Minimax Design of Stable IIR Filters with Sparse Coefficients

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***Abstract*—Coefficient sparsity of digital filters is of importance as it is directly related to implementation efficiency and cost. To date most work in the field has been focused on finite-impulse-response filters. In this paper, the design of sparse digital filters is investigated for a class of IIR filters. We propose a two-phase algorithm that promotes coefficient sparsity, maintains filter's stability, and optimizes its frequency response to approximate a desired frequency response in minimax sense. A design example is presented to illustrate the proposed algorithm and compare its performance relative to its nonsparse IIR and FIR counterparts.**

I. Introductions

Coefficient sparsity is an important issue in the analysis and design of digital filters as it is directly related to implementation efficiency and cost. Over the years, the issue has been investigated from different perspectives ranging from filter structures [1][2], filter classes [3][4], to sparsity-promoting designs of general FIR filters [5][6][7]. The recent studies of sparsity-promoting designs have been focused on FIR filters [4]–[7]. Among other things, a drawback of sparse FIR filters is larger group delay relative to their nonsparse counterparts.

In this paper, the design of sparse digital filters is examined for a class of IIR filters where numerator polynomials are sparse and denominator polynomials are of relatively lower order. An algorithm that promotes the coefficient sparsity of an IIR filter while approximating a desired frequency response in minimax sense with guaranteed stability is presented. A design experiment is included to demonstrate and compare the performance of a typical sparse IIR filter relative to its nonsparse counterparts.

II. A Class of Sparse IIR Filters and Design Problem

Consider an IIR transfer function

$$H(z) = \frac{a(z)}{z^{n-2r}d(z)} \tag{1a}$$

where

$$a(z) = \sum_{i=0}^{n} a_i z^{n-i}, \; d(z) = \prod_{i=1}^{r}(z^2 + d_{i1}z + d_{i2}) \tag{1b}$$

With (1), the IIR filter is of order n and possesses at most $2r$ non-trivial poles (i.e., poles not at the origin) and at least $n-2r$ poles at the origin. We remark that assigning certain number of poles at the origin is known to be beneficial for the design of several types of digital filters [8].

We also remark that there is an expression of $H(z)$ equivalent to (1) but in terms of z^{-1}, namely

$$H(z) = \frac{\sum_{i=0}^{n} a_i z^{-i}}{\prod_{i=1}^{r}(d_{i2}z^{-2} + d_{i1}z^{-1} + 1)} \tag{2}$$

In this paper, we are interested in a class of sparse IIR filters which assume the form of (1) with sparse numerator polynomial $a(z)$ and $2r \ll n$. A vector $\boldsymbol{a}$ of length $n+1$ is said to be k-sparse if the number of nonzero entries of $\boldsymbol{a}$ is no more than k with $k \ll n+1$. A polynomial $a(z)$ is said to be sparse if its coefficient vector $\boldsymbol{a} = [a_0 \;\; a_1 \;\; \cdots \;\; a_n]^T$ is sparse. Because of the sparsity of $\boldsymbol{a}$, the filter order n needs to be relatively large so as to produce satisfactory performance. By contrast, the denominator polynomial $d(z)$ is of low order (with r 2nd-order sections) and is not necessarily sparse.

Given a desired frequency response $H_d(\omega)$, filter order $(n, \; 2r)$, and sparsity k, our goal is to determine a stable IIR transfer function $H(z)$ of form (1) with a k-sparse numerator that approximates $H_d(\omega)$ in minimax sense.

III. Design Strategy and Algorithmic Details

The design strategy employed here can be described as a two-phase approach where phase one is aimed at identifying an index set, $\mathcal{I}^*$ with $|\mathcal{I}^*| = n+1-k$, for the coefficients of numerator $a(z)$ that are most suitable to be set to zero, while phase two is to optimize $H(z)$ so as to approximate $H_d(\omega)$ in minimax sense subject to sparsity and stability. The design approach is in spirit similar to that of [6], nevertheless the realization of the design idea here is considerably more involved primarily because of the nonconvex nature of the IIR design at hand. Our solution is in each design phase to develop a *convex* local model for the design and solve it sequentially, eventually leading to a solution of our design problem.

A. Phase One

Given order $(n, \; 2r)$, sparsity k and desired frequency $H_d(\omega)$, the purpose of phase 1 is to identify an index set of the most appropriate locations for the numerator polynomial $a(z)$ to be set to zero in order to satisfy a target sparsity. In a filter design context, this is done subject to (i) keeping closeness between $H(e^{j\omega})$ and $H_d(\omega)$ and (ii) stability of $H(z)$. The target sparsity of $a(z)$ is achieved by introducing the l_1-norm

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of $\boldsymbol{a}$ into objective function because minimizing the l_1-norm of $\boldsymbol{a}$ is known to promote its sparsity [9]. This yields a problem formulation below:

$$\begin{aligned} \text{minimize} \quad & \|H(e^{j\omega}, \boldsymbol{x}) - H_d(\omega)\|_\infty + \mu\|\boldsymbol{a}\|_1 & \text{(3a)} \\ \text{Subject to:} \quad & H(z) \text{ stable} & \text{(3b)} \end{aligned}$$

where $\|\boldsymbol{a}\|_1 = \sum_{i=0}^{h} |a_i|$, μ is a weight to control the regularization level, and the fidelity term is meant to be

$$\|H(e^{j\omega}, \boldsymbol{x}) - H_d(\omega)\|_\infty = \max_{\omega\in\Omega} \left| \frac{\sum_{i=0}^{n} a_i e^{-ji\omega}}{\prod_{i=1}^{r}(d_{i2}e^{-j2\omega} + d_{i1}e^{-j\omega} + 1)} - H_d(\omega) \right| \quad (4)$$

with $\boldsymbol{x} = [\boldsymbol{a}^T \ \boldsymbol{d}^T]^T$, $\boldsymbol{d} = [d_{11} \ d_{12} \ \dots \ d_{r1} \ d_{r2}]^T$. Evidently the problem in (3) is nonconvex because a part of the design variables is in the denominator. The problem is tackled by a local convex approximation of (3a) as follows. Suppose we are in the kth iteration with a known coefficient vector $\boldsymbol{x}_k$. In a vicinity of $\boldsymbol{x}_k$, $\boldsymbol{x}_k + \boldsymbol{\delta}$ with $\boldsymbol{\delta} = [\boldsymbol{\delta}_a^T \ \ \boldsymbol{\delta}_d^T]^T$, we use Taylor expansion of $H(\omega, \boldsymbol{x}_k + \boldsymbol{\delta})$ to write

$$H(\omega, \boldsymbol{x}_k + \boldsymbol{\delta}) \approx H(\omega, \boldsymbol{x}_k) + \boldsymbol{g}_k^T(\omega)\boldsymbol{\delta}$$

where $H(\omega, \boldsymbol{x}_k)$, $\boldsymbol{g}_k(\omega)$ as well as the desired frequency response $H_d(\omega)$ are complex-valued, which can be expressed as

$$\begin{aligned} H(\omega, \boldsymbol{x}_k) &= H_r(\omega, \boldsymbol{x}_k) + jH_i(\omega, \boldsymbol{x}_k) \\ \boldsymbol{g}_k(\omega) &= \boldsymbol{g}_{rk}(\omega) + j\boldsymbol{g}_{ik}(\omega) \\ H_d(\omega) &= H_{rd}(\omega) + jH_{id}(\omega) \end{aligned}$$

See Appendix for explicit expressions of gradient $\boldsymbol{g}_k(\omega)$. Under these circumstances, the objective function in (3a) admits a convex approximation as

$$(\max_{\omega} \ \|\tilde{\boldsymbol{G}}_k(\omega)\boldsymbol{\delta} + \boldsymbol{e}_k(\omega)\|_2) + \mu\|\boldsymbol{a}\|_1 \quad (5)$$

as long as $\boldsymbol{\delta}$ is small in magnitude, where

$$\tilde{\boldsymbol{G}}_k(\omega) = \begin{bmatrix} \boldsymbol{g}_{rk}^T(\omega) \\ \boldsymbol{g}_{ik}^T(\omega) \end{bmatrix}, \ \boldsymbol{e}_k(\omega) = \begin{bmatrix} H_r(\omega, \boldsymbol{x}_k) - H_{rd}(\omega) \\ H_i(\omega, \boldsymbol{x}_k) - H_{id}(\omega) \end{bmatrix}$$

Concerning the stability constraint in (3b), denote $\boldsymbol{x}_k = [\boldsymbol{a}_k^T \ \ \boldsymbol{d}_k^T]^T$ with $\boldsymbol{d}_k = [d_{11}^{(k)} \ d_{12}^{(k)} \ \cdots \ d_{r1}^{(k)} \ d_{r2}^{(k)}]^T$, then $\boldsymbol{d}_k$ represents a stable $d(z)$ if and only if [10]

$$\hat{\boldsymbol{C}}\boldsymbol{d}_i^{(k)} + \boldsymbol{e} > \boldsymbol{0} \qquad \text{for } 1 \le i \le r$$

where

$$\hat{\boldsymbol{C}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}, \ \boldsymbol{d}_i^{(k)} = \begin{bmatrix} d_{i1}^{(k)} \\ d_{i2}^{(k)} \end{bmatrix}, \ \boldsymbol{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

To ensure the stability of perturbed $\boldsymbol{d}_k + \boldsymbol{\delta}_d$ and prevent the filter's poles from being too close to the stability boundary, we impose the constraints

$$\hat{\boldsymbol{C}}(\boldsymbol{d}_i^{(k)} + \boldsymbol{\delta}_{di}) + (1-\tau)\boldsymbol{e} \ge \boldsymbol{0} \qquad \text{for } 1 \le i \le r \quad (6)$$

where $\boldsymbol{\delta}_{di}$ is the vector consisting of the ith pair of components of $\boldsymbol{\delta}_d$, and $\tau > 0$ is a parameter controlling the filter's stability margin. Putting together the r constraints in (6) leads to

$$\tilde{\boldsymbol{C}}\boldsymbol{\delta}_d + \boldsymbol{s}_k \ge \boldsymbol{0} \quad (7)$$

where $\tilde{\boldsymbol{C}} = \text{diag}\ \{\hat{\boldsymbol{C}}, \ \dots, \ \hat{\boldsymbol{C}}\} \in R^{3r\times 2r}$ and $\boldsymbol{s}_k$ is a $2r$-vector with $\hat{\boldsymbol{C}}\boldsymbol{d}_i^{(k)} + (1-\tau)\boldsymbol{e}$ as its ith pair of components. Based on the analysis above, the kth iterate $\boldsymbol{x}_k$ is updated to $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{\delta}^*$ where $\boldsymbol{\delta}^*$ solves

$$\begin{aligned} \underset{\boldsymbol{\delta}}{\text{minimize}}(\max_{\omega} \ \|\tilde{\boldsymbol{G}}_k(\omega)\boldsymbol{\delta} + \boldsymbol{e}_k(\omega)\|_2) + \mu\|\boldsymbol{a}_k + \boldsymbol{\delta}_a\|_1 \quad & \text{(8a)} \\ \text{subject to: } \tilde{\boldsymbol{C}}\boldsymbol{\delta}_d + \boldsymbol{s}_k \ge \boldsymbol{0} \quad & \text{(8b)} \\ \|\boldsymbol{\delta}\|_2 \le \beta \quad & \text{(8c)} \end{aligned}$$

To solve (8), we let η be an upper bound of $\|\tilde{\boldsymbol{G}}(\omega)\boldsymbol{\delta} + \boldsymbol{e}_k(\omega)\|$ over Ω — a set of K grid points in the frequency region of interest, namely $\Omega = \{\omega_i, \ 1 \le i \le K\}$. In addition, the non-differentiable l_1-norm in (8a) is handled by replacing it with $\sum_i u_i$ where each u_i acts as an upper bound of component $a_k(i) + \delta_a(i)$. By treating η and $\boldsymbol{u} = [u_0 \ u_1 \ \cdots \ u_n]^T$ as auxiliary design variables and denoting $\boldsymbol{y} = [\eta \ \boldsymbol{\delta}^T \ \boldsymbol{u}^T]^T$, problem (8) can be formulated as

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{f}^T\boldsymbol{y} & \text{(9a)} \\ \text{subject to:} \quad & \|\boldsymbol{G}_k(\omega)\boldsymbol{y} + \boldsymbol{e}_k(\omega)\|_2 \le \boldsymbol{b}^T\boldsymbol{y}, \ \omega \in \Omega & \text{(9b)} \\ & |a_k(i) + \delta_a(i)| \le u_i \quad \text{for } 0 \le i \le n & \text{(9c)} \\ & \boldsymbol{C}\boldsymbol{y} + \boldsymbol{s}_k \ge \boldsymbol{0} & \text{(9d)} \\ & \|\hat{\boldsymbol{I}}\boldsymbol{y}\|_2 \le \beta & \text{(9e)} \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{f} &= [1 \ \ 0 \ \ \cdots \ \ 0 \ \ \mu \ \cdots \ \mu]^T \in R^{1+(n+1+2r)+(n+1)} \\ \boldsymbol{b} &= [1 \ \ 0 \ \ \cdots \ \ 0]^T \in R^{2(n+r)+3} \\ \boldsymbol{G}_k(\omega) &= [\boldsymbol{0} \ \ \tilde{\boldsymbol{G}}_k(\omega) \ \ \boldsymbol{0}]^T \in R^{2\times[2(n+r)+3]} \\ \boldsymbol{C} &= [\boldsymbol{0} \ \ \tilde{\boldsymbol{C}} \ \ \boldsymbol{0}] \in R^{3r\times[2(n+r)+3]} \\ \hat{\boldsymbol{I}} &= [\boldsymbol{0} \ \ \boldsymbol{I}_{(n+1+2r)} \ \ \boldsymbol{0}] \in R^{(n+1+2r)\times[2(n+r)+3]} \end{aligned}$$

Problem (9) is a standard second-order cone program (SOCP) [11] and can be solved by efficient convex program solver such as SeDuMi.

Once the unique global solution $\boldsymbol{\delta}^*$ of (9) is obtained, $\boldsymbol{x}_k$ is updated to $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{\delta}^*$ and the kth iteration is complete. The process continues until certain termination criterion is met. Options of termination condition include a fixed number of iterations, or the smallness of the updating vector $\boldsymbol{\delta}^*$ against a prescribed tolerance. In this way, a solution point $\boldsymbol{x}^* = [\boldsymbol{a}^{*T} \ \ \boldsymbol{d}^{*T}]^T$ is found and hard-thresholding is applied to $\boldsymbol{a}^*$ to generate an index set

$$\mathcal{I}^* = \{i : \ |a_i^*| < \varepsilon_t\} \quad (10)$$

where ε_t is a threshold so tuned that the length of $\mathcal{I}^*$ meets the sparsity requirement $|\mathcal{I}^*| = n - k + 1$ to ensure that the number of nonzero coefficients in $a(z)$ is precisely equal to k.

B. Phase Two

With index set $\mathcal{I}^*$ identified, phase 2 of the design optimizes $H(z)$ so that it approximates $H_d(\omega)$ in minimax sense subject to $a_i = 0$ for $i \in \mathcal{I}^*$ and filter stability. The design is also carried out in an iterative manner with its kth iterate $\boldsymbol{x}_k = [\boldsymbol{a}_k \;\; \boldsymbol{d}_k]^T$ updated to $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{\delta}^{**}$ where $\boldsymbol{\delta}^{**}$ solves the convex problem

$$\begin{aligned} \underset{\boldsymbol{\delta}}{\text{minimize}} \;\; \max_{\omega} \quad & \|\tilde{\boldsymbol{G}}_k(\omega)\boldsymbol{\delta} + \boldsymbol{e}_k(\omega)\|_2 && \text{(11a)} \\ \text{subject to:} \quad & a_i = 0 \quad \text{for } i \in \mathcal{I}^* && \text{(11b)} \\ & \tilde{\boldsymbol{C}}\boldsymbol{\delta}_d + \boldsymbol{s}_k \geq \boldsymbol{0} && \text{(11c)} \\ & \|\boldsymbol{\delta}\|_2 \leq \beta && \text{(11d)} \end{aligned}$$

On comparing (11) with (8), we note that in (11a) the l_1-norm term is removed because the index set $\mathcal{I}^*$ has been identified, and that the constraints in (11b) are imposed in order to ensure the filter's sparsity.

Let $\bar{\boldsymbol{G}}_k(\omega)$ be the submatrix of $\tilde{\boldsymbol{G}}_k(\omega)$ consisting of k columns of $\tilde{\boldsymbol{G}}_k(\omega)$ whose indices are not included in $\mathcal{I}^*$ and $\bar{\boldsymbol{\delta}} = [\bar{\boldsymbol{\delta}}_a^T \;\; \boldsymbol{\delta}_d^T]^T$ with $\bar{\boldsymbol{\delta}}_a$ collecting the k components of $\boldsymbol{\delta}_a$ whose indices are not included in $\mathcal{I}^*$. It is evident that imposing the constraints in (11b) simplifies the term $\tilde{\boldsymbol{G}}_k(\omega)\boldsymbol{\delta}$ in (11a) and term $\boldsymbol{\delta}$ in (11d) to $\bar{\boldsymbol{G}}_k(\omega)\bar{\boldsymbol{\delta}}$ and $\bar{\boldsymbol{\delta}}$, respectively, and thus leads (11) to

$$\begin{aligned} \underset{\boldsymbol{\delta}}{\text{minimize}} \;\; \max_{\omega} \quad & \|\bar{\boldsymbol{G}}_k(\omega)\bar{\boldsymbol{\delta}} + \boldsymbol{e}_k(\omega)\|_2 && \text{(12a)} \\ & \tilde{\boldsymbol{C}}\boldsymbol{\delta}_d + \boldsymbol{s}_k \geq \boldsymbol{0} && \text{(12b)} \\ & \|\bar{\boldsymbol{\delta}}\|_2 \leq \beta && \text{(12c)} \end{aligned}$$

If we let $\bar{\eta}$ be an upper bound of $\|\bar{\boldsymbol{G}}_k(\omega)\bar{\boldsymbol{\delta}} + \boldsymbol{e}_k(\omega)\|_2$ over Ω and treat $\bar{\eta}$ as an auxiliary design variable, then (12) becomes a standard SOCP as

$$\begin{aligned} \text{minimize} \quad & \bar{\eta} && \text{(13a)} \\ \text{subject to:} \quad & \|\bar{\boldsymbol{G}}_k(\omega)\bar{\boldsymbol{\delta}} + \boldsymbol{e}_k(\omega)\|_2 \leq \bar{\eta} && \text{(13b)} \\ & \tilde{\boldsymbol{C}}\boldsymbol{\delta}_d + \boldsymbol{s}_k \geq \boldsymbol{0} && \text{(13c)} \\ & \|\bar{\boldsymbol{\delta}}\|_2 \leq \beta && \text{(13d)} \end{aligned}$$

Solving (13), the global and unique solution $\bar{\boldsymbol{\delta}}^{**} = [\bar{\boldsymbol{\delta}}_a^{**T} \;\; \boldsymbol{\delta}_d^{**T}]^T$ of dimension $k+2r$ is used to produce a vector $\boldsymbol{\delta}^{**}$ of dimension $n+1+2r$ in that $\boldsymbol{\delta}_d^{**}$ takes over the last $2r$ components and $\bar{\boldsymbol{\delta}}_a^{**}$ occupies the k components whose indices are not included in $\mathcal{I}^*$ while the remaining $n+1-k$ components are set to zero. If one starts above iterative procedure with an initial point $\boldsymbol{x}_0 = [\boldsymbol{a}_0^T \;\; \boldsymbol{d}_0^T]^T$ with $a_0(i) = 0$ for $i \in \mathcal{I}^*$, then the updated point $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{\delta}^{**}$ maintains the desired coefficient sparsity as the iteration continues. A solution $\boldsymbol{x}^* = \boldsymbol{x}_{k+1}$ is deemed found when a given number of iterations have been executed or the progress measured by $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\|_2$ becomes less than a prescribed tolerance.

IV. An Illustrative Example

Consider designing a lowpass IIR filter of order $(n = 35,\ 2r = 2)$ of form (1) where $a(z)$ possesses at least 16 zero coefficients i.e. $k = 20$. The desired frequency response is characterized by normalized passband edge $\omega_p = 0.2\pi$, stopband edge $\omega_a = 0.23\pi$, and passband group delay 21. A total of $K = 200$ grid points were uniformly placed over $[0, \omega_p]$ and $[\omega_a,\ \pi]$ to form set Ω. With $\beta = 0.1$, $\tau = 0.05$, $\mu = 0.005$, and $\varepsilon_t = 0.004$, 20 iterations were performed for phase-1 of the design, which produced an index set $\mathcal{I}^*$ of length 20 as $\mathcal{I}^* = \{1, 2, 3, 13, 14, 15, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 31, 35, 36\}$. With $\mathcal{I}^*$ identified, additional 20 iterations were performed for phase-2 of the design. The coefficients of $a(z)$ obtained are shown in Table 1. The denominator $d(z)$ is a second-order polynomial with $d_{11} = -1.5230137262$ and $d_{12} = 0.9500000026$. The magnitude of the two poles was found to be 0.9747 hence the filter was stable. The maximum passband ripple, minimum stopband attenuation, and relative maximum ripple in passband group delay were found to be 0.0325, 29.727 dB, and 0.1818, respectively. The amplitude response (solid line) of the filter is depicted in Fig. 1. For comparison, a stable non-sparse lowpass IIR filter of order $(n = 19,\ 2r = 2)$ with the same design specifications as the above IIR filter (except the passband group delay which was set to 12 for the best performance) was designed based on a formulation similar to that in (11) without the sparsity constraints in (11b). The reason of using $n = 19$ here is because the sparse $a(z)$ obtained above possesses only 20 nonzero coefficients. The maximum passband ripple, minimum stopband attenuation, and relative maximum ripple in passband group delay were found to be 0.0676, 23.4019 dB, and 0.4579, respectively. The amplitude response (dashed line) of the filter is depicted in Fig. 1. Comparisons of the sparse IIR filter with a minimax FIR filter were also made. With $\omega_p = 0.2\pi$ and $\omega_a = 0.23\pi$, a 88-tap equiripple FIR filter was designed using the Parks-McClellan algorithm to achieve an amplitude response comparable to that of the sparse IIR filter. The phase response of the FIR filter is perfectly linear, however its group delay is 43.5 compared to 21 offered by the sparse IIR filter. In addition, with 88 taps the FIR filter requires considerably more multiplications and additions per output relative to the sparse IIR filters with only 20 nonzero coefficients in $a(z)$ and 2 non-unity coefficients in $d(z)$.

Based on these experimental results, it is observed that sparse IIR filters have great potential to offer satisfactory filtering performance as well as implementation efficiency. It should be mentioned that a fairly large number of designs of various kinds of sparse IIR filters have been conducted and their design advantages similar to those reported above have also been observed.

V. Conclusions

A two-phase design technique for stable minimax IIR filters with sparse coefficients has been proposed. In each design phase, the technique is based on a local convex model which admits fast and reliable update for the design parameters in an iterative manner. A design example is presented to illustrate the proposed algorithm and its good performance relative to its nonsparse IIR and FIR counterparts.

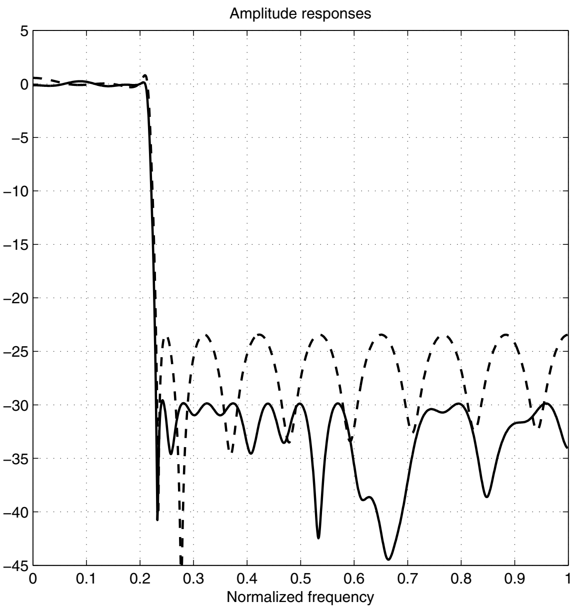

Fig. 1. Amplitude responses of (a) sparse IIR filter (solid line) and (b) an equivalent non-sparse IIR filter (dashed line).

Table 1: Coefficients of $a(z)$

0.011726332436338
−0.003775854426037
−0.005240441003938
0
0
0
0
0
0
0
0
0
0.000124549828270
−0.020388428612495
0.003112974735184
0
0
0.016810973368422
0.030391321509228
0.041323455889871
0.057160076121558
0.052063844635132
0.062030480871141
0.033756049688599
0.058377324058785
−0.002111926524235
0.030414917964838
0.063681289034808
0
0
−0.009069597853507
0
0
0
−0.007229880225230
0.008264197210904

APPENDIX

Let $\boldsymbol{x} = [\boldsymbol{a}^T \quad \boldsymbol{d}^T]^T$ and denote the ith pair of components of $\boldsymbol{d}$ by $\boldsymbol{d}_i$. The gradient of $H(\omega, \boldsymbol{x})$ is given by

$$g(\omega, \boldsymbol{x}) = \begin{bmatrix} \frac{\partial H(\omega,\boldsymbol{x})}{\partial \boldsymbol{a}} \\ \frac{\partial H(\omega,\boldsymbol{x})}{\partial \boldsymbol{d}_1} \\ \vdots \\ \frac{\partial H(\omega,\boldsymbol{x})}{\partial \boldsymbol{d}_r} \end{bmatrix}$$

where

$$\begin{aligned} \frac{\partial H(\omega, \boldsymbol{x})}{\partial \boldsymbol{a}} &= \frac{\boldsymbol{v}(\omega)}{d(e^{j\omega})} \\ \frac{\partial H(\omega, \boldsymbol{x})}{\partial \boldsymbol{d}_i} &= -H(\omega, \boldsymbol{x}) \frac{\boldsymbol{v}_2(\omega)}{1 + \boldsymbol{d}_i^T \boldsymbol{v}_2(\omega)} \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{v}(\omega) &= \boldsymbol{c}(\omega) - j\boldsymbol{s}(\omega) \\ \boldsymbol{c}(\omega) &= [1 \quad \cos\omega \ \cdots \ \cos n\omega]^T \\ \boldsymbol{s}(\omega) &= [0 \quad \sin\omega \ \cdots \ \sin n\omega]^T \\ \boldsymbol{v}_2(\omega) &= \begin{bmatrix} \cos\omega \\ \cos 2\omega \end{bmatrix} - j \begin{bmatrix} \sin\omega \\ \sin 2\omega \end{bmatrix} \end{aligned}$$

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