the definition of conditional entropy; (e) follows from the definition of conditional mutual information; (f) follows from Lemma 2; (g) follows from the definition of channel capacity, i.e., $I(S_t, R_t|R^{t-1}=r^{t-1}, G^t=g^t)\leq c_t$ and (h) follows from the fact that G_t is independent with X_t, we have $N(X_t|R^t,G^t)\geq\frac{1}{2\pi e}\mathbb{E}\{e^{-2C_t}e^{2H_{R^{t-1},G^{t-1}}(X_t)}\}=\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\}N(X_t|R^{t-1},G^{t-1})$.

2) Dynamics Update: Since $e^{2H(X_{t+1}|R^t=r^t,G^t=g^t)}=e^{2H(\lambda X_t+U_t|R^t=r^t,G^t=g^t)}\overset{(i)}{=}e^{2H(\lambda X_t|R^t=r^t,G^t=g^t)}\overset{(j)}{=}e^{2H(X_t|R^t=r^t,G^t=g^t)+2\ln|\lambda|}=\lambda^2e^{2H(X_t|R^t=r^t,G^t=g^t)}$ where (i) follows from the fact that $u_t=h_t(r^t,g^t)$ and (j) follows from Theorem 8.6.4 in [10], we have $N(X_{t+1}|R^t,G^t)\geq\mathbb{E}\Big\{\frac{1}{2\pi e}\lambda^2e^{2H_{R^t,G^t}(X_t)}\Big\}=\lambda^2N(X_t|R^t,G^t)$.

3) Proof of Necessity: Combining the results of communication update and dynamics update, we have $N(X_{t+1}|R^t,G^t)\geq\lambda^2\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\}N(X_t|R^{t-1},G^{t-1})$. In view of Lemma 1, $N(X_{t+1}|R^t,G^t)$ should converge to zero asymptotically. Thus $\lambda^2\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\}<1$, which is (4) and this proves the necessity.

B. Proof of Sufficiency

To prove the sufficiency, we need to construct a pair of encoder and decoder. The encoder and decoder are designed following an "estimation then control" strategy. The controller consecutively estimates the initial state x_0 by using the received information from the channel and then applies an equivalent control to the plant. The reason for adopting such strategy is explained as follows. The response of the linear system is $x_t=\lambda^t(x_0-\hat{x}_t)$ with $\hat{x}_t=-\sum_{i=0}^{t-1}\lambda^{-1-i}u_i$, which means $\mathbb{E}\{x_t^2\}=\lambda^{2t}\mathbb{E}\{(x_0-\hat{x}_t)^2\}$. We can treat $\hat{x}_t$ as an estimate of the controller for the initial state x_0. If the estimation error $\mathbb{E}\{(x_0-\hat{x}_t)^2\}$ converges to zero at a speed that is greater than λ^2, i.e., there exists $\eta>\lambda^2$ and $\alpha>0$, such that $\mathbb{E}\{(x_0-\hat{x}_t)^2\}\leq\frac{\alpha}{\eta^t}$, the mean square value of the system state would be bounded by $\mathbb{E}\{x_t^2\}\leq\alpha\left(\frac{\lambda^2}{\eta}\right)^t$. Thus $\lim_{t\to\infty}\mathbb{E}\{x_t^2\}=0$, i.e., system (3) is mean square stable. This intuition can be formalized using the following lemma.

Lemma 3 ([12]): If there exists an estimation scheme $\hat{x}_t$ for the initial system state x_0, such that the estimation error $e_t=\hat{x}_t-x_0$ satisfies the following property,

$$\mathbb{E}\{e_t\}=0 \tag{5}$$

$$\lim_{t\to\infty}A^t\mathbb{E}\{e_te_t'\}(A')^t=0 \tag{6}$$

then the system (1) can be mean square stabilized by the controller $u_t=K\left(A^t\hat{x}_t+\sum_{i=1}^{t}A^{t-i}Bu_{i-1}\right)$ with K being selected such that $A+BK$ is stable.

When g_t is known at the receiver, channel (2) resembles an AWGN channel. Shannon shows that when estimating a Gaussian random variable through an AWGN channel, the minimal mean square estimation error can be attained by using linear encoders and decoders, respectively [13]. And the minimal mean square error variance is given by $\frac{P\sigma_n^2}{\sigma_n^2+g_t^2P}$. Thus through one channel use, we can at best decrease the estimation error by a factor of $\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}$. Since g_t is i.i.d., we can transmit the estimation error from the decoder to the encoder and iteratively conduct the minimal mean square estimation process. Then the estimation error would decrease on average at a speed of $\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\}$. If $\lambda^2\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\}<1$, in view of Lemma 3, system (3) can be mean square stabilized. The estimation strategy actually follows the principle of the well-known scheme of Schalkwijk [14], which utilizes the noiseless feedback link to consecutively refine the estimation error. The detailed encoder/decoder design and stability analysis are given as follows.

1) Encoder/Decoder Design: Suppose the estimation of x_0 formed by the decoder is $\hat{x}_t$ at time t and the estimation error is $e_t = \hat{x}_t - x_0$. The encoder is designed as

$$\begin{aligned} s_0 &= \sqrt{\frac{P}{\sigma^2_{x_0}}} x_0 \\ s_t &= \sqrt{\frac{P}{\sigma^2_{e_{t-1}}}} \left(\hat{x}_{t-1} - x_0\right), \quad t \geq 1 \end{aligned} \tag{7}$$

The decoder is designed as

$$\begin{aligned} \hat{x}_0 &= \sqrt{\frac{\sigma^2_{x_0}}{P}} r_0 \\ \hat{x}_t &= \hat{x}_{t-1} - \frac{\mathbb{E}\{r_t e_{t-1}|g_t\}}{\mathbb{E}\{r_t^2|g_t\}} r_t, \quad t \geq 1 \end{aligned} \tag{8}$$

with $\sigma^2_{e_{t-1}}$ representing the variance of e_{t-1}.

2) Proof of Sufficiency: Since $r_0 = g_0 s_0 + n_0$, in view of (7) and (8), we have $e_0 = (g_0 - 1)x_0 + \sqrt{\frac{\sigma^2_{x_0}}{P}} n_0$. Because g_0, x_0, n_0 are independent and x_0, n_0 follows a zero mean Gaussian distribution, we know that the conditional probability distribution of e_0 given the event g_0 is Gaussian and $\mathbb{E}\{e_0|g_0\} = 0$, $\mathbb{E}\{e_0^2|g_0\} = (g_0 - 1)^2\sigma^2_{x_0} + \frac{\sigma^2_{x_0}\sigma^2_n}{P}$. Thus $\mathbb{E}\{e_0\} = \mathbb{E}\{\mathbb{E}\{e_0|g_0\}\} = 0$ and $\mathbb{E}\{e_0^2\} = \mathbb{E}\{\mathbb{E}\{e_0^2|g_0\}\} = \mathbb{E}\{(g_0 - 1)^2\}\sigma^2_{x_0} + \frac{\sigma^2_{x_0}\sigma^2_n}{P}$.

For $t \geq 1$, in view of (7) and (8), we have

$$\begin{aligned} e_t &= e_{t-1} - \frac{\mathbb{E}\{r_t e_{t-1}|g_t\}}{\mathbb{E}\{r_t^2|g_t\}} r_t \\ &= \Big(1 - g_t\sqrt{\frac{P}{\sigma^2_{e_{t-1}}}}\frac{\mathbb{E}\{r_t e_{t-1}|g_t\}}{\mathbb{E}\{r_t^2|g_t\}}\Big) e_{t-1} - \frac{\mathbb{E}\{r_t e_{t-1}|g_t\}}{\mathbb{E}\{r_t^2|g_t\}} n_t \end{aligned}$$

Thus the conditional probability distribution for e_t given the event g_t is Gaussian. We also have

$$\begin{aligned} \mathbb{E}\{e_t\} &= \mathbb{E}\{\mathbb{E}\{e_t|g_t\}\} \\ &= \mathbb{E}\Big\{\Big(1 - g_t\sqrt{\frac{P}{\sigma^2_{e_{t-1}}}}\frac{\mathbb{E}\{r_t e_{t-1}|g_t\}}{\mathbb{E}\{r_t^2|g_t\}}\Big)\mathbb{E}\{e_{t-1}|g_t\}\Big\} \\ &\overset{(a)}{=} \mathbb{E}\Big\{\Big(1 - g_t\sqrt{\frac{P}{\sigma^2_{e_{t-1}}}}\frac{\mathbb{E}\{r_t e_{t-1}|g_t\}}{\mathbb{E}\{r_t^2|g_t\}}\Big)\Big\}\mathbb{E}\{e_{t-1}\} \end{aligned}$$

where (a) follows from the fact that g_t is independent with e_{t-1}. Since $\mathbb{E}\{e_0\} = 0$, we further know that $\mathbb{E}\{e_t\} \equiv 0$. The sufficient condition (5) is satisfied.

Since e_{t-1}, g_t and n_t are independent, we have $\mathbb{E}\{e^2_{t-1}|g_t\} = \mathbb{E}\{e^2_{t-1}\}$ and $\mathbb{E}\{n_t^2|g_t\} = \mathbb{E}\{n_t^2\}$, which implies $\mathbb{E}\{r_t^2|g_t\} = \mathbb{E}\big\{\big(g_t\sqrt{\frac{P}{\sigma^2_{e_{t-1}}}}e_{t-1} + n_t\big)^2|g_t\big\} = \sigma_n^2 + g_t^2 P$ and $\mathbb{E}\{r_t e_{t-1}|g_t\} = \mathbb{E}\big\{e_{t-1}\big(g_t\sqrt{\frac{P}{\sigma^2_{e_{t-1}}}}e_{t-1} + n_t\big)|g_t\big\} = g_t\sqrt{P\sigma^2_{e_{t-1}}}$. Since $\mathbb{E}\{e_t^2|g_t\} = \mathbb{E}\{e^2_{t-1}|g_t\} - \frac{\mathbb{E}\{r_t e_{t-1}|g_t\}^2}{\mathbb{E}\{r_t^2|g_t\}}$, we also have $\mathbb{E}\{e_t^2|g_t\} = \mathbb{E}\{e^2_{t-1}\} - \frac{g_t^2 P\mathbb{E}\{e^2_{t-1}\}}{\sigma_n^2 + g_t^2 P} = \mathbb{E}\{e^2_{t-1}\}\frac{\sigma_n^2}{\sigma_n^2 + g_t^2 P}$, which implies $\mathbb{E}\{e_t^2\} = \mathbb{E}\{\mathbb{E}\{e_t^2|g_t\}\} = \mathbb{E}\{e^2_{t-1}\}\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2 + g_t^2 P}\}$. Thus if $\lambda^2\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2 + g_t^2 P}\} < 1$, the designed encoder/decoder pair can guarantee (6). In view of Lemma 3, the sufficiency of Theorem 1 is proved.

Remark 2: We can show that C_{MSC} is smaller than the Shannon capacity, which is $C_{\text{Shannon}} = \mathbb{E}\{c_t\}$ [9]. From Jensen's inequality, we know that $\mathbb{E}\{2^{-2c_t}\} \geq 2^{-2\mathbb{E}\{c_t\}}$ and the equality holds if and only if c_t is a constant. Thus it follows that $C_{\text{MSC}} = \frac{1}{2}\log\frac{1}{\mathbb{E}\{2^{-2c_t}\}} \leq \frac{1}{2}\log\frac{1}{2^{-2\mathbb{E}\{c_t\}}} = \mathbb{E}\{c_t\} = C_{\text{Shannon}}$ and the equality holds if and only if c_t is a constant.

Remark 3: By letting g_t in (4) be the Bernoulli distribution with failure probability ϵ, and taking the limit $\sigma_n^2 \to 0$ and $P \to \infty$, we can show that the necessary and sufficient condition to ensure mean square stabilizability for the real erasure channel is $\epsilon < \frac{1}{\lambda^2}$, which recovers the result in [5]. If we let g_t be a constant with $g_t = 1$, then the studied power constrained fading channel degenerates to the AWGN channel and the (4) degenerates to $\frac{1}{2}\log(1 + \frac{P}{\sigma_n^2}) < \log|\lambda|$, which recovers the result in [4], [6]. If $\sigma_n^2 = 0$ and the event $g_t = 0$ has zero probability measure, the right hand side of (4) becomes infinity. Then for any λ, (4) holds automatically. This is reasonable since we have assumed that g_t is known at the decoder side, thus if there is no additive noise, the channel resembles a perfect communication link. Since (3) is controllable, we can always find a pair of encoder and decoder to stabilize the system.

IV. Vector Systems

For vector systems, the situation becomes complicated by the fact that we have n sources $x_{i,0}$ and only one channel, where $x_{i,0}$ denotes the i-th element of x_0. Firstly, we would analyze the achievable minimal mean square estimation error for estimating x_0 over the channel (2) during one channel use. Consider the following Markov chain

$$X_0 \to S_t = f_t(X_0) \to R_t \to \hat{X}_t = h_t(R_t)$$

where $X_0 \in \mathbb{R}^n$ denotes the Gaussian initial state with covariance matrix Σ_{x_0}; $f_t(\cdot)$ is a scalar-valued function denoting the channel encoder for (2); R_t denotes the channel output and $\hat{X}_t$ is the estimation of X_0 formed by the decoder with decoding rule $h_t(\cdot)$.

Denote the estimation error as $e_t = X_0 - \hat{X}_t$, in view of Lemma 1, we have $\frac{1}{n}\text{tr}\mathbb{E}\{e_t e_t'\} \geq \frac{1}{2\pi e}e^{\frac{2}{n}H(e_t|R_t)}$. Since

$$\begin{aligned} H(e_t|R_t) &= H(X_0 - h_t(R_t)|R_t) = H(X_0|R_t) \\ &= H(X_0) - I(X_0; R_t) \\ &\overset{(a)}{=} H(X_0) - I(f_t(X_0); R_t) \\ &\geq \frac{1}{2}\ln((2\pi e)^n\det(\Sigma_{x_0})) - \frac{1}{2}\ln(1 + \frac{g_t^2 P}{\sigma_n^2}) \end{aligned}$$

where (a) follows from Lemma 2, thus we have

$$\text{tr}\mathbb{E}\{e_t e_t'\} \geq n\,\det(\Sigma_{x_0})\big(\frac{\sigma_n^2}{g_t^2 P + \sigma_n^2}\big)^{\frac{1}{n}}$$

From the above inequality, we know that the minimal mean square error is given in terms of $\frac{\sigma_n^2}{g_t^2 P + \sigma_n^2}$. However, this is only for the sum of the estimation errors $e_{i,t}$ with $e_{i,t}$ being the i-th element of e_t. There is no indication on the convergence speed for every single $e_{i,t}$. Lemma 3 implies that we should design the encoder/decoder to render that $\lim_{t\to\infty}\lambda_i^{2t}\mathbb{E}\{e_{i,t}^2\} = 0$ for all i, which places separate requirements for the convergence speed of each $e_{i,t}$. Thus we need to optimally allocate channel resources to each unstable state variable.

The previous analysis also implies that we should treat the unstable modes of A separately. Here we focus on the real Jordan canonical form of system (1). Let $\lambda_1,\ldots,\lambda_d$ be the distinct unstable eigenvalues (if λ_i is complex, we exclude from this list the complex conjugates λ_i^*) of A in (1), and let m_i be the algebraic multiplicity of each λ_i. The real Jordan canonical form J of A then has the block diagonal structure $J = \text{diag}(J_1,\ldots,J_d) \in \mathbb{R}^{n\times n}$, where the block $J_i \in \mathbb{R}^{\mu_i\times\mu_i}$ and $\det J_i = \lambda_i^{\mu_i}$, with

$$\mu_i = \begin{cases} m_i & \text{if } \lambda_i \in \mathbb{R} \\ 2m_i & \text{otherwise} \end{cases}$$

It is clear that we can equivalently study the following dynamical system instead of (1)

$$x_{k+1} = Jx_k + TBu_i \tag{9}$$

for some similarity matrix T. Let $\mathcal{U} = \{1,\ldots,d\}$ denote the index set of unstable eigenvalues.

Theorem 2: There exists a causal encoder/decoder pair $\{f_t\},\{h_t\}$, such that the LTI dynamics (1) can be stabilized over the communication channel (2) in mean square sense if

$$\sum_{i=1}^{d}\mu_i\log|\lambda_i| < -\frac{1}{2}\log\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\} \tag{10}$$

and only if $(\log|\lambda_1|,\ldots,\log|\lambda_d|) \in \mathbb{R}^d$ satisfy that for all $v_i \in \{0,\ldots,m_i\}$ and $i \in \mathcal{U}$

$$\sum_{i\in\mathcal{U}} a_i v_i\log|\lambda_i| < -\frac{v}{2}\log\mathbb{E}\{\big(\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\big)^{\frac{1}{v}}\} \tag{11}$$

where $v = \sum_{i\in\mathcal{U}} a_i v_i$, and $a_i = 1$ if $\lambda_i \in \mathbb{R}$, and $a_i = 2$ otherwise.

Proof: For the proof of necessity, notice that each block J_i has an invariant real subspace $\mathcal{A}_{v_i}$ of dimension $a_i v_i$, for any $v_i \in \{0,\ldots,m_i\}$. Consider the subspace $\mathcal{A}$ formed by taking the product of the invariant subspaces $\mathcal{A}_{v_i}$ for each real Jordan block. The total dimension of $\mathcal{A}$ is $v = \sum_{i\in\mathcal{U}} a_i v_i$. Denote by $x^{\mathcal{V}}$ of the components of x belonging to $\mathcal{A}$. Then $x^{\mathcal{V}}$ evolves as

$$x_{k+1}^{\mathcal{V}} = J^{\mathcal{V}} x_{k+1}^{\mathcal{V}} + QTu_k \tag{12}$$

where Q is a transformation matrix and $\det J^{\mathcal{V}} = \Pi_{i\in\mathcal{U}}\lambda_i^{a_i v_i}$. Since X_k is mean square stable, it is necessary that the subdynamics (12) is mean square stable. Similar to the necessity proof in Theorem 1, we may derive the necessary condition (11). And this completes the proof of necessity.

Here we prove the sufficiency using the idea of Time Division Multiple Access (TDMA). Based on the previous encoder/decoder design for scalar systems, the following information transmission strategy is designed for the vector system. Without loss of generality, here we assume that $\lambda_1,\ldots,\lambda_d$ are real and $m_i = 1$. For other cases, readers can refer to the analysis discussed in Chapter 2 of [1]. Specifically, under this assumption, J is a diagonal matrix and $d = n$. The sensor transmits periodically with a period of τ. During one channel use, the sensor only transmits the estimation error of the j-th value of x_0 using the scheme devised for scalar systems. The relative transmission frequency for the j-th value of x_0 is scheduled to be α_j among the τ transmission period with $\sum_{j=1}^n \alpha_j = 1$. The receiver maintains an array that represents the most recent estimation of x_0, which is set to 0 for $t = 0$. When the information about the j-th value of x_0 is transmitted, only the estimation of the j-th value of x_0 is updated at the decoder side, and the other estimation values remain unchanged. After updating the estimation, the controller takes action as the one designed in Lemma 3. If the diagonal elements of $A^t\mathbb{E}\{e_t e_t'\}(A')^t$ converge to zeros asymptotically, i.e., for $i = 1,\ldots,n$, $\lim_{t\to\infty}\lambda_i^{2t}\mathbb{E}\{e_{i,t}^2\} = 0$, the conditions in Lemma 3 can be satisfied. Since the transmission is scheduled periodically, we only need to require that $\lim_{k\to\infty}\lambda_i^{2k\tau}\mathbb{E}\{e_{i,k\tau}^2\} = 0$, $\forall i = 1,\ldots,n$. Following our designed transmission scheme, we have $\mathbb{E}\{e_{i,k\tau}^2\} = \mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\}^{\alpha_i k\tau}\mathbb{E}\{e_{i,0}^2\}$. If $\lambda_i^2\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\}^{\alpha_i} < 1$ for all $i = 1,\ldots n$, the sufficient condition in Lemma 3 can be satisfied. To complete the proof, we only need to show the equivalence between the requirement $\lambda_i^2\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\}^{\alpha_i} < 1$ for all $i = 1,\ldots n$ and (10). On one hand, since $\sum_{i=1}^n \alpha_i = 1$, if $\lambda_i^2\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\}^{\alpha_i} < 1$ for all $i = 1,\ldots n$, we know that (10) holds. On the other hand, if (10) holds, we can simply choose $\alpha_i = \frac{\log|\lambda_i|}{\sum_i\log|\lambda_i|}$, which satisfies the requirement that $\sum_{i=1}^n \alpha_i = 1$ and $\lambda_i^2\mathbb{E}\{\frac{\sigma_n^2}{\sigma_n^2+g_t^2P}\}^{\alpha_i} < 1$ for all $i = 1,\ldots,n$. The sufficiency is proved. ∎

V. Numerical Illustrations

A. Scalar Systems

The authors in [8] derive the mean square capacity of a power constrained fading channel with linear encoders/decoders. The necessary and sufficient condition for scalar systems is $\frac{1}{2}\log(1+\frac{\mu_g^2P}{\sigma_g^2P+\sigma_n^2}) > \log|\lambda|$ with μ_g and σ_g^2 being the mean and variance of g_t. We can similarly define the mean square capacity of the power constrained fading channel with linear encoders/decoders as $C_{\text{MSL}} = \frac{1}{2}\log(1+\frac{\mu_g^2P}{\sigma_g^2P+\sigma_n^2})$. Simply assume that the fading follows the Bernoulli distribution with failure probability ϵ, then the Shannon capacity, the mean square capacity achievable with causal encoders/decoders and the mean square capacity achievable with linear encoders/decoders are given as $C_{\text{Shannon}_{\text{BD}}} = \frac{1-\epsilon}{2}\log\big(1+\frac{P}{\sigma_n^2}\big)$, $C_{\text{MSC}_{\text{BD}}} = -\frac{1}{2}\log\big(\frac{\sigma_n^2+\epsilon P}{\sigma_n^2+P}\big)$, $C_{\text{MSL}_{\text{BD}}} = \frac{1}{2}\log\big(1+\frac{(1-\epsilon)^2P}{(1-\epsilon)\epsilon P+\sigma_n^2}\big)$. For fixed P and σ_n^2, the channel capacities are functions of ϵ. Let $P = 1$ and $\sigma_n^2 = 1$, the channel capacities in relation to the erasure probability are plotted in Fig. 2. It is clear that $C_{\text{Shannon}_{\text{BD}}} \geq C_{\text{MSC}_{\text{BD}}} \geq C_{\text{MSL}_{\text{BD}}}$ at any given erasure probability ϵ. This

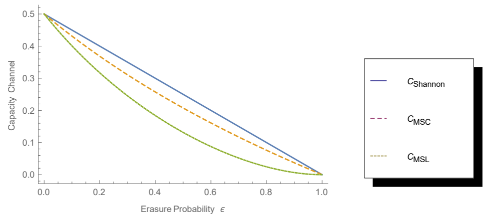

Fig. 2. Comparison of different channel capacities when $P = 1$, $\sigma_n^2 = 1$

result is obvious since we have proved that the Shannon capacity is no smaller than the mean square capacity with causal encoders/decoders. Besides, we have more freedom in designing the causal encoders/decoders compared with the linear encoders/decoders, thus allowing to achieve a higher capacity. The three kinds of capacity degenerate to the same when $\epsilon = 0$ and $\epsilon = 1$, which represent the AWGN channel case and the disconnected case respectively.

B. Vector Systems

Consider the two dimensional LTI system (9) with $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, and the communication channel is (2) in which the fading follows the Bernoulli distribution with failure probability ϵ. In view of Theorem 2, a sufficient condition to ensure mean square stabilizability is that $(\log|\lambda_1|, \log|\lambda_2|)$ should lie in the region of $\log|\lambda_1| + \log|\lambda_2| < C_{\mathrm{MSC_{BD}}}$. The necessary requirement is given by the following region in $(\log|\lambda_1|, \log|\lambda_2|)$ plane

$$\begin{cases} \log|\lambda_1| < C_{\mathrm{MSC_{BD}}}, \quad \log|\lambda_2| < C_{\mathrm{MSC_{BD}}} \\ \log|\lambda_1| + \log|\lambda_2| < -\log\left(\epsilon + (1-\epsilon)\left(\dfrac{\sigma_n^2}{\sigma_n^2 + P}\right)^{\frac{1}{2}}\right) \end{cases}$$

The necessary and sufficient condition to ensure mean square stability using linear encoders/decoders for this system is given in [8], which states that $(\log|\lambda_1|, \log|\lambda_2|)$ should be in the region constrained by $\log|\lambda_1| + \log|\lambda_2| < C_{\mathrm{MSL_{BD}}}$. Selecting $P = 1$, $\sigma_n^2 = 1$ and $\epsilon = 0.8$, we can plot the regions for $(\log|\lambda_1|, \log|\lambda_2|)$ indicated by the sufficiency and necessity in Theorem 2 and that indicated in Theorem 3.1 in [8] in Fig. 3. We can observe that the region of $(\log|\lambda_1|, \log|\lambda_2|)$ that can be stabilized with the designed causal encoders/decoders in Section IV is much larger than that can be stabilized by linear encoders/decoders in [8]. Thus by extending endocers/decoders from linear settings to causal requirements, we can tolerate more unstable systems.

VI. Conclusion

This paper characterized the requirement for a power constrained fading channel to allow the existence of a causal encoder/decoder pair that can mean square stabilize a discrete-time LTI system. The mean square capacity of the power constrained fading channel with causal encoders/decoders was given. It was shown that this mean square capacity is smaller than the Shannon capacity and they coincide with each other for some special situations. Throughout the paper, the capacity was derived with the assumption that there

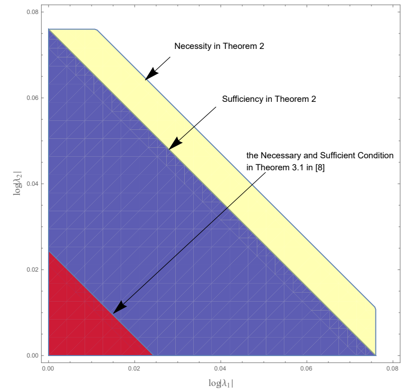

Fig. 3. Stability region of $(\log|\lambda_1|, \log|\lambda_2|)$ indicated by Theorem 2 for a vector system

exists a perfect feedback link from the channel output to the channel input. What would the capacity be for power constrained fading channels when there is no such feedback link or there is only a noisy feedback link is still under investigation.

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