Achievable Rate Regions for Cooperative Relay Broadcast Channels with Feedback

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***Abstract*—Achievable rate regions for cooperative relay broadcast channels with rate-limited feedback are proposed. Specifically, we consider two-receiver memoryless broadcast channels where each receiver sends feedback signals to the transmitter through a noiseless and rate-limited feedback link, meanwhile, acts as a relay to transmit cooperative information to the other receiver. It's shown that the proposed rate regions improve on the known regions that consider either relaying cooperation or feedback communication, but not both.**

I. Introduction

Relay broadcast channels (RBCs) describe communication networks where the transmitter sends information to a set of receivers with the help of relaying communication. In [1], [2], the dedicated-relay broadcast channel (BC) model was studied, where a relay node was introduced to the original BC to assist the cooperation between two receivers. Another RBC model, called cooperative RBC model was studied in [3], [4], where each receiver acts as a relay and sends cooperative information to the other receiver. It was shown that even partially cooperation (only one receiver relays cooperative information) still improves on the capacity region of original BC.

In a different line of work, many studies have been done on memoryless BCs with feedback, where the receivers send feedback signals to the transmitter through feedback links. In [5], it shows that feedback cannot increase the capacity region for all physically degraded BCs. The first simple example BC where feedback increases capacity was presented by Dueck [6]. Based on Dueck's idea, Shayevitz and Wigger [7] proposed an achievable region for BCs with generalized feedback. Other achievable regions for BCs with perfect or noisy feedback, have been proposed by Kramer [8] and Venkataramanan and Pradhan [9]. Most recently, Wu and Wigger [10], [11] showed that any positive feedback rate can increase the capacity region for a large class of BCs, called strictly essentially less noisy BCs, unless it is physically degraded.

Cooperative RBCs with prefect feedback was investigated in [3], where the capacity region was established for the case of perfect feedback from the receiver to the relay. In this paper, we consider the cooperative RBCs with rate-limited feedback from the receivers to the transmitter, i.e., each receiver sends feedback signals to the transmitter through a noiseless and rate-limited feedback link, and meanwhile, acts as a relay to transmit cooperative information to the other receiver.

In the first work, we first study the *partially* cooperative RBC with one-sided feedback (only one receiver sends feedback signals and relays cooperative information to the other receiver). We proposed a new coding scheme (Scheme 1) based on block-Markov coding, Marton's coding [12], partial decode-forward [13] and compress-forward strategies [13]. Specifically, in each block, the transmitter uses Marton's coding to send the source messages and *forward* the feedback message. The receiver who acts as relay performs combined partial decode-forward and compress-forward, and sends the compression message as feedback information. The other Receiver uses backward decoding to jointly decode its private message and the compression message. It is shown that when feedback rate is sufficiently large, our coding scheme strictly improves on Liang and Kramer's region [4], which is tight for the semideterministic partially cooperative RBCs and orthogonal partially cooperative RBCs.

In the second work, we study the *fully* cooperative RBCs with two-sided feedback (both receiver send feedback signals and relay cooperative information). Two block-Markov coding schemes (Scheme 2A and 2B) are proposed based on Scheme 1. Specifically, in each block, the transmitter uses Marton's coding to send the source messages and forward the feedback messages sent by both receivers. In Scheme 2A, both receivers apply compress-forward and backward decoding. Scheme 2B is similar to Scheme 1A except that one of the two receiver uses hybrid relaying strategy and sliding-window decoding. The resulting rate regions strictly improve on Wu and Wigger's region [10, Theorem 1], which shows that feedback strictly increases capacity region for a large class of BCs.

Note that in our coding schemes the transmitter can reconstruct the receivers' inputs due to a delicate design, which allows to superimpose the Marton's codes on the receivers inputs, and thus attains cooperation between the transmitter and the receivers.

This paper is organized as follows. Section II describes cooperative RBC with feedback and our main results are presented in Section III. Section IV compares various achievable rate regions and shows that our regions strictly improve the known rate regions that consider either relay cooperation or feedback communication, but not both. Sections VI and V contain the proofs of our results in Section III. Finally, Section

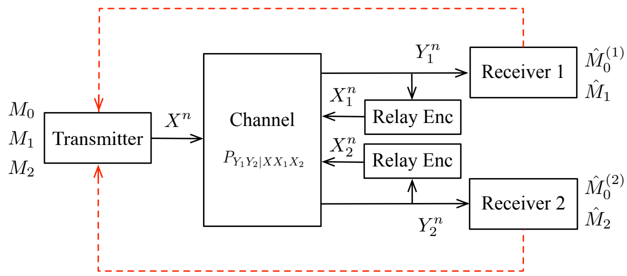

Fig. 1. Cooperative relay broadcast channel with feedback

VII concludes this paper.

Notations: We use capital letters to denote random variables and small letters for their realizations, e.g. X and x. For $j \in \mathbb{Z}^+$, we use the short hand notations X^j and x^j for the tuples $X^j := (X_1, \dots, X_j)$ and $x^j := (x_1, \dots, x_j)$. Given a positive integer n, let $\mathbf{1}_{[n]}$ denote the all-one tuple of length n, e.g., $\mathbf{1}_{[3]} = (1, 1, 1)$. The abbreviation i.i.d. stands for *independent and identically distributed*.

Given a distribution P_A over some alphabet $\mathcal{A}$, a positive real number $\varepsilon > 0$, and a positive integer n, let $\mathcal{T}_\varepsilon^n(P_A)$ denote the typical set in [14].

II. System Model

Consider 3-node cooperative RBC with feedback, as shown in Fig. 1. This setup is characterized by seven finite alphabets $\mathcal{X}, \mathcal{X}_k, \mathcal{Y}_k, \mathcal{F}_k$, for $k \in \{1, 2\}$, a channel law $P_{Y_1Y_2|XX_1X_2}$ and nonnegative feedback rates $R_{\text{fb},1}, R_{\text{fb},2}$. Specifically, at discrete-time $i \in [1:n]$, the transmitter sends input $x_i \in \mathcal{X}$. Receiver k observes output $y_{k,i} \in \mathcal{Y}_k$ and relays cooperative information $x_{k,i} \in \mathcal{X}_k$ to the other receiver. When both receiver relay information, it is called *fully* cooperative RBC. When only one receiver relays information, it is called *partially* cooperative RBC. After observing $y_{k,i}$, Receiver k also sends a feedback signal $f_{k,i} \in \mathcal{F}_{k,i}$ to the transmitter, where $\mathcal{F}_{k,i}$ denotes the finite alphabet of $f_{k,i}$. The feedback link between the transmitter and Receiver k is noiseless and *rate-limited* to $R_{\text{fb},k}$ bits per channel use. In other words, if the transmission takes place over a total blocklength n, then

$$|\mathcal{F}_{k,1}| \times \cdots \times |\mathcal{F}_{k,n}| \le 2^{nR_{\text{fb},k}}, \quad k \in \{1, 2\}. \tag{1}$$

In the communication, the transmitter wishes to send message $M_0 \in [1 : 2^{nR_0}]$ to both receivers, and message $M_k \in [1 : 2^{nR_k}]$ to Receiver k.

A $(2^{nR_0}, 2^{nR_1}, 2^{nR_2}, n)$ code for this channel consists of

- message sets $\mathcal{M}_0 := [1 : 2^{nR_0}]$ and $\mathcal{M}_k := [1 : 2^{nR_k}]$;
- a source encoder that maps (M_0, M_1, M_2) to a sequence $X_i\big(M_0, M_1, M_2, F_1^{i-1}, F_2^{i-1}\big)$;
- two receiver encoders where Receiver k maps Y_k^{i-1} to a sequence $X_{k,i}(Y_k^{i-1})$;
- two decoders where Receiver k estimates $(\hat{M}_0^{(k)}, \hat{M}_k)$ based on Y_k^n,

for each $i \in [1 : n]$ and $k \in \{1, 2\}$. Suppose M_0, M_1 and M_2 are uniformly distributed and independent with each other. A rate tuple (R_0, R_1, R_2) with average feedback rates $R_{\text{fb},k}$, for $k \in \{1, 2\}$, is called achievable if for every blocklength n, there exists a $(2^{nR_0}, 2^{nR_1}, 2^{nR_2}, n)$ code such that the average probability of error

$$P_e^{(n)} = \Pr[(\hat{M}_0^{(1)}, \hat{M}_0^{(2)}, \hat{M}_1, \hat{M}_2) \neq (M_0, M_0, M_1, M_2)]$$

tends to 0 as $n \to \infty$. The capacity region is all nonnegative rate tuples (R_0, R_1, R_2) such that $\lim_{n\to\infty} P_e^{(n)} = 0$.

III. Main Results

In this section, we present our main results as the following theorems. The proofs are given in Section V and Section VI.

Theorem 1: For the *partially* cooperative BRC with receiver-transmitter feedback, the capacity region includes the set $\mathcal{R}_1$ of all nonnegative rate tuples (R_0, R_1, R_2) that satisfy

$$R_0 + R_1 \le I(U_0, U_1; Y_1|X_1) \tag{2a}$$
$$R_0 + R_2 \le I(U_0, U_2, X_1; Y_2) - I(\hat{Y}_1; Y_1|U_0, U_2, X_1, Y_2) \tag{2b}$$
$$R_0 + R_1 + R_2 \le I(U_1; Y_1|U_0, X_1) + I(U_0, U_2, X_1; Y_2) - I(\hat{Y}_1; Y_1|U_0, U_2, X_1, Y_2) - I(U_1; U_2|U_0, X_1) \tag{2c}$$
$$R_0 + R_1 + R_2 \le I(U_0, U_1; Y_1|X_1) + I(U_2; \hat{Y}_1, Y_2|U_0, X_1) - I(U_1; U_2|U_0, X_1) \tag{2d}$$
$$2R_0 + R_1 + R_2 \le I(U_0, U_1; Y_1|X_1) + I(U_0, U_2, X_1; Y_2) - I(\hat{Y}_1; Y_1|U_0, U_2, X_1, Y_2) - I(U_1; U_2|U_0, X_1) \tag{2e}$$

for some pmf $P_{U_0U_1U_2X_1}P_{\hat{Y}_1|U_0X_1Y_1}$ and function $X = f(U_0, U_1, U_2)$ such that

$$I(\hat{Y}_1; Y_1|U_0, X_1) \le R_{\text{fb},1}. \tag{2f}$$

Proof: See Section V. ■

Remark 1: The rate constraint (2f) can be relaxed as

$$I(\hat{Y}_1; Y_1|U_0, X_1, Y_2) \le R_{\text{fb},1} \tag{3}$$

by using a trick in [11, Section V], where the receivers use the feedback links to send Wyner-Ziv compression messages about their previously observed outputs to the transmitter.

Remark 2: If $\hat{Y}_1 = \emptyset$, i.e., no feedback signal is sent by Receiver 1, then rate region $\mathcal{R}_1$ reduces to $\mathcal{R}_{\text{Liang}}$, which is the set of all nonnegative rate tuples (R_0, R_1, R_2) satisfying

$$R_0 + R_1 \le I(U_0, U_1; Y_1|X_1) \tag{4a}$$
$$R_0 + R_2 \le I(U_0, U_2, X_1; Y_2) \tag{4b}$$
$$R_0 + R_1 + R_2 \le I(U_1; Y_1|U_0, X_1) + I(U_0, U_2, X_1; Y_2) - I(U_1; U_2|U_0, X_1) \tag{4c}$$
$$R_0 + R_1 + R_2 \le I(U_0, U_1; Y_1|X_1) + I(U_2; Y_2|U_0, X_1) - I(U_1; U_2|U_0, X_1) \tag{4d}$$
$$2R_0 + R_1 + R_2 \le I(U_0, U_1; Y_1|X_1) + I(U_0, U_2, X_1; Y_2) - I(U_1; U_2|U_0, X_1) \tag{4e}$$

for some pmf $P_{U_0U_1U_2X_1}$ and function $X = f(U_0, U_1, U_2)$. This rate region was proposed by Liang and Kramer [4, Theorem 2], and was shown to be the capacity region for semideterministic partially cooperative RBCs and orthogonal partially cooperative RBCs.

Theorem 2: For the *fully* cooperative BRC with two-sided and rate-limited feedback, the capacity region includes the set $\mathcal{R}_2$ of all nonnegative rate tuples (R_0, R_1, R_2) that satisfy

$$\begin{aligned}
R_0 + R_1 &\le I(U_0, U_1; \hat{Y}_2, Y_1|X_1, X_2) + \Delta_1 && \text{(5a)}\\
R_0 + R_2 &\le I(U_0, U_2; \hat{Y}_1, Y_2|X_1, X_2) + \Delta_2 && \text{(5b)}\\
R_0 + R_1 + R_2 &\le I(U_0, U_1; \hat{Y}_2, Y_1|X_1, X_2) + \Delta_1\\
&\quad + I(U_2; Y_2, \hat{Y}_1|U_0, X_1, X_2)\\
&\quad - I(U_1; U_2|U_0, X_1, X_2) && \text{(5c)}\\
R_0 + R_1 + R_2 &\le I(U_0, U_2; \hat{Y}_1, Y_2|X_1, X_2) + \Delta_2\\
&\quad + I(U_1; Y_1, \hat{Y}_2|U_0, X_1, X_2)\\
&\quad - I(U_1; U_2|U_0, X_1, X_2) && \text{(5d)}\\
2R_0 + R_1 + R_2 &\le I(U_0, U_1; \hat{Y}_2, Y_1|X_1, X_2) + \Delta_1\\
&\quad + I(U_0, U_2; \hat{Y}_1, Y_2|X_1, X_2) + \Delta_2\\
&\quad - I(U_1; U_2|U_0, X_1, X_2) && \text{(5e)}
\end{aligned}$$

for some pmf $P_{X_1}P_{X_2}P_{U_0U_1U_2|X_1X_2}P_{\hat{Y}_1|X_1Y_1}P_{\hat{Y}_2|X_2Y_2}$ and function $X = f(U_0, U_1, U_2)$ such that

$$I(\hat{Y}_1; Y_1|X_1) \le R_{\text{fb},1} \quad \text{and} \quad I(\hat{Y}_2; Y_2|X_2) \le R_{\text{fb},2} \quad \text{(5f)}$$

where

$$\begin{aligned}
\Delta_1 &= \min\{0, I(X_2; Y_1|X_1) - I(\hat{Y}_2; Y_2|X_1, X_2, Y_1)\}\\
\Delta_2 &= \min\{0, I(X_1; Y_2|X_2) - I(\hat{Y}_1; Y_1|X_1, X_2, Y_2)\}.
\end{aligned}$$

Proof: See Section VI-A. ■

Remark 3: If $R_0 = 0$ and $X_1 = X_2 = \emptyset$, i.e., both receivers send feedback signals without relaying cooperative information, by relaxing rate constraint (2f) as in Remark 1, the rate region $\mathcal{R}_1$ reduces to $\mathcal{R}_{\text{Wu}}$, which is the set of all nonnegative rate tuples (R_0, R_1, R_2) satisfying

$$\begin{aligned}
R_1 &\le I(U_0, U_1; Y_1, \hat{Y}_2) - I(\hat{Y}_2; Y_2|Y_1) && \text{(6a)}\\
R_2 &\le I(U_0, U_2; Y_2, \hat{Y}_1) - I(\hat{Y}_1; Y_1|Y_2) && \text{(6b)}\\
R_1 + R_2 &\le I(U_0, U_1; Y_1, \hat{Y}_2) - I(\tilde{Y}_2; Y_2|Y_1)\\
&\quad + I(U_2; Y_2, \hat{Y}_1|U_0) - I(U_1; U_2|U_0) && \text{(6c)}\\
R_1 + R_2 &\le I(U_0, U_2; Y_2, \hat{Y}_1) - I(\hat{Y}_1; Y_1|Y_2)\\
&\quad + I(U_1; Y_1, \hat{Y}_2|U_0) - I(U_1; U_2|U_0) && \text{(6d)}\\
R_1 + R_2 &\le I(U_0, U_1; Y_1, \hat{Y}_2) - I(\hat{Y}_2; Y_2|Y_1) - I(U_1; U_2|U_0)\\
&\quad + I(U_0, U_2; Y_2, \tilde{Y}_1) - I(\hat{Y}_1; Y_1|Y_2) && \text{(6e)}
\end{aligned}$$

for some pmf $P_{U_0U_1U_2}P_{\tilde{Y}_1|Y_1}P_{\tilde{Y}_2|Y_2}$ and function $X = f(U_0, U_1, U_2)$ such that

$$I(\tilde{Y}_1; Y_1|Y_2) \le R_{\text{Fb},1} \quad \text{and} \quad I(\tilde{Y}_2; Y_2|Y_1) \le R_{\text{Fb},2}. \quad \text{(6f)}$$

This rate region coincides with Wu and Wigger's region in [10, Corrollary 1], which shows feedback can strictly increase the entire capacity region for a large class of BCs, called strictly essentially less noisy BCs, unless it is physically degraded.

In the scheme for Theorem 2, both receivers apply compress-forward. If one of the two receivers uses a hybrid relaying strategy that combines partially decode-forward and compress-forward, we obtain a new achievable region below.

Theorem 3: For the *fully* cooperative BRC with two-sided and rate-limited feedback, the capacity region includes the set $\mathcal{R}_3^{(1)}$ of all nonnegative rate tuples (R_0, R_1, R_2) that satisfy

$$\begin{aligned}
R_0 &\le I(U_0; Y_1|X_1, X_2) + \Delta && \text{(7a)}\\
R_0 + R_1 &\le I(U_0; Y_1|X_1, X_2) + \Delta + I_1 && \text{(7b)}\\
R_0 + R_2 &\le I(U_0; Y_1|X_1, X_2) + \Delta + I_2 && \text{(7c)}\\
R_0 + R_2 &\le I(U_0, U_2, X_1; Y_2|X_2) && \text{(7d)}\\
&\quad - I(\hat{Y}_1; Y_1|U_0, U_2, X_1, X_2, Y_2) && \text{(7e)}\\
R_0 + R_1 + R_2 &\le I(U_0; Y_1|X_1, X_2) + I(U_0, U_2, X_1; Y_2|X_2)\\
&\quad + \Delta - I(\hat{Y}_1; Y_1|U_0, U_2, X_1, X_2, Y_2)\\
&\quad - I(U_1; U_2|U_0, X_1, X_2) && \text{(7f)}\\
R_0 + R_1 + R_2 &\le I(U_0; Y_1|X_1, X_2) + \Delta\\
&\quad + I_1 + I_2 - I(U_1; U_2|U_0, X_1, X_2) && \text{(7g)}
\end{aligned}$$

for some pmf $P_{X_1}P_{X_2}P_{U_0U_1U_2|X_1X_2}P_{\hat{Y}_1|U_0X_1X_2Y_1}P_{\hat{Y}_2|X_2Y_2}$ and function $X = f(U_0, U_1, U_2)$ such that

$$\begin{aligned}
I(\hat{Y}_1; Y_1|U_0, X_1, X_2, Y_2) &\le R_{\text{fb},1} && \text{(7h)}\\
I(\hat{Y}_2; Y_2|U_0, X_1, X_2, Y_1) &\le R_{\text{fb},2} && \text{(7i)}
\end{aligned}$$

where

$$\begin{aligned}
\Delta &= \min\{0, I(X_2; Y_1|X_1) - I(\hat{Y}_2; Y_2|U_0, X_1, X_2, Y_1)\}\\
I_1 &= I(U_1; \hat{Y}_2, Y_1|U_0, X_1, X_2)\\
&\quad + \min\{0, R_{\text{fb},2} - I(\hat{Y}_2; Y_2|U_0, X_1, X_2, Y_1)\}\\
I_2 &= I(U_2; \hat{Y}_1, Y_2|U_0, X_1, X_2)\\
&\quad + \min\{0, R_{\text{fb},1} - I(\hat{Y}_1; Y_1|U_0, X_1, X_2, Y_2)\}.
\end{aligned}$$

Proof: See Section VI-B ■

Remark 4: The region $\mathcal{R}_3^{(2)}$ is also achievable by exchanging indices 1 and 2 in the above definition of $\mathcal{R}_3^{(1)}$. The convex hull of the union of $\mathcal{R}_3^{(1)}$ and $\mathcal{R}_3^{(2)}$ leads to a potentially larger rate region.

IV. Comparisons among $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_{\text{Liang}}$ and $\mathcal{R}_{\text{Wu}}$

We compare our regions $\mathcal{R}_1$, $\mathcal{R}_2$ with the known rate regions $\mathcal{R}_{\text{Liang}}$ and $\mathcal{R}_{\text{Wu}}$. Note that $\mathcal{R}_{\text{Liang}}$ is for RBCs without feedback and $\mathcal{R}_{\text{Wu}}$ is for BCs with feedback, while $\mathcal{R}_1$ and $\mathcal{R}_2$ include both feedback communication and relay cooperation.

A. $\mathcal{R}_{\text{Liang}}$ *versus* $\mathcal{R}_1$

In Remark 2, it showed that our rate region $\mathcal{R}_1$ includes Liang and Kramer's region $\mathcal{R}_{\text{Liang}}$. In this subsection, we will prove that when the feedback rate is sufficiently large, this inclusion is strict for some channels, i.e.

$$\mathcal{R}_{\text{Liang}} \subset \mathcal{R}_1. \quad \text{(8)}$$

Suppose $R_0 = 0$ for simplicity. To prove (8), in view of Remark 2, it's sufficient to show there exists some rate pair (R_1^*, R_2^*) in $\mathcal{R}_1$ lying strictly outside of $\mathcal{R}_{\text{Liang}}$. Consider the corner point $(0, R_{2,\text{Liang}}^*)$ on the boundary of $\mathcal{R}_{\text{Liang}}$ in (4), where the transmitter spends all power to send message M_2 to Receiver 2, i.e., $U_1 = \emptyset$ and $U_2 = X$. Thus, we have

$$R^*_{2,\text{Liang}} \leq I(X, X_1; Y_2) \tag{9a}$$
$$R^*_{2,\text{Liang}} \leq I(U_0; Y_1|X_1) + I(X_1; Y_2|X_1, U_0) \tag{9b}$$

for some pmf P_{XX_1U}, which is the partial decode-forward lower bound of relay channel [13].

Now consider $\mathcal{R}_1$ in (2). Let $R_0 = R_1 = 0$ and $U_1 = \emptyset$ and $U_2 = X$, then the marginal rate R_2 is achievable if

$$R^*_{2,\text{Scheme1}} \leq I(X, X_1; Y_2) - I(\hat{Y}_1; Y_1|U_0, X, X_1, Y_2) \tag{10a}$$
$$R^*_{2,\text{Scheme1}} \leq I(U_0; Y_1|X_1) + I(X; \hat{Y}_1, Y_2|U_0, X_1) \tag{10b}$$

for some pmf $P_{U_0X_1X}P_{\hat{Y}_1|U_0X_1Y_1}$ satisfying

$$I(\hat{Y}_1; Y_1|U_0, X_1) \leq R_{\text{fb},1}. \tag{10c}$$

If feedback rate is sufficiently large such that rate constraint (10c) is inactive, then (10) turns to be Gabbai and Bross's rate in [15, Theorem 3]. In their work, they evaluated the rates (9) and (10) for the Gaussian and Z relay channels, and showed that $R^*_{2,\text{Scheme1}} > R^*_{2,\text{Liang}}$. In view of this fact and from Remark 2, we have

Corollary 1: $\mathcal{R}_{\text{Liang}} \subset \mathcal{R}_1$ holds when $R_{\text{fb},1}$ satisfies (10c).

B. $\mathcal{R}_{\text{Wu}}$ versus $\mathcal{R}_2$

Remark 3 states that $\mathcal{R}_{\text{Wu}} \subseteq \mathcal{R}_2$. Here we prove that $\mathcal{R}_{\text{Wu}} \subset \mathcal{R}_2$. To prove the strict inclusion, we follow similar procedures in Section IV-A and show that there exists some rate pair (R^*_1, R^*_2) inside $\mathcal{R}_2$ lying strictly outside of $\mathcal{R}_{\text{Wu}}$.

Consider the corner point $(0, R^*_{2,\text{Wu}})$ on the boundary of $\mathcal{R}_{\text{Wu}}$. From (6), it's easy to check that

$$R^*_{2,\text{Wu}} \leq I(X; Y_2) \tag{11}$$

for some P_X, which is the capacity of the link from the transmitter to Receiver 2.

Now consider $\mathcal{R}_2$ in (5). Let $R_0 = R_1 = 0$ and $U_0 = U_1 = \hat{Y}_2 = \emptyset$, then the marginal rate R_2 is achievable if

$$R^*_{2,\text{CF}} \leq I(X; \hat{Y}_1, Y_2|X_1) \tag{12a}$$
$$R^*_{2,\text{CF}} \leq I(X, X_1; Y_2) - I(\hat{Y}_1; Y_1|X, X_1, Y_2) \tag{12b}$$

for some pmf $P_X P_{X_1} P_{\hat{Y}_1|X_1Y_1}$, which is the compress-forward lower bound of the relay channel [13]. It's well known that introducing a compress-forward relay to the point-to-point channel, such as Gaussian channel, can strictly increase the capacity (11). Thus, we have

Corollary 2: $\mathcal{R}_{\text{Wu}} \subset \mathcal{R}_2$.

C. Example

Consider the Gaussian relay broadcast channel with perfect feedback from Receiver 1 to the transmitter, see Fig. 2. The channel outputs are:

$$Y_1 = g_{01}X + Z_1,$$
$$Y_2 = g_{02}X + g_{12}X_1 + Z_2$$

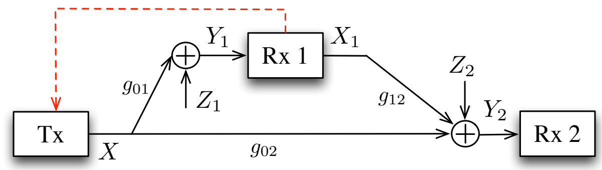

Fig. 2. Gaussian RBC with relay-transmitter feedback

TABLE I
MARGNIAL RATE R^*_2 ACHIEVED BY VARIOUS CODING SCHEMES

d	$R^*_{2,\text{Liang}}$	$R^*_{2,\text{Scheme1}}$	$R^*_{2,\text{Wu}}$	$R^*_{2,\text{CF}}$
0.73	1.6881	1.7069	1.2925	1.6908
0.74	1.6703	1.7111	1.2925	1.6971
0.75	1.6529	1.7153	1.2925	1.7033
0.76	1.6358	1.7195	1.2925	1.7094

where g_{01}, g_{02} and g_{12} are channel gains, and $Z_1 \sim \mathcal{N}(0, 1)$ and $Z_2 \sim \mathcal{N}(0, 1)$ are independent Gaussian noise variables. The input power constraints are $\mathbb{E}|X^2| \leq P$ and $\mathbb{E}|X_1^2| \leq P_1$.

Table I compares $R^*_{2,\text{Liang}}$, $R^*_{2,\text{Scheme1}}$, $R^*_{2,\text{Wu}}$, and $R^*_{2,\text{CF}}$, see (9)–(12), for this channel with $g_{01} = 1/d, g_{02} = 1$, $g_{12} = 1/|1-d|$, and $P = 5, P_1 = 1$. It can be seen that $R^*_{2,\text{Scheme1}} > R^*_{2,\text{CF}} > R^*_{2,\text{Liang}} > R^*_{2,\text{Wu}}$, which means that our rate regions $\mathcal{R}_1$ and $\mathcal{R}_2$ can strictly improve on $\mathcal{R}_{\text{Liang}}$ and $\mathcal{R}_{\text{Wu}}$, respectively.

V. Coding Scheme for Partially Cooperative BRCs with Rate-Limited Feedback

In this section we present a block-Markov coding scheme for partially cooperative BRCs with relay-transmitter and rate-limited feedback. Assume only Receiver 1 relays cooperative information X_1 without loss of generality. In the transmission, a sequence of B i.i.d message tuples $(m_{0,b}, m_{1,b}, m_{2,b})$, $b \in [1:B]$, are sent over $B+1$ blocks, each consisting of n transmissions.

Split message $m_{k,b}$ into common and private parts: $m_{k,b} = (m_{c,k,b}, m_{p,k,b})$, where $m_{c,k,b} \in [1 : 2^{nR_{c,k}}]$, $m_{p,k,b} \in [1 : 2^{nR_{p,k}}]$ and $R_k = R_{c,k} + R_{p,k}$. Define $\mathbf{m}_{c,b} := (m_{0,b}, m_{c,1,b}, m_{c,2,b})$ and $R_c := R_0 + R_{c,1} + R_{c,2}$.

In each block $b \in [1 : B+1]$, after obtaining feedback message $m_{\text{fb},1,b-1}$, the transmitter uses Marton's coding to send $(\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})$ in the cloud centre $u^n_{0,b}$, and $m_{p,1,b}, m_{p,2,b}$ in two different satellites $u^n_{1,b}, u^n_{2,b}$, respectively. Receiver 1 first jointly decodes $(\mathbf{m}_{c,b}, m_{p,1,b})$, and then compress its channel outputs $y^n_{1,b}$. Finally, it sends the compression message $m_{\text{fb},1,b}$ as feedback information and $x^n_{1,b+1}(\mathbf{m}_{c,b}, m_{\text{fb},1,b})$ as channel inputs in next bock. Receiver 2 uses backward decoding to jointly decode $(\mathbf{m}_{c,b-1}, m_{p,2,b}, m_{\text{fb},1,b-1})$. Note that the transmitter knows $(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})$, from which it can reconstruct Receiver 1's input $x^n_{1,b}$, thus we superimpose $(u^n_{0,b}, u^n_{1,b}, u^n_{2,b})$ on $x^n_{1,b}$ that attains cooperation between the transmitter and Receiver 1. Coding is explained with the help of Table II.

TABLE II
SCHEME 1 FOR PARTIALLY COOPERATIVE BRCS WITH RATE-LIMITED FEEDBACK

Block	1	2	...	b	...
X_1	$x_{1,1}^n(1,1)$	$x_{1,2}^n(\mathbf{m}_{c,1}, m_{\text{fb},1,1})$	...	$x_{1,b}^n(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})$	...
U_0	$u_{0,1}^n(\mathbf{m}_{c,1}\|1,1)$	$u_{0,2}^n(\mathbf{m}_{c,2}\|\mathbf{m}_{c,1}, m_{\text{fb},1,1})$	...	$u_{0,b}^n(\mathbf{m}_{c,b}\|\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})$	...
U_k	$u_{k,1}^n(m_{p,k,1}, v_{k,1}\|\mathbf{m}_{c,1}, 1, 1)$	$u_{k,2}^n(m_{p,k,2}, v_{k,2}\|\mathbf{m}_{c,2}, \mathbf{m}_{c,1}, m_{\text{fb},1,1})$	...	$u_{k,b}^n(m_{p,k,b}, v_{k,b}\|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})$	...
$\hat{Y}_1$	$\hat{y}_{1,1}^n(m_{\text{fb},k,1}\|\mathbf{m}_{c,1}, 1, 1)$	$\hat{y}_{1,2}^n(m_{\text{fb},1,2}\|\mathbf{m}_{c,2}, \mathbf{m}_{c,1}, m_{\text{fb},1,1})$	...	$\hat{y}_{1,b}^n(m_{\text{fb},1,b}\|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})$	...
Y_1	$(\hat{\mathbf{m}}_{c,1}^{(1)}, \hat{m}_{p,1,1}, \hat{v}_{1,1})$	$(\hat{\mathbf{m}}_{c,2}^{(1)}, \hat{m}_{p,1,2}, \hat{v}_{1,2}) \rightarrow$	...	$(\hat{\mathbf{m}}_{c,b}^{(1)}, \hat{m}_{p,1,b}, \hat{v}_{1,b}) \rightarrow$	...
Y_2	$(\hat{m}_{p,2,1}, \hat{v}_{2,1})$	$\leftarrow (\hat{\mathbf{m}}_{c,1}^{(2)}, \hat{m}_{p,2,2}, \hat{v}_{2,2}, \hat{m}_{\text{fb},1,1})$	...	$\leftarrow (\hat{\mathbf{m}}_{c,b-1}^{(2)}, \hat{m}_{p,2,b}, \hat{v}_{2,b}, \hat{m}_{\text{fb},1,b-1})$	...

1) Code construction: Fix pmf $P_{U_0U_1U_2X_1}P_{\hat{Y}_1|U_0X_1Y_1}$ and a function $X = f(U_0, U_1, U_2)$. For each block $b \in [1 : B+1]$, randomly and independently generate $2^{n(R_c+\hat{R}_1)}$ sequences $x_{1,b}^n(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}) \sim \prod_{i=1}^n P_{X_1}(x_{1,b,i})$, $\mathbf{m}_{c,b-1} \in [1 : 2^{nR_c}]$ and $m_{\text{fb},1,b-1} \in [1 : 2^{n\hat{R}_1}]$. For each $(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})$, randomly and independently generate 2^{nR_c} sequences $u_{0,b}^n(\mathbf{m}_{c,b}|\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}) \sim \prod_{i=1}^n P_{U_0|X_1}(u_{0,b,i}|x_{1,b,i})$. For each $(\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})$, randomly and independently generate $2^{n(R_{p,k}+R_k')}$ sequences $u_{k,b}^n(m_{p,k,b}, v_{k,b}|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}) \sim \prod_{i=1}^n P_{U_k|U_0X_1}(u_{k,b,i}|u_{0,b,i}, x_{1,b,i})$, $m_{p,k,b} \in [1 : 2^{nR_{p,k}}]$ and $v_{k,b} \in [1 : 2^{nR_k'}]$. For each $(\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})$, randomly and independently generate $2^{n\hat{R}_1}$ sequences $\hat{y}_{1,b}^n(m_{\text{fb},1,b}|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}) \sim \prod_{i=1}^n P_{\hat{Y}_1|U_0X_1}(\hat{y}_{1,b,i}|u_{0,b,i}, x_{1,b,i})$.

2) Encoding: In each block $b \in [1 : B+1]$, assume that the transmitter already knows $m_{\text{fb},1,b-1}$ through feedback link. It first looks for a pair of indices $(v_{1,b}, v_{2,b})$ such that

$$\big(u_{1,b}^n(m_{p,1,b}, v_{1,b}|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}), u_{0,b}^n(\mathbf{m}_{c,b}|\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}), x_{1,b}^n(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}), u_{2,b}^n(m_{p,2,b}, v_{2,b}|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})\big) \in \mathcal{T}_\epsilon^n(P_{U_0U_1U_2X_1}).$$

Then in block b it sends x_b^n with $x_{b,i} = f(u_{0,b,i}, u_{1,b,i}, u_{2,b,i})$.

By covering lemma [14], this is successful with high probability for sufficiently large n if

$$R_1' + R_2' \geq I(U_1; U_2|U_0, X_1) \tag{13}$$

3) Receiver 1's decoding: In each block $b \in [1 : B+1]$, Receiver 1 looks for $(\hat{\mathbf{m}}_{c,b}^{(1)}, \hat{m}_{p,1,b}, \hat{v}_{1,b})$ such that

$$\big(x_{1,b}^n(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}), y_{1,b}^n, u_{1,b}^n(\hat{m}_{p,1,b}, \hat{v}_{1,b}|\hat{\mathbf{m}}_{c,b}^{(1)}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}), u_{0,b}^n(\hat{\mathbf{m}}_{c,b}^{(1)}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})\big) \in \mathcal{T}_\epsilon^n(P_{X_1U_0U_1Y_1}).$$

It then compresses $y_{1,b}^n$ by finding $m_{\text{fb},1,b}$ satisfying

$$\big(x_{1,b}^n(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}), u_{0,b}^n(\mathbf{m}_{c,b}|\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}), y_{1,b}^n, \hat{y}_{1,b}^n(m_{\text{fb},1,b}|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})\big) \in \mathcal{T}_\epsilon^n(P_{\hat{Y}_1X_1U_0Y_1}).$$

Finally, it sends $m_{\text{fb},1,b}$ as feedback message to the transmitter at rate

$$\hat{R}_1 \leq R_{\text{fb},1}, \tag{14}$$

and sends $x_{1,b+1}^n(\mathbf{m}_{c,b}, m_{\text{fb},1,b})$ as channel inputs in block $b+1$.

By the independence of the codebooks, the Markov lemma [14], packing lemma [14] and the induction on backward decoding, these steps are successful with high probability if

$$R_{p,1} + R_1' < I(U_1; Y_1|U_0, X_1) \tag{15a}$$

$$R_{p,1} + R_1' + R_c < I(U_0, U_1; Y_1|X_1) \tag{15b}$$

$$\hat{R}_1 > I(\hat{Y}_1; Y_1|U_0, X_1) \tag{15c}$$

4) Receiver 2's decoding: Receiver 2 performs backward decoding. In each block $b \in [1 : B+1]$, It looks for $(\hat{\mathbf{m}}_{c,b-1}^{(2)}, \hat{m}_{p,2,b}, \hat{v}_{2,b}, \hat{m}_{\text{fb},1,b-1})$ such that

$$\big(x_{1,b}^n(\hat{\mathbf{m}}_{c,b-1}^{(2)}, \hat{m}_{\text{fb},1,b-1}), \hat{y}_{1,b}^n(m_{\text{fb},1,b}|\mathbf{m}_{c,b}, \hat{\mathbf{m}}_{c,b-1}^{(2)}, \hat{m}_{\text{fb},1,b-1}), u_{2,b}^n(\hat{m}_{p,2,b}, \hat{v}_{2,b}|\mathbf{m}_{c,b}, \hat{\mathbf{m}}_{c,b-1}^{(2)}, \hat{m}_{\text{fb},1,b-1}), y_{2,b}^n, u_{0,b}^n(\mathbf{m}_{c,b}, \hat{\mathbf{m}}_{c,b-1}^{(2)}, \hat{m}_{\text{fb},1,b-1})\big) \in \mathcal{T}_\epsilon^n(P_{X_1U_0U_2Y_2\hat{Y}_1}).$$

By the independence of the codebooks, the Markov lemma, packing lemma and the induction on backward decoding, these steps are successful with high probability if

$$R_{p,2} + R_2' < I(U_2; Y_2, \hat{Y}_1|U_0, X_1) \tag{16a}$$

$$R_{p,2} + R_2' + R_c + \hat{R}_1 < I(U_0, U_2, X_1; Y_2) + I(\hat{Y}_1; U_2, Y_2|U_0, X_1) \tag{16b}$$

Combine (13–16) and use Fourier-Motzkin elimination to eliminate $R_1', R_2', \hat{R}_1, \hat{R}_2$, then we obtain Theorem 1.

VI. ACHIEVABLE RATES FOR FULLY COOPERATIVE RBC WITH RATE-LIMITED FEEDBACK

In this section we present two block-Markov coding schemes for fully cooperative BRC with relay/receiver-transmitter and rate-limited feedback.

A. Scheme 2A: Compress-forward relaying and backward decoding

In this subsection we propose a block-Markov coding scheme where a sequence of B i.i.d message tuples $(m_{0,b}, m_{1,b}, m_{2,b})$ are sent over $B+1$ blocks, each consisting of n transmissions. Split the message $m_{k,b}$ in the same way as Section V and define $\mathbf{m}_{\text{fb},b} := (m_{\text{fb},1,b}, m_{\text{fb},2,b})$.

In each block $b \in [1 : B+1]$, after obtaining feedback messages $\mathbf{m}_{\text{fb},b-1}$, the transmitter uses Marton's coding to send

TABLE III
SCHEME 2A FOR FULLY COOPERATIVE BRCS WITH RATE-LIMITED FEEDBACK

Block	1	2	...	b	...
X_k	$x^n_{k,1}(1)$	$x^n_{k,2}(m_{\text{fb},k,1})$	...	$x^n_{k,b}(m_{\text{fb},k,b\text{–}1})$	...
U_0	$u^n_{0,1}(\mathbf{m}_{c,1}\|1,1)$	$u^n_{0,2}(\mathbf{m}_{c,2}\|\mathbf{m}_{\text{fb},1})$	...	$u^n_{0,b}(\mathbf{m}_{c,b}\|\mathbf{m}_{\text{fb},b\text{–}1})$	...
U_k	$u^n_{k,1}(m_{p,k,1},v_{k,1}\|\mathbf{m}_{c,1},1,1)$	$u^n_{k,2}(m_{p,k,2},v_{k,2}\|\mathbf{m}_{c,2},\mathbf{m}_{\text{fb},1})$	...	$u^n_{k,b}(m_{p,k,b},v_{k,b}\|\mathbf{m}_{c,b},\mathbf{m}_{\text{fb},b\text{–}1})$	...
$\hat{Y}_k$	$\hat{y}^n_{k,1}(m_{\text{fb},k,1}\|1)$	$\hat{y}^n_{k,2}(m_{\text{fb},k,2}\|m_{\text{fb},k,1})$	...	$\hat{y}^n_{k,b}(m_{\text{fb},k,b}\|m_{\text{fb},k,b\text{–}1})$	...
Y_1	$(\hat{\mathbf{m}}^{(1)}_{c,1},\hat{m}_{p,1,1},\hat{v}_{1,1})$	$\leftarrow(\hat{\mathbf{m}}^{(1)}_{c,2},\hat{m}_{p,1,2},\hat{v}_{1,2},\hat{m}_{\text{fb},2,1})$	...	$\leftarrow(\hat{\mathbf{m}}^{(1)}_{c,b},\hat{m}_{p,1,b},\hat{v}_{1,b},\hat{m}_{\text{fb},2,b\text{–}1})$	...
Y_2	$(\hat{\mathbf{m}}^{(2)}_{c,1},\hat{m}_{p,2,1},\hat{v}_{2,1})$	$\leftarrow(\hat{\mathbf{m}}^{(2)}_{c,2},\hat{m}_{p,2,2},\hat{v}_{2,2},\hat{m}_{\text{fb},1,1})$	...	$\leftarrow(\hat{\mathbf{m}}^{(2)}_{c,b},\hat{m}_{p,2,b},\hat{v}_{2,b},\hat{m}_{\text{fb},1,b\text{–}1})$	...

$(\mathbf{m}_{c,b},\mathbf{m}_{\text{fb},b-1})$ in the cloud centre $u^n_{0,b}$, and $m_{p,1,b},m_{p,2,b}$ in two different satellites $u^n_{1,b},u^n_{2,b}$, respectively. Receiver $k\in\{1,2\}$ first uses backward decoding to decode $(\mathbf{m}_{c,b},m_{p,k,b})$ and reconstructs the other receiver's compression message. Then, it compresses its channel outputs $y^n_{k,b}$. Finally, it sends $m_{\text{fb},k,b}$ as feedback message and $x^n_{k,b+1}(m_{\text{fb},k,b})$ as channel inputs in next bock. Here $(u^n_{0,b},u^n_{1,b},u^n_{2,b})$ are superimposed on $(x^n_{1,b},x^n_{2,b})$ that attains cooperation between the transmitter and the receivers. Coding is explained with the help of Table III.

1) Code construction: Fix pmf

$$P_{X_1}P_{X_2}P_{U_0U_1U_2|X_1X_2}P_{\hat{Y}_1|X_1Y_1}P_{\hat{Y}_2|X_2Y_2}$$

and a function $X=f(U_0,U_1,U_2)$. For each block $b\in[1\!:\!B+1]$ and $k\in\{1,2\}$, randomly and independently generate $2^{n\hat{R}_k}$ sequences $x^n_{k,b}(m_{\text{fb},k,b-1})\sim\prod_{i=1}^n P_{X_k}(x_{k,b,i})$, $m_{\text{fb},k,b-1}\in[1:2^{n\hat{R}_k}]$. For each $m_{\text{fb},k,b-1}$, randomly and independently generate $2^{n\hat{R}_k}$ sequences $\hat{y}^n_{k,b}(m_{\text{fb},k,b}|m_{\text{fb},k,b-1})\sim\prod_{i=1}^n P_{\hat{Y}_k|X_k}(\hat{y}_{k,b,i}|x_{k,b,i})$. For each $\mathbf{m}_{\text{fb},b-1}$, randomly and independently generate 2^{nR_c} sequences $u^n_{0,b}(\mathbf{m}_{c,b}|\mathbf{m}_{\text{fb},b-1})\sim\prod_{i=1}^n P_{U_0|X_1X_2}(u_{0,b,i}|x_{1,b,i},x_{2,b,i})$, $\mathbf{m}_{c,b}\in[1:2^{nR_c}]$. For each $(\mathbf{m}_{c,b},\mathbf{m}_{\text{fb},b-1})$, randomly and independently generate $2^{n(R_{p,k}+R'_k)}$ sequences $u^n_{k,b}(m_{p,k,b},v_{k,b}|\mathbf{m}_{c,b},\mathbf{m}_{\text{fb},b-1})\sim\prod_{i=1}^n P_{U_k|U_0X_1X_2}(u_{k,b,i}|u_{0,b,i},x_{1,b,i},x_{2,b,i})$, $m_{p,k,b}\in[1:2^{nR_{p,k}}]$ and $v_{k,b}\in[1:2^{nR'_k}]$.

2) Encoding: In each block $b\in[1:B+1]$, assume that the transmitter already knows $\mathbf{m}_{\text{fb},b-1}$ through feedback links. It first looks for a pair of indices $(v_{1,b},v_{2,b})$ such that

$$\begin{aligned}&\big(u^n_{0,b}(\mathbf{m}_{c,b}|\mathbf{m}_{\text{fb},b\text{–}1}),x^n_{1,b}(m_{\text{fb},1,b\text{–}1}),\\&\quad u^n_{1,b}(m_{p,1,b},v_{1,b}|\mathbf{m}_{c,b},\mathbf{m}_{\text{fb},b\text{–}1}),x^n_{2,b}(m_{\text{fb},2,b\text{–}1}),\\&\qquad u^n_{2,b}(m_{p,2,b},v_{2,b}|\mathbf{m}_{c,b},\mathbf{m}_{\text{fb},b\text{–}1})\big)\in\mathcal{T}^n_\epsilon(P_{U_0U_1U_2X_1X_2})\end{aligned}$$

Then in block b it sends x^n_b with $x_{b,i}=f(u_{0,b,i},u_{1,b,i},u_{2,b,i})$.

By covering lemma, this is successful with high probability for sufficiently large n if

$$R'_1+R'_2\geq I(U_1;U_2|U_0,X_1,X_2). \tag{17}$$

3) Decoding: Both receivers perform backward decoding and compress-forward strategy. In each block $b\in[1:B+1]$, Receiver 1 looks for $(\hat{\mathbf{m}}^{(1)}_{c,b},\hat{m}_{p,1,b},\hat{v}_{1,b},\hat{m}_{\text{fb},2,b\text{–}1})$ such that

$$\begin{aligned}&\big(x^n_{1,b}(m_{\text{fb},1,b\text{–}1}),x^n_{2,b}(\hat{m}_{\text{fb},2,b\text{–}1}),\hat{y}^n_{2,b}(m_{\text{fb},2,b}|\hat{m}_{\text{fb},2,b\text{–}1}),\\&\quad u^n_{1,b}(\hat{m}_{p,1,b},\hat{v}_{1,b}|\hat{\mathbf{m}}_{c,b},m_{\text{fb},1,b\text{–}1},\hat{m}_{\text{fb},2,b\text{–}1}),y^n_{1,b},\\&\qquad u^n_{0,b}(\hat{\mathbf{m}}_{c,b}|m_{\text{fb},1,b\text{–}1},\hat{m}_{\text{fb},2,b\text{–}1})\big)\in\mathcal{T}^n_\epsilon(P_{X_1X_2U_0U_1Y_1\hat{Y}_2}).\end{aligned}$$

It then compresses $y^n_{1,b}$ by finding a unique index $m_{\text{fb},1,b}$ such that

$$\big(x^n_{1,b}(m_{\text{fb},1,b\text{–}1}),\hat{y}^n_{1,b}(m_{\text{fb},1,b}|m_{\text{fb},1,b\text{–}1}),y^n_{1,b}\big)\in\mathcal{T}^n_\epsilon(P_{\hat{Y}_1X_1Y_1}).$$

Finally, in block $b+1$ it sends $x^n_{1,b+1}(m_{\text{fb},1,b})$ as channel input and forwards $m_{\text{fb},1,b}$ through the feedback link at rate:

$$\hat{R}_1\leq R_{\text{fb},1}. \tag{18}$$

Receiver 2 performs in a similar way with exchanging indices of 1 and 2 in above steps.

By the independence of the codebooks, the Markov lemma, packing lemma and the induction on backward decoding, these steps are successful with high probability if

$$\hat{R}_1>I(\hat{Y}_1;Y_1|X_1) \tag{19a}$$
$$\hat{R}_2>I(\hat{Y}_2;Y_2|X_2) \tag{19b}$$
$$R_{p,1}+R'_1<I(U_1;Y_1,\hat{Y}_2|U_0,X_1,X_2) \tag{19c}$$
$$R_{p,2}+R'_2<I(U_2;Y_2,\hat{Y}_1|U_0,X_1,X_2) \tag{19d}$$
$$R_{p,1}+R'_1+R_c<I(U_0,U_1;\hat{Y}_2,Y_1|X_1,X_2) \tag{19e}$$
$$R_{p,2}+R'_2+R_c<I(U_0,U_2;\hat{Y}_1,Y_2|X_1,X_2) \tag{19f}$$
$$\begin{aligned}R_{p,1}+R'_1+R_c+\hat{R}_2<{}&I(U_0,U_1,X_2;Y_1|X_1)\\&+I(\hat{Y}_2;U_0,U_2,Y_1,X_1|X_2)\end{aligned} \tag{19g}$$
$$\begin{aligned}R_{p,2}+R'_2+R_c+\hat{R}_1<{}&I(U_0,U_2,X_1;Y_2|X_2)\\&+I(\hat{Y}_1;U_0,U_1,Y_2,X_2|X_1).\end{aligned} \tag{19h}$$

Combine (17–19) and use Fourier-Motzkin elimination to eliminate $R'_1,R'_2,\hat{R}_1,\hat{R}_2$, then we obtain Theorem 2.

B. Scheme 2B: Hybrid relaying strategy and sliding-window decoding

In Scheme 2A both receivers apply compress-forward. In this subsection, we propose a coding scheme where one of the two receivers, called Receiver 1 without loss of generality, applies a hybrid relaying strategy that combines partially decode-forward and compress-forward. More specifically, Receiver 1 first decodes the cloud center containing $(\mathbf{m}_{c,b},m_{\text{fb},2,b\text{–}1})$, then

TABLE IV
SCHEME 2B FOR FULLY COOPERATIVE BRCS WITH RATE-LIMITED FEEDBACK

Block	1	2	$\dots$	b	$\dots$
X_1	$x_{1,1}^n(\mathbf{1}_{[3]}, 1)$	$x_{1,2}^n(\mathbf{m}_{c,1}, m_{\text{fb},1,1})$	$\dots$	$x_{1,b}^n(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1})$	$\dots$
X_2	$x_{2,1}^n(1)$	$x_{2,2}^n(m_{\text{fb},2,1})$	$\dots$	$x_{2,b}^n(m_{\text{fb},2,b-1})$	$\dots$
U_0	$u_{0,1}^n(\mathbf{m}_{c,1}\lvert\mathbf{1}_{[3]}, \mathbf{1}_{[2]})$	$u_{0,2}^n(\mathbf{m}_{c,2}\lvert\mathbf{m}_{c,1}, \mathbf{m}_{\text{fb},1})$	$\dots$	$u_{0,b}^n(\mathbf{m}_{c,b}\lvert\mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1})$	$\dots$
U_k	$u_{k,1}^n(m_{k,1}, v_{k,1}\lvert\mathbf{m}_{c,1}, \mathbf{1}_{[3]}, \mathbf{1}_{[2]})$	$u_{k,2}^n(m_{k,2}, v_{k,2}\lvert\mathbf{m}_{c,2}, \mathbf{m}_{c,1}, \mathbf{m}_{\text{fb},1})$	$\dots$	$u_{k,b}^n(m_{k,b}, v_{k,b}\lvert\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1})$	$\dots$
$\hat{Y}_1$	$\hat{y}_{1,1}^n(m_{\text{fb},1,1}, j_{1,1}\lvert\mathbf{m}_{c,1}, \mathbf{1}_{[3]}, \mathbf{1}_{[2]})$	$\hat{y}_{1,2}^n(m_{\text{fb},1,2}, j_{1,2}\lvert\mathbf{m}_{c,2}, \mathbf{m}_{c,1}, \mathbf{m}_{\text{fb},1})$	$\dots$	$\hat{y}_{1,b}^n(m_{\text{fb},1,b}, j_{1,b}\lvert\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1})$	$\dots$
$\hat{Y}_2$	$\hat{y}_{2,2}^n(m_{\text{fb},2,2}, j_{2,2}\lvert m_{\text{fb},2,1})$	$\hat{y}_{2,1}^n(m_{\text{fb},2,1}, j_{2,1}\lvert 1)$	$\dots$	$\hat{y}_{2,b}^n(m_{\text{fb},2,b}, j_{2,b}\lvert m_{\text{fb},2,b-1})$	$\dots$
Y_1	$\hat{\mathbf{m}}_{c,1}^{(1)} \rightarrow$	$(\hat{\mathbf{m}}_{c,2}^{(1)}, \hat{m}_{\text{fb},2,1}), (j_{2,1}, \hat{m}_{p,1,1}, \hat{v}_{1,1}) \rightarrow$	$\dots$	$(\hat{\mathbf{m}}_{c,b}^{(1)}, \hat{m}_{\text{fb},2,b-1}), (j_{2,b-1}, \hat{m}_{p,1,b-1}, \hat{v}_{1,b-1}) \rightarrow$	$\dots$
Y_2	$(\hat{m}_{p,2,1}, \hat{v}_{2,1}, \hat{j}_{1,b})$	$\leftarrow (\hat{\mathbf{m}}_{c,1}^{(2)}, \hat{m}_{p,2,2}, \hat{v}_{2,2}, \hat{m}_{\text{fb},1,1}, \hat{j}_{1,2})$	$\dots$	$\leftarrow (\hat{\mathbf{m}}_{c,b-1}^{(2)}, \hat{m}_{p,2,b}, \hat{v}_{2,b}, \hat{m}_{\text{fb},1,b-1}, \hat{j}_{1,b})$	$\dots$

reconstructs Receiver 2's compression outputs $\hat{y}_{2,b-1}^n$ and decodes $m_{p,1,b-1}$ based on the enhanced outputs $(\hat{y}_{2,b-1}^n, y_{1,b-1}^n)$. Finally it compresses $y_{1,b}^n$, and sends the compression message $m_{\text{fb},1,b}$ as feedback and $x_{1,b+1}^n(\mathbf{m}_{c,b}, m_{\text{fb},1,b})$ as channel inputs in block $b+1$. Note that Receiver 1 needs to decode $\mathbf{m}_{c,b}$ before sending $x_{1,b+1}^n$, thus it has to use sliding-window decoding instead of backward decoding. The transmitter and the other receiver perform similarly as Scheme 1A. Coding is explained with the help of Table IV.

1) Code construction: Fix pmf $P_{X_1}P_{X_2}P_{U_0U_1U_2|X_1X_2}P_{\hat{Y}_1|U_0X_1X_2Y_1}P_{\hat{Y}_2|X_2Y_2}$ and a function $X = f(U_0, U_1, U_2)$. For each block $b \in [1 : B+1]$, randomly and independently generate $2^{n(R_c+\hat{R}_1)}$ sequences $x_{1,b}^n(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}) \sim \prod_{i=1}^n P_{X_1}(x_{1,b,i})$, for $\mathbf{m}_{c,b-1} \in [1 : 2^{nR_c}]$ and $m_{\text{fb},1,b-1} \in [1 : 2^{n\hat{R}_1}]$. Randomly and independently generate $2^{n\hat{R}_2}$ sequences $x_{2,b}^n(m_{\text{fb},2,b-1}) \sim \prod_{i=1}^n P_{X_2}(x_{2,b,i})$, for $m_{\text{fb},2,b-1} \in [1 : 2^{n\hat{R}_k}]$. For each $(\mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1})$, randomly and independently generate 2^{nR_c} sequences $u_{0,b}^n(\mathbf{m}_{c,b}|\mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1}) \sim \prod_{i=1}^n P_{U_0|X_1X_2}(u_{0,b,i}|x_{1,b,i}, x_{2,b,i})$, $\mathbf{m}_{c,b} \in [1 : 2^{nR_c}]$. For each $(\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1})$, randomly and independently generate $2^{n(R_{p,k}+R'_k)}$ sequences $u_{k,b}^n(m_{p,k,b}, v_{k,b}|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1}) \sim \prod_{i=1}^n P_{U_k|U_0X_1X_2}(u_{k,b,i}|u_{0,b,i}, x_{1,b,i}, x_{2,b,i})$, $m_{p,k,b} \in [1 : 2^{nR_{p,k}}]$ and $v_{k,b} \in [1 : 2^{nR'_k}]$. For each $m_{\text{fb},2,b-1}$, randomly and independently generate $2^{n(\hat{R}_2+\tilde{R}_2)}$ sequences $\hat{y}_{2,b}^n(m_{\text{fb},2,b}, j_{2,b}|m_{\text{fb},2,b-1}) \sim \prod_{i=1}^n P_{\hat{Y}_2|X_2}(\hat{y}_{2,b,i}|x_{2,b,i})$, $j_{2,b} \in [1 : 2^{n\tilde{R}_2}]$. For each $(\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1})$, randomly and independently generate $2^{n(\hat{R}_1+\tilde{R}_1)}$ sequences $\hat{y}_{1,b}^n(m_{\text{fb},1,b}, j_{1,b}|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1}) \sim \prod_{i=1}^n P_{\hat{Y}_1|U_0X_1X_2}(\hat{y}_{1,b,i}|u_{0,b,i}, x_{1,b,i}, x_{2,b,i})$, $j_{1,b} \in [1 : 2^{n\tilde{R}_1}]$.

2) Encoding: In each block $b \in [1 : B+1]$, assume that the transmitter already knows $\mathbf{m}_{\text{fb},b-1}$ through feedback links. It first looks for a pair of indices $(v_{1,b}, v_{2,b})$ such that

$$\begin{aligned}&\big(u_{0,b}^n(\mathbf{m}_{c,b}|\mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1}), x_{1,b}^n(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}),\\ &\ u_{1,b}^n(m_{p,1,b}, v_{1,b}|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1}), x_{2,b}^n(m_{\text{fb},2,b-1}),\\ &\ \ u_{2,b}^n(m_{p,2,b}, v_{2,b}|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1})\big) \in \mathcal{T}_\epsilon^n(P_{U_0U_1U_2X_1X_2}).\end{aligned}$$

Then in block b it sends x_b^n with $x_{b,i} = f(u_{0,b,i}, u_{1,b,i}, u_{2,b,i})$.

By covering lemma, this is successful with high probability for sufficiently large n if

$$R'_1 + R'_2 \geq I(U_1; U_2|U_0, X_1, X_2). \tag{20}$$

3) Receiver 1's Decoding: In each block $b \in [1 : B+1]$, Receiver 1 first decodes cloud centre $u_{0,b}^n$ by looking for $(\hat{\mathbf{m}}_{c,b}^{(1)}, \hat{m}_{\text{fb},2,b-1})$ such that

$$\begin{aligned}&\big(u_{0,b}^n(\hat{\mathbf{m}}_{c,b}^{(1)}|\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}, \hat{m}_{\text{fb},2,b-1}), x_{2,b}^n(\hat{m}_{\text{fb},2,b-1}),\\ &\quad x_{1,b}^n(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}), y_{1,b}^n\big) \in \mathcal{T}_\epsilon^n(P_{U_0X_1X_2Y_1}).\end{aligned}$$

It then decodes $(\hat{y}_{2,b-1}^n, u_{1,b-1}^n)$ by looking for $(\hat{j}_{2,b-1}, \hat{m}_{1,b-1}, \hat{v}_{1,b-1})$ such that

$$\begin{aligned}&\big(u_{0,b-1}^n(\mathbf{m}_{c,b-1}|\mathbf{m}_{c,b-2}, \mathbf{m}_{\text{fb},b-2}), x_{1,b-1}^n(\mathbf{m}_{c,b-2}, m_{\text{fb},1,b-2}),\\ &\ \hat{y}_{2,b-1}^n(m_{\text{fb},2,b-1}, \hat{j}_{2,b-1}|m_{\text{fb},1,b-1}),\\ &\quad u_{1,b-1}^n(\hat{m}_{p,1,b-1}, \hat{v}_{1,b-1}|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1}),\\ &\qquad x_{2,b-1}^n(m_{\text{fb},2,b-2}), y_{2,b-1}^n\big) \in \mathcal{T}_\epsilon^n(P_{U_0U_1X_1X_2\hat{Y}_2Y_1}).\end{aligned}$$

Then, it compresses $y_{1,b}^n$ by looking for a unique pair $(m_{\text{fb},1,b}, j_{1,b})$ such that

$$\begin{aligned}&\big(u_{0,b}^n(\mathbf{m}_{c,b}|\mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1}), x_{1,b}^n(\mathbf{m}_{c,b-1}, m_{\text{fb},1,b-1}),\\ &\quad \hat{y}_{1,b}^n(m_{\text{fb},1,b}, j_{1,b}|\mathbf{m}_{c,b}, \mathbf{m}_{c,b-1}, \mathbf{m}_{\text{fb},b-1}),\\ &\qquad x_{2,b}^n(m_{\text{fb},2,b-1}), y_{1,b}^n\big) \in \mathcal{T}_\epsilon^n(P_{\hat{Y}_1U_0X_1X_2Y_1}).\end{aligned}$$

Finally, in block $b+1$ it sends $x_{1,b+1}^n(\mathbf{m}_{c,b}, m_{\text{fb},1,b})$ as channel input and forwards $m_{\mathbf{fb},1,b}$ through the feedback link at rate:

$$\hat{R}_1 \leq R_{\text{fb},1}. \tag{21}$$

By the independence of the codebooks, the Markov lemma, packing lemma and the induction on backward decoding, these steps are successful with high probability if

$$R_c < I(U_0; Y_1|X_1, X_2) \tag{22a}$$

$$R_c + \hat{R}_2 < I(U_0, X_2; Y_1|X_1) \tag{22b}$$

$$\tilde{R}_2 < I(\hat{Y}_2; U_0, X_1, Y_1|X_2) \tag{22c}$$

$$R_{p,1} + R'_1 < I(U_1; Y_1, \hat{Y}_2|U_0, X_1, X_2) \tag{22d}$$

$$\begin{aligned}R_{p,1} + R'_1 + \tilde{R}_2 < {}& I(U_1; Y_1, \hat{Y}_2|U_0, X_1, X_2)\\ &+ I(\hat{Y}_2; U_0, X_1, Y_1|X_2)\end{aligned} \tag{22e}$$

$$\hat{R}_1 + \tilde{R}_1 > I(\hat{Y}_1; Y_1|U_0, X_1, X_2). \tag{22f}$$

4) Receiver 2's Decoding: Receiver 2 performs backward decoding. In each block $b \in [1 : B+1]$, Receiver 2 looks for $(\hat{\mathbf{m}}^{(2)}_{c,b-1}, \hat{m}_{p,2,b}, \hat{v}_{2,b}, \hat{m}_{\text{fb},1,b\text{–}1}, \hat{j}_{1,b})$ such that

$$\begin{aligned}&\big(u^n_{0,b}(\mathbf{m}_{c,b}|\hat{\mathbf{m}}^{(2)}_{c,b-1}, \hat{m}_{\text{fb},1,b-1}, m_{\text{fb},2,b-1}), x^n_{2,b}(m_{\text{fb},2,b-1}),\\&\quad \hat{y}^n_{1,b}(m_{\text{fb},1,b}, \hat{j}_{1,b}|\mathbf{m}_{c,b}, \hat{\mathbf{m}}^{(2)}_{c,b-1}, \hat{m}_{\text{fb},1,b-1}), y^n_{2,b},\\&\quad\quad u^n_{2,b}(\hat{m}_{p,2,b}, \hat{v}_{2,b}|\mathbf{m}_{c,b}, \hat{\mathbf{m}}^{(2)}_{c,b-1}, \hat{m}_{\text{fb},1,b-1}, m_{\text{fb},2,b-1}),\\&\quad\quad\quad x^n_{1,b}(\hat{\mathbf{m}}^{(2)}_{c,b-1}, \hat{m}_{\text{fb},1,b-1})\big) \in \mathcal{T}^n_\epsilon(P_{X_1X_2U_0U_1Y_2\hat{Y}_1}).\end{aligned}$$

Also, it compresses $y^n_{2,b}$ by looking for a unique pair $(m_{\text{fb},2,b}, j_{2,b})$ such that .

$$\begin{aligned}&\big(x^n_{2,b}(m_{\text{fb},1,b-1}), y^n_{2,b},\\&\quad \hat{y}^n_{2,b}(m_{\text{fb},2,b}, j_{2,b}|m_{\text{fb},2,b-1})\big) \in \mathcal{T}^n_\epsilon(P_{\hat{Y}_2X_2Y_2}).\end{aligned}$$

Finally, in block $b+1$ it sends $x^n_{2,b+1}(m_{\text{fb},2,b})$ as channel input and forwards $m_{\mathbf{fb},2,b}$ through the feedback link at rate:

$$\hat{R}_2 \leq R_{\text{fb},2}. \tag{23}$$

By the independence of the codebooks, the Markov lemma, packing lemma and the induction on backward decoding, these steps are successful with high probability if

$$\begin{aligned}\tilde{R}_1 &< I(\hat{Y}_1; Y_2, U_2|U_0, X_1, X_2) && \text{(24a)}\\ R_{p,2}+R'_2 &< I(U_2; Y_2, \hat{Y}_1|U_0, X_1, X_2) && \text{(24b)}\\ R_{p,2}+R'_2+\tilde{R}_1 &< I(U_2; Y_2, \hat{Y}_1|U_0, X_1, X_2)\\ &\quad +I(\hat{Y}_1; Y_2|U_0, X_1, X_2) && \text{(24c)}\\ R_c+\hat{R}_1+R_{p,2}+R'_2+\tilde{R}_1 &< I(\hat{Y}_1; Y_2, U_2|U_0, X_1, X_2)\\ &\quad +I(U_0, U_2, X_1; Y_2|X_2) && \text{(24d)}\\ \hat{R}_2+\tilde{R}_2 &> I(\hat{Y}_2; Y_2|X_2). && \text{(24e)}\end{aligned}$$

Combine (20–24) and use Fourier-Motzkin elimination to eliminate $R'_1, R'_2, \hat{R}_1, \hat{R}_2, \tilde{R}_1, \tilde{R}_2$, then we obtain Theorem 3.

VII. Conclusion

In this paper, we studied partially and fully cooperative RBCs with relay/receiver-transmitter and rate-limited feedback. New coding schemes have been proposed to improve on the known rate regions that consider either feedback or relay cooperation, but not both. Specifically, our first rate region strictly improves on Liang and Kramer's region for the partially cooperative RBCs without feedback, and our second rate region strictly improves Wu and Wigger's region for the BCs with feedback but in the absence of relay cooperation. These two results together demonstrates that using feedback and relay simultaneously is a powerful tool to improve the rate performance of networks.

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