

# An Upper Bound on Secret Key Rates for General Multiterminal Wiretap Channels

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## Abstract

An upper bound is derived on the secret key rates of a general multiterminal wiretap channel. The bound unifies and generalizes some of the previously known bounds. Additionally, a multivariate dependence balance bound is introduced that is of independent interest.

## 1 Introduction

Consider a general multiterminal network with the channel  $p(y_1, y_2, \dots, y_k | x_1, x_2, \dots, x_k)$  where  $x_i$  and  $y_i$  are the respective channel inputs and outputs of the  $i$ -th transceiver. The cut-set upper bound limits the communication rates among the terminals [1, Theorem 15.10.1]. In this paper we are interested in *common/shared randomness* that can be generated among different terminals. Note that producing common randomness is more general than communicating messages: a message sent from one terminal to another (in either forward or backward directions) contributes to the common randomness that is shared between them. However, shared randomness may also be produced via correlated channel noise at the output of different terminals.

In studying common randomness, we also allow for a (passive) wiretapper and include one more variable in our network model  $p(y_1, y_2, \dots, y_k, z | x_1, x_2, \dots, x_k)$  where  $z$  is the wiretapper output. The common randomness should be kept hidden from the wiretapper, i.e., the common randomness shared among a group of terminals serves as a secret key. The problem of generating multiple keys among different sets of terminals has been studied before, and we refer to [2]. While final results are known for special cases, e.g. [3], no general outer bound (for an arbitrary network) on the trade-off of different key rates is known.

Our goal is to provide an upper bound that unifies several results in the literature. Some of these results involve communication channels with feedback, e.g., two basic models in information-theoretic security are known as the *source model* and the *channel model* and they involve noiseless public feedback links, see [4, Chapter 22]. To incorporate feedback, we consider parallel channels where, in addition to the main network  $p(y_1, y_2, \dots, y_k, z | x_1, x_2, \dots, x_k)$ , there are  $L$  parallel channels  $q_\ell(y_1, y_2, \dots, y_k, z | x_1, x_2, \dots, x_k)$  for  $\ell = 1, 2, \dots, L$  that the legitimate terminals can use.

Our contributions can be summarized as follows: we give a new upper bound for the shared secret key rates among different terminals of a general network. This upper bound recovers the best-known bounds for the *source model* and the *channel model* [5, 6]. The bound also recovers the bound on the key agreement bound from [7] for wiretap channels with a secure rate-limited feedback link.

Our bound uses the idea of auxiliary receivers which can be applied to many settings in network information theory [8]. The bound also uses a notion of multivariate information among  $k$  random variables. To derive the upper bound, we introduce a new dependence-balance bound [9] for our notion of information that recovers the refined dependence balance equations of [10] for Shannon mutual information. We illustrate its utility by deriving new constraints on the rate region of the two-user multiple-access channel (MAC) with generalized feedback discussed in [11].

**Notation:** We adopt most of the notation from [4]. The set  $\{1, \dots, k\}$  is denoted by  $[k]$ . For a set  $\mathcal{U}$ , we use  $Y_{\mathcal{U}}$  to denote  $(Y_i : i \in \mathcal{U})$ . In particular,  $x_{[k]} = (x_1, \dots, x_k)$ . We use  $Y^i$  to denote the sequence  $(Y_1, Y_2, \dots, Y_i)$ , and  $Y_i^j$  to denote  $(Y_i, Y_{i+1}, \dots, Y_j)$ . We say that  $X \dashv\!\!\dashv Y \dashv\!\!\dashv Z$  forms a Markov chain if  $I(X; Z|Y) = 0$ .

## 2 System Model

The main channel  $p(y_{[k]}, z|x_{[k]})$  has input alphabet sets  $\mathcal{X}_i$  and output alphabet sets  $\mathcal{Y}_i$  and  $\mathcal{Z}$ . In addition, we consider  $L$  parallel channels  $q_\ell(y_{[k]}, z|x_{[k]})$  that the terminals can use. These are described by input alphabet sets  $\mathcal{X}_i^{(\ell)}$  and output alphabet sets  $\mathcal{Y}_i^{(\ell)}$  and  $\mathcal{Z}^{(\ell)}$ ,  $\ell \in [L]$ , where  $x_i \in \mathcal{X}_i^{(\ell)}$ ,  $y_i \in \mathcal{Y}_i^{(\ell)}$  and  $z \in \mathcal{Z}^{(\ell)}$ . For instance, a noiseless public channel can be modeled by the parallel channel  $Y_1 = Y_2 = \dots = Y_k = Z = X_{[k]}$ .

A code of length  $n$  is defined as follows: at time instance  $j \in [n]$ , the  $i$ -th legitimate terminal uses local (private) random strings  $W_i$  and transmits the symbol

$$X_{ij} = f_{ij}(W_i, Y_{i[j-1]}), \quad j \in [n] \quad (1)$$

over the main channel  $p(y_{[k]}, z|x_{[k]})$  or over one of the parallel channels  $q_i(y_{[k]}, z|x_{[k]})$  (the identity of the channel to be used at time instance  $j$  is known and fixed a priori by the code). Here,  $n$  is the number of transmissions and  $f_{ij}(\cdot)$  is the encoding function at terminal  $i$  for time  $j$ , and  $Y_{ij}$  is the channel output symbol seen by terminal  $i$  at time  $j$ . The random variable  $Y_{i[j-1]}$  (also sometimes denoted by  $Y_i^{j-1}$ ) is the collection of past outputs of terminal  $i$  at time  $j$ . Suppose the main channel is used  $m \leq n$  times during the  $n$  transmissions, while the channel  $q_\ell(y_{[k]}, z|x_{[k]})$  is used  $m_\ell$  times for  $\ell = 1, 2, \dots, L$ . Thus,  $m + \sum_{\ell=1}^L m_\ell = n$ . We call

$$\alpha_\ell = \frac{m_\ell}{m} \quad (2)$$

the *rate of channel use* for  $q_\ell(y_{[k]}, z|x_{[k]})$ .

After transmission, every subset  $\mathcal{V} \subseteq [k]$  of terminals ( $|\mathcal{V}| \geq 2$ ) generates a shared key of rate  $R_{\mathcal{V}}$ . Keys generated by different subsets must be mutually independent of each other and of  $Z^n$ . More specifically, the  $i$ -th terminal generates  $S_{i,\mathcal{V}} = g_{i,\mathcal{V}}(W_i, Y_{i[n]})$  for every  $\mathcal{V}$  containing  $i$  where  $S_{i,\mathcal{V}} \in [2^{mR_{\mathcal{V}}}]$ . In an  $(n, \epsilon)$  code, we require existence of a random variable

$$S_{\mathcal{V}} \in [2^{mR_{\mathcal{V}}}]$$

for every subset  $\mathcal{V}$  such that random variables  $(S_{\mathcal{V}} : \mathcal{V} \subseteq [k])$  are mutually independent. Moreover, the following reliability and security conditions hold:

$$\begin{aligned} \frac{1}{m} H(S_{\mathcal{V}}) &\geq R_{\mathcal{V}} - \epsilon, \\ \mathbb{P}[\forall i \in \mathcal{V} : S_{i,\mathcal{V}} = S_{\mathcal{V}}] &\geq 1 - \epsilon, \\ \frac{1}{m} I(S_{\mathcal{V}}; Z^n, \{S_{\mathcal{V}'} : \mathcal{V}' \neq \mathcal{V}\}) &\leq \epsilon. \end{aligned} \quad (3)$$

Observe that we use the normalization factor  $1/m$  and not  $1/n$  in the above definition. The number  $R_{\mathcal{V}}$  is called the *group secret key rate* for the subset  $\mathcal{V}$ . Given channel-use rates  $\alpha_\ell \geq 0$  for  $\ell \in [L]$ , we are interested in the rates  $R_{\mathcal{V}}$  that can be achieved for any  $\epsilon > 0$  as  $m$  tends to infinity.

An important special case is when there is only one subset of terminals – without loss of generality assumed to be the first  $r$  terminals – the generate the secret keys, i.e.,  $R_{\mathcal{V}} = 0$  when  $\mathcal{V} \neq [r]$ . In this example, while terminals  $r+1, r+2, \dots, k$  do not generate secret keys, they have channel inputs and can participate as *helper terminals*. If we wish to keep the secret key generated by the first  $r$  terminals private from a collection of helper terminals, the outputs of these terminals could be included as part of the eavesdropper's  $Z$ .

Our model includes several models as special cases.

- **Source model:** consider  $k = 2$  and assume that in the main channel  $X_1$  and  $X_2$  are constants. Add another channel to allow for public discussion, and let  $\alpha_1$  (as defined in (2)) tend to infinity (which means that the public discussion is free and unrestricted). Similarly, the multiuser case studied in [5, 12] is a special case of our model. The exact capacity of the source model problem is open in general; see [13–15] for some recent results.
- **Channel model:** consider  $k = 2$  and assume that in the main channel  $X_2$  and  $Y_1$  are constants. Add another channel to allow for public discussion and let  $\alpha_1$  tend to infinity. Similarly, the multiuser case in [5, 12] is a special case of our model. Also, the MAC model ([16, 17]) for which each legitimate terminal is either a receiver or transmitter can be included by choosing the alphabets of either  $X_i$  or  $Y_i$  to have only one element.

- Wiretap channels with a private feedback link: For a secure rate-limited feedback link as in [7], we can set  $k = 2$  and consider a parallel channel where  $Y_2$  and  $Z$  are constant while  $p(y_1|x_2)$  has a capacity equal to the desired feedback rate.
- The channel model of [18] reduces to the model considered here if the parallel channels model a public channel available to all parties.

### 3 Multivariate Information

This section studies a notion of multivariate information using fractional partitions.

**Definition 1** (Fractional Partition). *Let  $\mathcal{B}$  be the set of all non-empty proper subsets of  $[k]$ . A fractional partition of  $[k]$  is a collection of non-negative weights assigned to non-empty subsets of  $[k]$ , i.e.,  $\lambda_{\mathcal{B}}$  for every  $\mathcal{B} \in \mathcal{B}$  such that*

$$\sum_{\mathcal{B} \in \mathcal{B}: i \in \mathcal{B}} \lambda_{\mathcal{B}} = 1, \quad \forall i \in [k]. \quad (4)$$

**Definition 2** (Multivariate Information). *Let  $(\lambda_{\mathcal{B}} : \mathcal{B} \in \mathcal{B})$  be an arbitrary fractional partition of  $[k]$ . The  $\lambda$ -multivariate information among variables  $X_i$ ,  $i \in [k]$ , conditioned on another random variable  $T$  is*

$$I_{\lambda}(X_1; X_2; \dots; X_k | T) = H(X_{[k]} | T) - \sum_{\mathcal{B}} \lambda_{\mathcal{B}} H(X_{\mathcal{B}} | X_{\mathcal{B}^c}, T). \quad (5)$$

Basic properties of  $I_{\lambda}$  are discussed in Appendix A.

**Remark 1.** *The above definition of multivariate information first appeared implicitly in [12, Equation 6]. However, this paper does not formally view  $I_{\lambda}$  as a multivariate notion of information. In [12], the minimum of  $I_{\lambda}$  over all fractional partitions  $\lambda$  is shown to be related to the secret key rate. After establishing the tightness of an upper bound in [12], the authors in [19] called the minimum of  $I_{\lambda}$  over all fractional partitions  $\lambda$  a multivariate information. The same minimum over all fractional partitions  $\lambda$  is called shared information in [20, Remark 3.11]*

*In this paper, we view  $I_{\lambda}$  for a fixed (and arbitrary) choice of  $\lambda$  as a multivariate information. In particular, fixing  $\lambda$  (and not taking the minimum over  $\lambda$ ) allows proving a dependence balance bound for the multivariate information, see Section 5).*

Observe that when  $k = 2$ ,  $\lambda_{\{1\}} = \lambda_{\{2\}} = 1$  and  $\lambda_{\{1,2\}} = 0$ , the  $\lambda$ -multivariate information reduces to the ordinary conditional mutual information. Next, we have

$$I_{\lambda}(X_1; X_2; \dots; X_k) = (1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}) H(X_{[k]}) + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} H(X_{\mathcal{B}}). \quad (6)$$

Since  $\lambda_{\mathcal{B}} \geq 0$  and  $\sum_{\mathcal{B}} \lambda_{\mathcal{B}} \geq 1$ , the coefficient of  $H(X_{[k]})$  is non-positive while the coefficient of  $H(X_{\mathcal{B}})$  for any proper subset  $\mathcal{B}$  is non-negative. Consequently, we cannot express  $I(X_1; X_2) = H(X_1) + H(X_2) - H(X_1, X_2)$  as special case of  $I_{\lambda}(X_1; X_2; \dots; X_k)$  for some fractional partition of  $[k]$  if  $k > 2$  (as the coefficient of  $H(X_1, X_2)$  is negative). Therefore, to define a more general notion of multivariate information, we should consider a weighted version of  $I_{\lambda}$  for different subsets of the variables. This is formalized next.

**Definition 3.** *For every subset  $\mathcal{U} \subset [k]$ , take some fractional partition  $\lambda_{\mathcal{B}}^{\mathcal{U}}$  ( $\mathcal{B}$  is a proper non-empty subset of  $\mathcal{U}$ ) such that*

$$\sum_{\mathcal{B} \subset \mathcal{U}: i \in \mathcal{B}} \lambda_{\mathcal{B}}^{\mathcal{U}} = 1, \quad \forall i \in \mathcal{U}. \quad (7)$$

*For every non-empty set  $\mathcal{U} = \{i_1, i_2, \dots, i_u\} \subset [k]$ , the corresponding multi-variate information using the fractional partition  $\lambda_{\mathcal{B}}^{\mathcal{U}}$  equals*

$$I_{\lambda^{\mathcal{U}}}(X_{i_1}; X_{i_2}; \dots; X_{i_u}).$$

*Let  $\omega_{\mathcal{U}}$  be a non-negative weight assigned to set  $\mathcal{U}$  such that*

$$\sum_{\mathcal{U}} \omega_{\mathcal{U}} = 1.$$

*Then, the  $(\omega, \lambda)$  multivariate information among  $X_1, \dots, X_k$  is defined as follows:*

$$I_{\omega, \lambda}(X_1; X_2; \dots; X_k) \triangleq \sum_{\mathcal{U}} \omega_{\mathcal{U}} \times I_{\lambda^{\mathcal{U}}}(X_{i_1}; X_{i_2}; \dots; X_{i_u}).$$

We utilize the  $(\omega, \lambda)$  multivariate information to obtain tight upper bounds for the special case of the source model problem with silent terminals.

## 4 New Upper Bound

Before stating our main result, we need the following definition.

**Definition 4.** Given a joint distribution  $p_{A,B,C}$ , let

$$S(A \rightarrow B \| C) = \max[I(V; B|U) - I(V; C|U)] \quad (8)$$

where the maximum is over all Markov chains  $(U, V) \text{---} A \text{---} (B, C)$ . The term  $S(A \rightarrow B \| C)$  is the one-way secrecy capacity in the source model problem. It is known that  $S(A \rightarrow B \| C) \leq I(A; B|C)$  as  $I(A; B|C)$  is an upper bound on the secrecy rate in the source model problem. In particular, we have  $S(A \rightarrow B \| C) = 0$  when  $B = C$ .

Consider an auxiliary random variable  $T$  with alphabet set  $\mathcal{T}$  defined by a conditional distribution  $q(t|y_{[k]}, z, x_{[k]})$ . We call  $T$  an auxiliary receiver.

**Definition 5.** Take some arbitrary choice of  $(\omega, \lambda)$  as defined in Definition 3. Given a conditional distribution  $q(t, y_{[k]}, z|x_{[k]})$  let

$$\begin{aligned} V_{\omega, \lambda}(q(t, y_{[k]}, z|x_{[k]})) = \max & \left[ I_{\omega, \lambda}(X_1 Y_1; X_2 Y_2; \dots; X_k Y_k | T) - I_{\omega, \lambda}(X_1; X_2; \dots; X_k) \right. \\ & - \sum_{\mathcal{U}} \omega_{\mathcal{U}} \left( 1 - \sum_{\mathcal{B} \subseteq \mathcal{U}} \lambda_{\mathcal{B}}^{\mathcal{U}} \right) I(X_{[k]}; Y_{\mathcal{U}}, T | X_{\mathcal{U}}) \\ & \left. + S(X_{[k]} Y_{[k]} \rightarrow T \| Z) \right] \end{aligned}$$

where the maximum is over all  $p(x_{[k]})$ .

**Remark 2.** Consider the special case  $T = Z$ ,  $\omega_{[k]} = 1$  and  $\omega_{\mathcal{U}} = 0$  when  $\mathcal{U} \neq [k]$ . Let  $\lambda$  be a fractional partition corresponding to  $[k]$ . Then, we obtain

$$V_{\omega, \lambda}(q(t, y_{[k]}, z|x_{[k]})) = \max[I_{\lambda}(X_1 Y_1; X_2 Y_2; \dots; X_k Y_k | Z) - I_{\lambda}(X_1; X_2; \dots; X_k)]$$

where the maximum is over all  $p(x_{[k]})$ .

We are now ready to state our main upper bound. We use an auxiliary receiver described by  $p(t|y_{[k]}, z, x_{[k]})$  for the main channel, and an auxiliary receiver with alphabet  $\mathcal{T}^{(\ell)}$  described by  $q_{\ell}(t|y_{[k]}, z, x_{[k]})$  for the parallel channels.

**Theorem 1.** Consider the main channel  $p(y_{[k]}, z|x_{[k]})$  and  $L$  parallel channels  $q_{\ell}(y_{[k]}, z|x_{[k]})$ ,  $\ell \in [L]$ , along with channel use rates  $\alpha_{\ell}$  as defined in (2). Take auxiliary receivers  $p(t|y_{[k]}, z, x_{[k]})$  and  $q_{\ell}(t|y_{[k]}, z, x_{[k]})$  ( $\ell = 1, 2, \dots, L$ ) for the main channel and the parallel channels, respectively. The group secret key rates  $R_{\mathcal{V}}$  for  $\mathcal{V} \subset [k]$  are achievable only if for any any  $(\omega, \lambda)$  (see Definition 3) we have

$$\begin{aligned} & \sum_{\mathcal{V}} R_{\mathcal{V}} \left( \sum_{\mathcal{U}: \mathcal{V} \cap \mathcal{U} \neq \emptyset} \omega_{\mathcal{U}} \left( 1 - \sum_{\mathcal{B} \subseteq \mathcal{U}: \mathcal{V} \cap (\mathcal{U} - \mathcal{B}) = \emptyset} \lambda_{\mathcal{B}}^{\mathcal{U}} \right) \right) \\ & \leq V_{\omega, \lambda}(p(y_{[k]}, z|x_{[k]})p(t|x_{[k]}, y_{[k]}, z)) + \sum_{\ell=1}^L \alpha_{\ell} V_{\omega, \lambda}(q_{\ell}(y_{[k]}, z|x_{[k]})q_{\ell}(t|x_{[k]}, y_{[k]}, z)). \quad (9) \end{aligned}$$

Intuitively,  $V_{\omega, \lambda}(p(y_{[k]}, z|x_{[k]})p(t|x_{[k]}, y_{[k]}, z))$  is an upper bound on the contribution of the main channel to the total secret key, while  $V_{\omega, \lambda}(q_{\ell}(y_{[k]}, z|x_{[k]})q_{\ell}(t|x_{[k]}, y_{[k]}, z))$  is an upper bound on the contribution of the  $\ell$ -th parallel channel.

**Corollary 1.** Consider the special case  $\omega_{[k]} = 1$  and  $\omega_{\mathcal{U}} = 0$  when  $\mathcal{U} \neq [k]$ . Let  $\lambda$  be a fractional partition for  $[k]$ . Then the group secret key rates  $R_{\mathcal{V}}$  for  $\mathcal{V} \subset [k]$  are achievable only if

$$\begin{aligned} & \sum_{\mathcal{V}} R_{\mathcal{V}} \left( 1 - \sum_{\mathcal{B}: \mathcal{V} \subset \mathcal{B} \subsetneq [k]} \lambda_{\mathcal{B}} \right) \\ & \leq V_{\omega, \lambda} (p(y_{[k]}, z|x_{[k]})p(t|x_{[k]}, y_{[k]}, z)) + \sum_{\ell=1}^L \alpha_{\ell} V_{\omega, \lambda} (q_{\ell}(y_{[k]}, z|x_{[k]})q_{\ell}(t|x_{[k]}, y_{[k]}, z)). \end{aligned} \quad (10)$$

where

$$V_{\omega, \lambda} (q(t, y_{[k]}, z|x_{[k]})) = \max[I_{\lambda}(X_1 Y_1; X_2 Y_2; \dots; X_k Y_k | T) - I_{\lambda}(X_1; X_2; \dots; X_k) + S(X_{[k]} Y_{[k]} \rightarrow T | Z)]$$

where the maximum is over all  $p(x_{[k]})$ . For  $T = Z$  and  $\mathcal{U} = [k]$ , we get a bound that can also be deduced from Theorem 3.1 of [21].

*Proof of Theorem 1.* Take an arbitrary auxiliary channel  $p(t|x_{[k]}, y_{[k]}, z)$  for the main channel and  $q_{\ell}(t|x_{[k]}, y_{[k]}, z)$  for the parallel channel  $\ell \in [L]$ . Take some arbitrary code and let  $T^n$  be defined as follows:

$$P_{T^n | X_{[k]}^n, Y_{[k]}^n, Z^n} = \prod_{j=1}^n P_{T_j | X_{[k]j}, Y_{[k]j}, Z_j}$$

where  $P_{T_j | X_{[k]j}, Y_{[k]j}, Z_j}$  is the auxiliary channel corresponding to the  $j$ -th time instance, i.e., it is equal to  $p(t|x_{[k]}, y_{[k]}, z)$  if we use the main channel at time instance  $j$ , or  $q_{\ell}(t|x_{[k]}, y_{[k]}, z)$  if we use the  $\ell$ -th parallel channel at time instance  $j$ .

Let  $\mathbf{M}_i = (S_{i, \mathcal{V}} : \mathcal{V})$  be the set of keys generated by the  $i$ -th terminal. From Fano's inequality and (3), we have

$$\begin{aligned} \frac{1}{m} I_{\omega, \lambda} (\mathbf{M}_1; \mathbf{M}_2; \dots; \mathbf{M}_k) &= \frac{1}{m} \sum_{\mathcal{U}} \omega_{\mathcal{U}} \left( H(\mathbf{M}_{\mathcal{U}}) - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}}^{\mathcal{U}} H(\mathbf{M}_{\mathcal{B}} | \mathbf{M}_{\mathcal{U}-\mathcal{B}}) \right) \\ &\geq -k_1(\epsilon) + \sum_{\mathcal{U}} \omega_{\mathcal{U}} \left( \sum_{\mathcal{V}: \mathcal{V} \cap \mathcal{U} \neq \emptyset} R_{\mathcal{V}} - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}}^{\mathcal{U}} \sum_{\mathcal{V}: \mathcal{V} \cap \mathcal{B} \neq \emptyset, \mathcal{V} \cap (\mathcal{U}-\mathcal{B}) = \emptyset} R_{\mathcal{V}} \right) \end{aligned}$$

for some  $k_1(\epsilon)$  that tends to zero as  $\epsilon$  tends to zero. Thus, we have

$$\sum_{\mathcal{V}} R_{\mathcal{V}} \left( \sum_{\mathcal{U}: \mathcal{V} \cap \mathcal{U} \neq \emptyset} \omega_{\mathcal{U}} \left( 1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}: \mathcal{V} \cap (\mathcal{U}-\mathcal{B}) = \emptyset} \lambda_{\mathcal{B}}^{\mathcal{U}} \right) \right) \leq \frac{1}{m} I_{\omega, \lambda} (\mathbf{M}_1; \mathbf{M}_2; \dots; \mathbf{M}_k) + k_1(\epsilon).$$

Next, using the conditioning inequality for  $I_{\lambda}$ , as shown in Proposition 1 in Appendix A we have

$$\begin{aligned} I_{\omega, \lambda} (\mathbf{M}_1; \mathbf{M}_2; \dots; \mathbf{M}_k) &\leq I_{\omega, \lambda} (\mathbf{M}_1; \mathbf{M}_2; \dots; \mathbf{M}_k | T^n) + I(\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k; T^n) \\ &\leq I_{\omega, \lambda} (\mathbf{M}_1; \mathbf{M}_2; \dots; \mathbf{M}_k | T^n) \\ &\quad + I(\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k; T^n) - I(\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k; Z^n) + nk_2(\epsilon) \end{aligned}$$

for some  $k_2(\epsilon)$  that tend to zero as  $\epsilon$  tends to zero. Observe that

$$I(\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k; T^n) - I(\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k; Z^n) \quad (11)$$

$$= \sum_{j=1}^n I(\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k; T_j | Z_{j+1}^n, T^{j-1}) - I(\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k; Z_j | Z_{j+1}^n, T^{j-1}) \quad (12)$$

$$= \sum_{j=1}^n I(V_j; T_j | U_j A_j) - I(V_j; Z_j | U_j A_j) \quad (13)$$

where  $V_j = (\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k)$ ,  $U_j = Z_{j+1}^n$  and  $A_j = T^{j-1}$ . Note that

$$A_j \oplus X_{[k]j} \oplus Y_{[k]j} T_j Z_j$$

$$U_j V_j A_j \text{---} X_{[k]j} Y_{[k]j} \text{---} T_j Z_j$$

form Markov chains. Next, we have

$$I_{\omega, \lambda}(\mathbf{M}_1; \mathbf{M}_2; \dots; \mathbf{M}_k | T^n) \leq I_{\omega, \lambda}(W_1 Y_1^n; W_2 Y_2^n; \dots; W_k Y_k^n | T^n) \quad (14)$$

$$\begin{aligned} &= I_{\omega, \lambda}(W_1 Y_1^n; W_2 Y_2^n; \dots; W_k Y_k^n | T^n) - I_{\omega, \lambda}(W_1; W_2; \dots; W_k) \\ &\leq \sum_{j=1}^n I_{\omega, \lambda}(X_{1j} Y_{1j}; X_{2j} Y_{2j}; \dots; X_{kj} Y_{kj} | T_j, T^{j-1}) - \sum_{j=1}^n I_{\omega, \lambda}(X_{1j}; X_{2j}; \dots; X_{kj} | T^{j-1}) \\ &\quad - \sum_{j=1}^k \sum_{\mathcal{U}} \omega_{\mathcal{U}} \left( 1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}}^{\mathcal{U}} \right) I(X_{[k]j}; Y_{\mathcal{U}, j}, T_j | X_{\mathcal{U}, j}, T^{j-1}) \end{aligned} \quad (15)$$

where (14) follows from the data processing inequality for  $I_{\lambda}$ , as shown in Proposition 1 in Appendix A, and (15) follows from Lemma 1 in Section 5, and  $k_3(\epsilon)$  is a function that tends to zero as  $\epsilon$  tends to zero.

Collecting the above results, we obtain

$$\begin{aligned} &\sum_{\mathcal{V}} R_{\mathcal{V}} \left( \sum_{\mathcal{U}: \mathcal{V} \cap \mathcal{U} \neq \emptyset} \omega_{\mathcal{U}} \left( 1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}: \mathcal{V} \cap (\mathcal{U} - \mathcal{B}) = \emptyset} \lambda_{\mathcal{B}}^{\mathcal{U}} \right) \right) - k_1(\epsilon) - k_2(\epsilon) \\ &\leq \frac{1}{m} \sum_{j=1}^n \left[ I_{\omega, \lambda}(X_{1j} Y_{1j}; X_{2j} Y_{2j}; \dots; X_{kj} Y_{kj} | T_j, A_j) - I_{\omega, \lambda}(X_{1j}; X_{2j}; \dots; X_{kj} | A_j) \right. \\ &\quad - \sum_{\mathcal{U}} \omega_{\mathcal{U}} \left( 1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}}^{\mathcal{U}} \right) I(X_{[k]j}; Y_{\mathcal{U}, j}, T_j | X_{\mathcal{U}, j}, A_j) \\ &\quad \left. + I(V_j; T_j | U_j A_j) - I(V_j; Z_j | U_j A_j) \right]. \end{aligned} \quad (16)$$

Consider set of  $m$  indices  $j_1, j_2, \dots, j_m \in [n]$  where the main channel is used. We have

$$\begin{aligned} &\sum_{b=1}^m \left[ I_{\omega, \lambda}(X_{1j_b} Y_{1j_b}; X_{2j_b} Y_{2j_b}; \dots; X_{kj_b} Y_{kj_b} | T_{j_b}, A_{j_b}) - I_{\omega, \lambda}(X_{1j_b}; X_{2j_b}; \dots; X_{kj_b} | A_{j_b}) \right. \\ &\quad - \sum_{\mathcal{U}} \omega_{\mathcal{U}} \left( 1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}}^{\mathcal{U}} \right) I(X_{[k]j_b}; Y_{\mathcal{U}, j_b}, T_{j_b} | X_{\mathcal{U}, j_b}) \\ &\quad \left. + I(V_{j_b}; T_{j_b} | U_{j_b} A_{j_b}) - I(V_{j_b}; Z_{j_b} | U_{j_b} A_{j_b}) \right] \end{aligned} \quad (17)$$

$$\leq m \cdot V_{\omega, \lambda}(p(y_{[k]}, z | x_{[k]}) p(t | x_{[k]}, y_{[k]}, z)). \quad (18)$$

A similar argument shows that sum of the terms in (16), where the parallel channel  $q_{\ell}(y_{[k]}, z | x_{[k]})$  is used, is bounded from above by

$$m \cdot \alpha_{\ell} \cdot V_{\omega, \lambda}(q_{\ell}(y_{[k]}, z | x_{[k]}) q_{\ell}(t | x_{[k]}, y_{[k]}, z)).$$

□

## 4.1 Relation with previously known results

### 4.1.1 The cut-set bound

The bound in Corollary 1 can be written in a "cut-set form" when  $T = Z = \emptyset$ , i.e., when the secrecy aspect of the problem is removed. In this case, the problem reduces to generating common randomness among different subsets of terminals at given rates. Assume that there are no parallel channels so that  $L = 0$  and

$$\begin{aligned} &I_{\lambda}(X_1 Y_1; X_2 Y_2; \dots; X_k Y_k) - I_{\lambda}(X_1; X_2; \dots; X_k) \\ &= I_{\lambda}(Y_1; Y_2; \dots; Y_k | X_{[k]}) + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} I(X_{\mathcal{B}}; Y_{\mathcal{B}^c} | X_{\mathcal{B}^c}). \end{aligned} \quad (19)$$

Thus, common randomness generation at rate  $R_{\mathcal{V}}$  for subset  $\mathcal{V}$  is possible only if

$$\sum_{\mathcal{V}} R_{\mathcal{V}} \left( 1 - \sum_{\mathcal{B}: \mathcal{V} \subseteq \mathcal{B} \subseteq [k]} \lambda_{\mathcal{B}} \right) \leq I_{\lambda}(Y_1; Y_2; \dots; Y_k | X_{[k]}) + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} I(X_{\mathcal{B}}; Y_{\mathcal{B}^c} | X_{\mathcal{B}^c}) \quad (20)$$

for some  $p(x_{[k]})$ . As a special case, consider  $k = 2$  and a two-way channel  $p(y_1, y_2 | x_1, x_2)$ . The rate of the shared randomness that can be produced between the two terminals is at most

$$I(X_1; Y_2 | X_2) + I(X_2; Y_1 | X_1) + I(Y_1; Y_2 | X_1, X_2) \quad (21)$$

for some  $p(x_1, x_2)$ . The terms  $I(X_1; Y_2 | X_2)$  and  $I(X_2; Y_1 | X_1)$  correspond to the cut-set terms for generating common randomness by communicating bits from one terminal to the other, and  $I(Y_1; Y_2 | X_1, X_2)$  can be interpreted as an upper bound on the randomness generated through the noise of the channel. A similar interpretation holds for a general network  $p(y_{[k]} | x_{[k]})$ : assuming that user  $i$  communicates to user  $j$  at rate  $r_{ij}$ , pairwise shared randomness at rate  $R_{\{i,j\}} = r_{ij} + r_{ji}$  will be produced between terminals  $i$  and  $j$  as a result of the communication. Let  $R_{\mathcal{V}} = 0$  if  $|\mathcal{V}| > 2$ . We have

$$\begin{aligned} \sum_{\mathcal{V}} R_{\mathcal{V}} \left( 1 - \sum_{\mathcal{B}: \mathcal{V} \subseteq \mathcal{B} \subseteq [k]} \lambda_{\mathcal{B}} \right) &= \sum_{i \neq j} (r_{ij} + r_{ji}) \left( 1 - \sum_{\mathcal{B}: i, j \in \mathcal{B}} \lambda_{\mathcal{B}} \right) \\ &= \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \sum_{i \in \mathcal{B}, j \in \mathcal{B}^c} r_{i,j}, \end{aligned} \quad (22)$$

where we used (4). Therefore, we obtain

$$\sum_{\mathcal{B} \subseteq [k]} \lambda_{\mathcal{B}} \sum_{i \in \mathcal{B}, j \in \mathcal{B}^c} r_{i,j} \leq I_{\lambda}(Y_1; Y_2; \dots; Y_k | X_{[k]}) + \sum_{\mathcal{B} \subseteq [k]} \lambda_{\mathcal{B}} I(X_{\mathcal{B}}; Y_{\mathcal{B}^c} | X_{\mathcal{B}^c}) \quad (23)$$

for some  $p(x_{[k]})$ . The expression  $I(X_{\mathcal{B}}; Y_{\mathcal{B}^c} | X_{\mathcal{B}^c})$  can be interpreted as a cut-set upper bound on the information flow  $\sum_{i \in \mathcal{B}, j \in \mathcal{B}^c} r_{i,j}$ , and  $I_{\lambda}(Y_1; Y_2; \dots; Y_k | X_{[k]})$  can be interpreted as an upper bound on the randomness generated through the noise of the channel.

#### 4.1.2 Two-terminal source model problem

The bound in Corollary 1 recovers the best known upper bound for the source model [5]. Suppose  $k = 2$  and  $X_1$  and  $X_2$  are constants. Choosing  $\lambda_{\{1\}} = \lambda_{\{2\}} = 1$ , the  $\lambda$ -multivariate information reduces to the ordinary conditional mutual information. Take some arbitrary  $p(t|y_1, y_2, z)$ . Then we obtain

$$V_{\omega, \lambda}(p(t, y_1, y_2, z | x_1, x_2)) = \max[I(X_1 Y_1; X_2 Y_2 | T) - I(X_1; X_2) + I(V; T | U) - I(V; Z | U)] \quad (24)$$

where the maximum is over all  $p(x_{[k]})$  and auxiliary random variables  $U, V$  satisfying

$$p_{X_1, X_2} p_{Y_1, Y_2, T, Z} p_{U, V | X_1, X_2, Y_1, Y_2}.$$

Since  $X_1$  and  $X_2$  are constants, we have

$$I(X_1 Y_1; X_2 Y_2 | T) - I(X_1; X_2) = I(Y_1; Y_2 | T) \quad (25)$$

and

$$V_{\omega, \lambda}(p(t, y_1, y_2, z | x_1, x_2)) = I(Y_1; Y_2 | T) + \max_{(V, U) \oplus (Y_1, Y_2) \oplus (T, Z)} [I(V; T | U) - I(V; Z | U)]. \quad (26)$$

Next, consider one parallel channel of the form  $Y_1 = Y_2 = Z = (X_1, X_2)$  where  $X_1$  and  $X_2$  are binary (each use of the parallel channel is equivalent to broadcasting one bit). For this parallel channel, we utilize the auxiliary receiver  $T = Z$ . Since  $H(X_{[k]}, Y_{[k]} | Z) = 0$  in the parallel channel, we have  $V_{\omega, \lambda}(q_{\ell}(y_{[k]}, z | x_{[k]}) q_{\ell}(t | x_{[k]}, y_{[k]}, z)) \leq 0$  and

$$\begin{aligned} R_{[k]} &\leq V_{\omega, \lambda}(p(y_{[k]}, z | x_{[k]}) p(t | x_{[k]}, y_{[k]}, z)) \\ &= I(Y_1; Y_2 | T) + \max_{(V, U) \oplus (Y_1, Y_2) \oplus (T, Z)} [I(V; T | U) - I(V; Z | U)]. \end{aligned} \quad (27)$$

Note that the channel-use rate  $\alpha_1$  does not appear in the upper bound and can be set to infinity (allowing infinite public discussion). This recovers the best known upper bound for the source model problem in the two-user case [22].

### 4.1.3 Two-terminal channel model problem

Suppose that in the main channel  $X_2$  and  $Y_1$  are constants. This case is similar to the one discussed above. Take some arbitrary  $p(t|x_1, y_2, z)$ . Then we obtain

$$V_{\omega, \lambda}(p(t, y_1, y_2, z|x_1, x_2)) = \max[I(X_1; Y_2|T) + I(V; T|U) - I(V; Z|U)] \quad (28)$$

where the maximum is over  $p(x_1)$  and all auxiliary random variables  $U$  and  $V$  satisfying

$$p_{X_1} p_{Y_2, T, Z|X_1} p_{U, V|X_1, Y_2}.$$

As in the previous subsection, the corresponding term for the parallel channel (the public channel) vanishes. This recovers the best known upper bound for the channel model problem in the two-user case [6].

### 4.1.4 $k$ -terminal source model problem

Next, consider a  $k$  terminal network  $p(y_{[k]}, z|x_{[k]})$  where  $|\mathcal{X}_i| = 1$  in the main network, i.e., the inputs are constant and the main network is described by  $p(y_{[k]}, z)$ . Consider the case  $H(Z|Y_i) = 0$  for  $i = 1, 2, \dots, k$ , and  $R_{\mathcal{V}} = 0$  when  $\mathcal{V} \neq [k]$ . In other words, the terminals aim to create a common secret key. Only the first  $u$  terminals can participate in public discussion while the terminals  $u+1, u+2, \dots, k$  are silent. This public discussion can be modeled by the parallel channel  $Y_1 = Y_2 = \dots = Y_k = Z = X_{[u]}$  with  $X_{u+1}, \dots, X_k$  being constants.

Deriving the capacity in this case requires using the general version of the upper bound with suitable weights  $\omega_{\mathcal{U}}$ . It cannot be obtained by the weaker bound in Corollary 1, as discussed in Appendix C. Here we consider the special case of having no silent terminal, i.e.,  $u = k$  and all terminals can speak and the public discussion can be modeled by the parallel channel  $Y_1 = Y_2 = \dots = Y_k = Z = X_{[k]}$ . Using the private key capacity result of [23], we obtain the maximum value for  $R_{\mathcal{V}}$  as

$$\min_{\lambda} I_\lambda(Y_1; Y_2; \dots; Y_k|Z). \quad (29)$$

To recover this value from our upper bound in Corollary 1, we choose the auxiliary receiver  $T = Z$  for the main channel. Since  $X_i$ 's are constants, after some simplification we obtain

$$V_{\omega, \lambda}(p(t, y_{[k]}, z|x_{[k]})) = I_\lambda(Y_1; Y_2; \dots; Y_k|Z). \quad (30)$$

Next, consider the parallel channel  $Y_1 = Y_2 = \dots = Y_k = Z = X_{[k]}$  with density  $q_1(y_{[k]}, z|x_{[k]})$ . We use the auxiliary receiver  $T = Z$  for the parallel channel. Since  $I_\lambda(X_1 Y_1; X_2 Y_2; \dots; X_k Y_k|Z, A) = 0$  it is immediate that  $V_{\omega, \lambda}(q_1(t, y_{[k]}, z|x_{[k]})) \leq 0$ . As before, the auxiliary channel-use rate  $\alpha_1$  does not appear in the upper bound and can be set to infinity (infinite public discussion). Since  $\lambda$  was arbitrary, we obtain the upper bound  $\min_{\lambda} I_\lambda(Y_1; Y_2; \dots; Y_k|Z)$ .

The connection between our bound and wiretap channels with rate-limited secure feedback is discussed in Appendix D.

## 5 A Dependence Balance Bound for $I_\lambda$

A key lemma for proving our main result is as follows.

**Lemma 1.** *Given random variables  $W_i, X_{ij}, Y_{ij}$  and  $Z_j$  for  $i \in [k], j \in [n]$  satisfying*

$$X_{ij} = f_{ij}(W_i, Y_{i[j-1]}), \quad i \in [k], j \in [n] \quad (31)$$

and the Markov chain

$$W_{[k]} Y_{[k]}^{j-1} Z^{j-1} \text{---} X_{[k]j} \text{---} Z_j Y_{[k]j}, \quad j \in [n]$$

we have

$$\begin{aligned} & I_\lambda(W_1 Y_1^n; W_2 Y_2^n; \dots; W_k Y_k^n | Z^n) - I_\lambda(W_1; W_2; \dots; W_k) \\ & \leq \sum_{j=1}^n I_\lambda(X_{1j} Y_{1j}; X_{2j} Y_{2j}; \dots; X_{kj} Y_{kj} | Z_j, Z^{j-1}) - \sum_{j=1}^n I_\lambda(X_{1j}; X_{2j}; \dots; X_{kj} | Z^{j-1}) \end{aligned} \quad (32)$$



for any fractional partition  $\lambda$  of  $[k]$ . More generally, let  $\mathcal{U}$  to be a non-empty subset of  $[k]$ . Assume that  $|\mathcal{U}| = u$  and  $\mathcal{U} = \{i_1, i_2, \dots, i_u\}$ . Let  $(\lambda_{\mathcal{B}} : \mathcal{B} \subsetneq \mathcal{U})$  be an arbitrary fractional partition for the  $u$  elements in  $\mathcal{U}$ . Then we have

$$\begin{aligned}
& I_\lambda(W_{i_1}Y_{i_1}^n; W_{i_2}Y_{i_2}^n; \dots; W_{i_u}Y_{i_u}^n | Z^n) - I_\lambda(W_{i_1}; W_{i_2}; \dots; W_{i_u}) \\
& \leq \sum_{j=1}^n I_\lambda(X_{i_1j}Y_{i_1j}; X_{i_2j}Y_{i_2j}; \dots; X_{i_uj}Y_{i_uj} | Z_j, Z^{j-1}) \\
& \quad - \sum_{j=1}^n I_\lambda(X_{i_1j}; X_{i_2j}; \dots; X_{i_uj} | Z^{j-1}) \\
& \quad - \left(1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}}\right) \sum_j I(X_{[k],j}; Y_{\mathcal{U},j}, Z_j | X_{\mathcal{U},j}, Z^{j-1})
\end{aligned} \tag{33}$$

where  $Y_{\mathcal{U},j} = (Y_{i_1j}, Y_{i_2j}, \dots, Y_{i_uj})$  and similarly for  $X_{\mathcal{U},j}$ .

*Proof of Lemma 1.* Without loss of generality assume that  $\mathcal{U} = [u]$ . We have

$$\begin{aligned}
& I_\lambda(W_1Y_1^n; W_2Y_2^n; \dots; W_uY_u^n | Z^n) - I_\lambda(W_1; W_2; \dots; W_u) \\
& = \sum_{j=1}^n \left[ I_\lambda(W_1Y_1^j; W_2Y_2^j; \dots; W_uY_u^j | Z^j) - I_\lambda(W_1Y_1^{j-1}; W_2Y_2^{j-1}; \dots; W_uY_u^{j-1} | Z^{j-1}) \right] \\
& = \sum_{j=1}^n \left[ \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) \left( H(W_{[u]}Y_{[u]}^j | Z^j) - H(W_{[u]}Y_{[u]}^{j-1} | Z^{j-1}) \right) \right. \\
& \quad \left. + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \left( H(W_{\mathcal{B}^c}Y_{\mathcal{B}^c}^j | Z^j) - H(W_{\mathcal{B}^c}Y_{\mathcal{B}^c}^{j-1} | Z^{j-1}) \right) \right] \\
& = \sum_{j=1}^n \left[ \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) \left( H(Y_{[u],j} | X_{[u],j} Z^j W_{[u]}Y_{[u]}^{j-1}) - I(Z_j; X_{[u],j} W_{[u]}Y_{[u]}^{j-1} | Z^{j-1}) \right) \right. \\
& \quad \left. + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \left( H(Y_{\mathcal{B}^c,j} | X_{\mathcal{B}^c,j} W_{\mathcal{B}^c}Y_{\mathcal{B}^c}^{j-1} Z^j) - I(Z_j; X_{\mathcal{B}^c,j} W_{\mathcal{B}^c}Y_{\mathcal{B}^c}^{j-1} | Z^{j-1}) \right) \right] \\
& = \sum_{j=1}^n \left[ \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) \left( H(Y_{[u],j} | X_{[u],j} Z^j) - I(Z_j; X_{[u],j} | Z^{j-1}) \right) \right. \\
& \quad - \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) I(W_{[u]}Y_{[u]}^{j-1}; Y_{[u],j} | X_{[u],j} Z^j) \\
& \quad - \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) I(W_{[u]}Y_{[u]}^{j-1}; Z_j | X_{[u],j} Z^{j-1}) \\
& \quad \left. + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \left( H(Y_{\mathcal{B}^c,j} | X_{\mathcal{B}^c,j} W_{\mathcal{B}^c}Y_{\mathcal{B}^c}^{j-1} Z^j) - I(Z_j; X_{\mathcal{B}^c,j} W_{\mathcal{B}^c}Y_{\mathcal{B}^c}^{j-1} | Z^{j-1}) \right) \right] \\
& = \sum_{j=1}^n \left[ \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) \left( H(Y_{[u],j} | X_{[u],j} Z^j) - I(Z_j; X_{[u],j} | Z^{j-1}) \right) \right. \\
& \quad \left. + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \left( H(Y_{\mathcal{B}^c,j} | X_{\mathcal{B}^c,j} W_{\mathcal{B}^c}Y_{\mathcal{B}^c}^{j-1} Z^j) - I(Z_j; X_{\mathcal{B}^c,j} W_{\mathcal{B}^c}Y_{\mathcal{B}^c}^{j-1} | Z^{j-1}) \right) \right. \\
& \quad \left. - \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) I(W_{[u]}Y_{[u]}^{j-1}; Z_j Y_{[u],j} | X_{[u],j} Z^{j-1}) \right] \\
& \leq \sum_{j=1}^n \left[ \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) \left( H(Y_{[u],j} | X_{[u],j} Z^j) - I(Z_j; X_{[u],j} | Z^{j-1}) \right) \right. \\
& \quad \left. + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \left( H(Y_{\mathcal{B}^c,j} | X_{\mathcal{B}^c,j} Z^j) - I(Z_j; X_{\mathcal{B}^c,j} | Z^{j-1}) \right) \right. \\
& \quad \left. - \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) I(X_{[k],j}; Z_j, Y_{[u],j} | X_{[u],j} Z^{j-1}) \right]
\end{aligned} \tag{35}$$

$$\begin{aligned}
&= \sum_{j=1}^n \left[ \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) \left( H(X_{[u],j} Y_{[u],j} | Z^j) - H(X_{[u],j} | Z^{j-1}) \right) \right. \\
&\quad \left. + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \left( H(X_{\mathcal{B}^c,j} Y_{\mathcal{B}^c,j} | Z^j) - H(X_{\mathcal{B}^c,j} | Z^{j-1}) \right) \right. \\
&\quad \left. - \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) I(X_{[k],j}; Z_j Y_{[u],j} | X_{[u],j} Z^{j-1}) \right] \\
&= \sum_{j=1}^n \left[ I_\lambda(X_{1j} Y_{1j}; X_{2j} Y_{2j}; \dots; X_{uj} Y_{uj} | Z_j, Z^{j-1}) - I_\lambda(X_{1j}; X_{2j}; \dots; X_{uj} | Z^{j-1}) \right. \\
&\quad \left. - \left(1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\right) I(X_{[k],j}; Z_j, Y_{[u],j} | X_{[u],j} Z^{j-1}) \right]
\end{aligned}$$

where (34) follows from (31) and (35) follows from the fact that conditioning reduces the entropy,  $1 - \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \leq 0$  and the Markov chain

$$(W_{[u]}, Y_{[u]}^{j-1}, Z^{j-1}) \text{---} X_{[k],j} \text{---} (Y_{[u],j}, Z_j).$$

□

This lemma is of independent interest. For example, one can apply Lemma 1 to a  $k$ -user MAC with generalized feedback

$$p(y_{F_1}, y_{F_2}, \dots, y_{F_k}, y | x_1, x_2, \dots, x_k). \quad (36)$$

Here the  $X_i$ 's are channel inputs and the  $Y_{F_i}$ 's represent noisy feedback. The receiver sees  $Y$ . The transmitter  $i$  wants to send a message of rate  $R_i$  to the receiver. Lemma 1 (with  $Z = Y$  and  $Y_i = Y_{F_i}$  in the lemma) yields the following bound for which the full proof is given in Appendix B.

**Theorem 2.** *Consider a  $k$ -user MAC with generalized output feedback. Then any achievable rate vector  $(R_1, R_2, \dots, R_k)$  satisfies*

$$R_{\mathcal{S}} \leq I(X_{\mathcal{S}}; Y | X_{\mathcal{S}^c}, T), \quad \forall \mathcal{S} \subset [k] \quad (37)$$

for some  $p_{T, X_{[k]}} p_{Y_{F_1}, Y_{F_2}, \dots, Y_{F_k}, Y | X_{[k]}}$  such that for any  $\mathcal{U} \subset [k]$  and any fractional partition  $\lambda$  for indices in  $\mathcal{U}$  we have

$$\begin{aligned}
&I_\lambda(X_{i_1} Y_{F_{i_1}}; X_{i_2} Y_{F_{i_2}}; \dots; X_{i_u} Y_{F_{i_u}} | T, Y) \\
&\geq I_\lambda(X_{i_1}; X_{i_2}; \dots; X_{i_u} | T) + \left(1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}}\right) I(X_{[k]}; Y Y_{F_1} Y_{F_2} \dots Y_{F_k} | X_{\mathcal{U}}, T). \quad (38)
\end{aligned}$$

In Section B, we show that for  $\mathcal{U} = [k]$  the above bound recovers the refined dependence balance equations of [10]. Moreover, an improved version of Theorem 1 of [11] for MACs with generalized feedback is also discussed.

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## A Properties of $\lambda$ -multivariate information

The following proposition essentially follows from the arguments in [12].

**Proposition 1.**  *$\lambda$ -multivariate information satisfies the following properties:*

- (Nonnegativity):  $I_\lambda(X_1; X_2; \dots; X_k) \geq 0$
- (Conditioning): We have

$$I_\lambda(X_1; X_2; \dots; X_k) - I_\lambda(X_1; X_2; \dots; X_k|T) \leq I(X_{[k]}; T)$$

- (Data processing): We have

$$I_\lambda(X_1; X_2; \dots; X_k) \geq I_\lambda(X'_1; X'_2; \dots; X'_k)$$

$$\text{if } p(x'_{[k]}, x_{[k]}) = p(x_{[k]}) \prod_{i=1}^k p(x'_i | x_i)$$

*Proof.* For non-negativity, we have

$$\begin{aligned} H(X_1 X_2 \dots X_k) &= \sum_i H(X_i | X^{i-1}) \\ &= \sum_i \left( \sum_{\mathcal{B}: i \in \mathcal{B}} \lambda_{\mathcal{B}} \right) H(X_i | X^{i-1}) \\ &= \sum_{\mathcal{B}} \sum_{i \in \mathcal{B}} \lambda_{\mathcal{B}} H(X_i | X^{i-1}) \\ &\geq \sum_{\mathcal{B}} \sum_{i \in \mathcal{B}} \lambda_{\mathcal{B}} H(X_i | X_{[i-1] \cap \mathcal{B}} X_{\mathcal{B}^c}) \\ &= \sum_{\mathcal{B}} \lambda_{\mathcal{B}} H(X_{\mathcal{B}} | X_{\mathcal{B}^c}). \end{aligned}$$

The second part follows from the identity

$$I_\lambda(X_1; X_2; \dots; X_k) - I_\lambda(X_1; X_2; \dots; X_k | T) = I(X_{[k]}; T) - \sum_{\mathcal{B}} \lambda_{\mathcal{B}} I(X_{\mathcal{B}}; T | X_{\mathcal{B}^c}).$$

Finally, using functional representation, one can find random variables  $Y_{[k]}$ , mutually independent of each other and of  $X_{[k]}$  such that  $H(X'_i | X_i, Y_i) = 0$ . Since adding private noise  $Y_i$  to  $X_i$  does not change the  $\lambda$ -multivariate information

$$I_\lambda(X_1; X_2; \dots; X_k) = I_\lambda(X_1 Y_1; X_2 Y_2; \dots; X_k Y_k)$$

it suffices to show that

$$I_\lambda(X'_1 X_1 Y_1; X'_2 X_2 Y_2; \dots; X'_k X_k Y_k) \geq I_\lambda(X'_1; X'_2; \dots; X'_k).$$

This inequality follows from

$$\begin{aligned} &I_\lambda(X'_1 X_1 Y_1; X'_2 X_2 Y_2; \dots; X'_k X_k Y_k | T) - I_\lambda(X'_1; X'_2; \dots; X'_k) \\ &= I_\lambda(X_1 Y_1; X_2 Y_2; \dots; X_k Y_k | X'_{[k]}) + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} I(X'_{\mathcal{B}}; X_{\mathcal{B}^c} Y_{\mathcal{B}^c} | X'_{\mathcal{B}^c}). \end{aligned}$$

□

It is shown in [12, Lemma A.1] that  $I_\lambda(X_1; X_2; \dots; X_k)$  is concave in  $p(x_1)$  for a fixed  $p(x_2, x_3, \dots, x_k | x_1)$ . An interactive communication property of  $I_\lambda$  can be also deduced from Lemma 6 of [17].

## A.1 Relation to another definition of multivariate information

Define another multivariate information for the random variables  $X_1, X_2, \dots, X_k$  as

$$J(X_1; X_2; \dots; X_k) = -H(X_1 \dots X_k) + \sum_i H(X_i).$$

Let  $\lambda_{\mathcal{B}} = 0$  if  $|\mathcal{B}| \neq k-1$ , and  $\lambda_{\mathcal{B}} = \frac{1}{k-1}$  otherwise. We have

$$\begin{aligned} I_\lambda(X_1; X_2; \dots; X_k) &= H(X_1 X_2 \dots X_k) - \frac{1}{k-1} \sum_i H(X_{[k]-i} | X_i) \\ &= \frac{1}{k-1} \left( -H(X_1 \dots X_k) + \sum_i H(X_i) \right) \\ &= \frac{1}{k-1} J(X_1; X_2; \dots; X_k). \end{aligned}$$

Next, let  $\Pi = (\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_r)$  be a partition of  $[k]$  into  $r \geq 2$  sets. Let  $\lambda_B = \frac{1}{r-1}$  if  $B = [k] - \mathcal{P}_i$  for some  $i \in [r]$ , and  $\lambda_B = 0$  otherwise. We have

$$I_\lambda(X_1; X_2; \dots; X_k) = \frac{1}{r-1} J(X_{\mathcal{P}_1}; X_{\mathcal{P}_2}; \dots; X_{\mathcal{P}_r}).$$

Consequently,

$$\min_{\lambda} I_\lambda(X_1; X_2; \dots; X_k) \leq \min_{\Pi} \frac{1}{r-1} J(X_{\mathcal{P}_1}; X_{\mathcal{P}_2}; \dots; X_{\mathcal{P}_r}) \quad (39)$$

where the minimum is over all  $r \geq 2$  and over all partitions  $\Pi = (\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_r)$  of  $[k]$  into  $r$  sets.

The following theorem complements Example A.1, showing that (39) holds with equality.

**Theorem 3.** [19, Theorem 4.1] For any fractional partition  $\lambda_B$  and any  $X_1, X_2, \dots, X_k$ , we have

$$I_\lambda(X_1; X_2; \dots; X_k) \geq \min_{\Pi} \frac{1}{r-1} J(X_{\mathcal{P}_1}; X_{\mathcal{P}_2}; \dots; X_{\mathcal{P}_r})$$

where the minimum is over all  $r \geq 2$  and over all partitions  $\Pi = (\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_r)$  of  $[k]$  into  $r$  sets.

## B Application of the Dependence Balance Bound for $I_\lambda$ to MAC with generalized feedback

*Proof of Theorem 2.* Let  $M_i$  be the message of transmitter  $i$ . Then, for any arbitrary  $\mathcal{U} \subset [k]$  and fractional partition  $\lambda$  of the entries in  $\mathcal{U}$ , Lemma 1 implies

$$\begin{aligned} 0 &\leq I_\lambda(M_{i_1} Y_{F_{i_1}}^n; M_{i_2} Y_{F_{i_2}}^n; \dots; M_{i_u} Y_{F_{i_u}}^n | Y^n) - I_\lambda(M_{i_1}; M_{i_2}; \dots; M_{i_u}) \\ &\leq \sum_{j=1}^n I_\lambda(X_{i_1 j} Y_{F_{i_1 j}}; X_{i_2 j} Y_{F_{i_2 j}}; \dots; X_{i_u j} Y_{F_{i_u j}} | Y_j, Y^{j-1}) - \sum_{j=1}^n I_\lambda(X_{i_1 j}; X_{i_2 j}; \dots; X_{i_u j} | Y^{j-1}) \\ &\quad - \sum_{j=1}^n \left( 1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}} \right) I(X_{[k]j}; Y_j Y_{F_{1j}} Y_{F_{2j}} \dots Y_{F_{kj}} | X_{\mathcal{U}j}, Y^{j-1}). \end{aligned}$$

Next, we also have

$$\begin{aligned} n \sum_{i \in \mathcal{S}} R_i &= H(M_{\mathcal{S}} | X_{\mathcal{S}^c}^n) \\ &= I(M_{\mathcal{S}}; Y^n | X_{\mathcal{S}^c}^n) + nk(\epsilon) \\ &= \sum_{j=1}^n I(M_{\mathcal{S}}; Y_j | Y^{j-1}, X_{\mathcal{S}^c}^n) \\ &\leq \sum_{j=1}^n I(M_{\mathcal{S}} Y_{F_{\mathcal{S}}}^{j-1}; Y_j | Y^{j-1}, X_{\mathcal{S}^c}^n) \\ &= \sum_{j=1}^n I(M_{\mathcal{S}} Y_{F_{\mathcal{S}}}^{j-1} X_{\mathcal{S}j}; Y_j | Y^{j-1}, X_{\mathcal{S}^c}^n) \\ &= \sum_{j=1}^n I(X_{\mathcal{S}j}; Y_j | Y^{j-1}, X_{\mathcal{S}^c}^n) \\ &\leq \sum_{j=1}^n I(X_{\mathcal{S}^c}^n, X_{\mathcal{S}j}; Y_j | Y^{j-1}, X_{\mathcal{S}^c}^n) \\ &= \sum_{j=1}^n I(X_{\mathcal{S}j}; Y_j | Y^{j-1}, X_{\mathcal{S}^c}^n). \end{aligned}$$

Letting  $T = (Q, Y^{Q-1})$  for a time sharing variable  $Q$  gives the desired bound.  $\square$

To see the benefit of using a set  $\mathcal{U} \neq [k]$ , consider the following example. Assume that the  $k$ -th terminal's input variable  $X_k$  does not affect the channel outputs significantly; for simplicity, assume that  $X_k$  is constant. However, assume that the feedback that the  $k$ -th terminal receives  $Y_{F_k}$  is  $(X_{[k-1]}, Y_{F_1}, Y_{F_2}, \dots, Y_{F_{k-1}})$ , which is very informative. In this case, the choice of  $\mathcal{U} = [k]$  can lead to weak bounds since  $X_k Y_k$  is very informative (even though  $X_k$  is a constant). On the other hand, consider the choice of  $\mathcal{U} = [k-1]$  which leaves out the  $k$ -th terminal. For this choice, the term  $I(X_{[k]}; Y | X_{\mathcal{U}}, T)$  vanishes since  $X_k$  is a constant. Moreover,  $I_{\lambda}(X_1 Y_{F_1}; X_2 Y_{F_2}; \dots; X_{k-1} Y_{F_{k-1}} | T, Y)$  does not include  $Y_{F_k}$ .

As another example, let  $Y_{F_i} = Y$  for all  $i$ . We show that in this case setting  $\mathcal{U} = [k]$  is optimal. The constraint in this case can be written as

$$\begin{aligned} & \left(1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}}\right) H(X_{\mathcal{U}} | T, Y) + \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}} H(X_{\mathcal{U}-\mathcal{B}} | T, Y) \\ & \geq \left(1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}}\right) H(X_{\mathcal{U}} | T) + \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}} H(X_{\mathcal{U}-\mathcal{B}} | T) + \left(1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}}\right) I(X_{[k]}; Y | X_{\mathcal{U}}, T) \end{aligned}$$

which simplifies as

$$I(X_{[k]}; Y | T) \leq \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}} I(X_{\mathcal{B} \cup \mathcal{U}^c}; Y | T, X_{\mathcal{U}-\mathcal{B}}) \quad (40)$$

for any fractional partition of the indices in  $\mathcal{U}$ . We argue that the bound for  $\mathcal{U} = [k]$  yields the strongest bound. Note that one can convert the fractional partition of  $\mathcal{U}$  to a fractional partition of  $[k]$  as follows: let  $i_1 \in \mathcal{U}$ . For any  $\mathcal{B} \subset \mathcal{U}$ , let  $\lambda'_{\mathcal{B}} = \lambda_{\mathcal{B}}$  if  $i_1 \notin \mathcal{B}$ . For  $i_1 \in \mathcal{B}$ , let  $\lambda'_{\mathcal{B} \cup \mathcal{U}^c} = \lambda_{\mathcal{B}}$ . Finally, assign  $\lambda'_{\mathcal{B}'} = 0$  for all the other sets  $\mathcal{B}' \subset [k]$  that are not of the above two forms. The right hand side of (40) for fractional partition  $\lambda'$  is greater than or equal to that for  $\lambda$ .

For the case of  $\mathcal{U} = [k]$ , we recover the refined dependence balance equations of [10] as we vary  $\lambda$  according to the choices in Example A.1.

Equation (32) can be also used to improve the upper bound given in Theorem 1 of [11] (which uses more than one auxiliary random variables) as follows.

**Theorem 4.** Consider a two-user MAC with generalized feedback of the form  $Y_{F_1} = (Y, \tilde{Y}_{F_1})$  and  $Y_{F_2} = (Y, \tilde{Y}_{F_2})$ . Then any achievable rate pair  $(R_1, R_2)$  satisfies

$$\{(R_1, R_2) : R_1 \leq I(X_1; Y, \tilde{Y}_{F_2} | X_2, T_1, T_2) \quad (41)$$

$$R_2 \leq I(X_2; Y, \tilde{Y}_{F_1} | X_1, T_1, T_2) \quad (42)$$

$$R_1 + R_2 \leq I(X_1, X_2; Y, \tilde{Y}_{F_1}, \tilde{Y}_{F_2} | T_1, T_2) \quad (43)$$

$$R_1 + R_2 \leq I(X_1, X_2; Y | T_1) \quad (44)$$

for some  $p(t_1, t_2, x_1, x_2)$  satisfying

$$I(X_1; X_2 | T_1, T_2) \leq I(X_1; X_2 | Y, \tilde{Y}_{F_1}, \tilde{Y}_{F_2}, T_1, T_2) \quad (45)$$

$$I(X_1; X_2 | T_1) \leq I(X_1, \tilde{Y}_{F_1}; X_2, \tilde{Y}_{F_2} | T_1, Y). \quad (46)$$

Moreover, one can choose  $|\mathcal{T}_1| \leq 5$  and  $|\mathcal{T}_2| \leq |\mathcal{X}_1| |\mathcal{X}_2| + 3$ .

The bound is proved by choosing  $T_1 = (Q, Y^{Q-1})$  and  $T_2 = (\tilde{Y}_{F_1}^{Q-1}, \tilde{Y}_{F_2}^{Q-1})$  where  $Q$  is the time-sharing variable. Equations (41)-(43) and (45) follow from Theorem 1 of [11]. Equations (44) and (46) are new.

## C $k$ -terminal source model problem with silent nodes

This appendix treats the  $k$ -terminal source model problem with silent nodes when  $H(Z | Y_i) = 0$  for  $i = 1, 2, \dots, k$  and where the first  $u$  terminals use the public channel. The paper [5, Theorem 6] showed that the maximum value for  $R_{[k]}$  is

$$H(Y_{[u]} | Z) - \min_{(r_1, r_2, \dots, r_u) \in \mathcal{R}} \sum_i r_i$$

where  $\mathcal{R}$  is the set of tuples  $(r_1, r_2, \dots, r_u)$  such that for any proper set  $\mathcal{B}$  satisfying  $\mathcal{B} \cap [u] \neq \emptyset$  we have

$$\sum_{j \in \mathcal{B} \cap [u]} r_j \geq H(Y_{\mathcal{B} \cap [u]} | Y_{\mathcal{B}^c} Z).$$

If  $\mathcal{B} \cap [u] \neq [u]$ , it is best to include  $[k] - [u]$  in  $\mathcal{B}$ . Thus, in this case, for any  $\mathcal{B} \subsetneq [u]$  we have

$$\sum_{j \in \mathcal{B}} r_j \geq H(Y_{\mathcal{B}} | Y_{[u] - \mathcal{B}} Z).$$

For the case  $\mathcal{B} \cap [u] = [u]$ , we obtain the following bound

$$\sum_{i \in [u]} r_i \geq H(Y_{[u]} | Y_j Z), \quad \forall j \in [k] - [u].$$

By writing the dual of the above linear program, we obtain the expression:

$$R_{[k]} = \min \left[ H(Y_{[u]} | Z) - \sum_{\mathcal{B} \subsetneq [u]} \zeta_{\mathcal{B}} H(Y_{\mathcal{B} \cap [u]} | Y_{[u] - \mathcal{B}} Z) - \sum_{j \in [k] - [u]} \zeta_{\{j\}} H(Y_{[u]} | Y_j Z) \right]$$

where the minimum is over non-negative  $\zeta_{\mathcal{B}} : \mathcal{B} \subsetneq [u]$  and  $\zeta_{\{j\}}$  for  $j > u$  satisfying

$$\sum_{\mathcal{B}: i \in \mathcal{B}} \zeta_{\mathcal{B}} + \sum_{j > u} \zeta_{\{j\}} = 1, \quad \forall i \in [u].$$

To obtain this bound, we choose

$$\begin{aligned} \omega_{[u]} &= \sum_{\mathcal{B}: i \in \mathcal{B}} \zeta_{\mathcal{B}} \\ \omega_{[u] \cup \{j\}} &= \zeta_{\{j\}}, \quad \forall j \in [k] - [u] \\ \omega_{\mathcal{U}} &= 0 \text{ otherwise.} \end{aligned}$$

For the set  $[u]$ , we define

$$\lambda_{\mathcal{B}}^{[u]} = \frac{\zeta_{\mathcal{B}}}{1 - \sum_{j > u} \zeta_{\{j\}}}, \quad \forall \mathcal{B} \subset [u].$$

For the set  $[u] \cup \{j\}$  for  $j > u$ , we define  $\lambda_{\mathcal{B}}^{[u] \cup \{j\}} = 1$  if  $\mathcal{B} = [u]$  or  $\mathcal{B} = \{j\}$  and  $\lambda_{\mathcal{B}}^{[u] \cup \{j\}} = 0$  for all the other sets  $\mathcal{B}$ . This choice of  $\omega_{\mathcal{U}}$  and  $\lambda_{\mathcal{B}}^{\mathcal{U}}$  yields the desired bound if we use the auxiliary receiver  $T = Z$  for the main channel and the parallel channel. Note that the parallel channel is  $Y_1 = Y_2 = \dots = Y_k = Z = X_{[u]}$  with  $X_{u+1}, \dots, X_k$  being constants. The proof of  $V_{\omega, \lambda}(q_1(t, y_{[k]}, z | x_{[k]})) \leq 0$  for the parallel channel is similar to the one discussed in Section 4.1.4; the only extra step is to show that

$$- \sum_{\mathcal{U}} \omega_{\mathcal{U}} \left( 1 - \sum_{\mathcal{B} \subsetneq \mathcal{U}} \lambda_{\mathcal{B}}^{\mathcal{U}} \right) I(X_{[k]}; Y_{\mathcal{U}}, Z | X_{\mathcal{U}}) = 0. \quad (47)$$

Note that for the sets  $\mathcal{U}$  where  $\omega_{\mathcal{U}} > 0$  we have  $[u] \subset \mathcal{U}$ . Then, the term  $I(X_{[k]}; Y_{\mathcal{U}}, Z | X_{\mathcal{U}})$  vanishes since  $X_j$  is a constant for  $j \notin [u]$ . Therefore, (47) holds.

## D Wiretap channel with rate-limited secure feedback

Consider  $k = 2$  and suppose that in the main channel  $X_2$  and  $Y_1$  are constants so we obtain a wiretap channel  $p(y_2, z | x_1)$ . For the parallel channel, we assume a secure rate-limited feedback link as in [7]. We can model this by a parallel channel where  $Y_2$  and  $Z$  are constant while  $Y_1 = X_2$  with the desired feedback rate  $R_f$ . We also set the parallel channel-use rate  $\alpha_1 = 1$ . The main result of [7] is the following upper bound on the rate of secure and reliable communication from the first terminal to the second terminal:

$$R \leq \max_{p(x_1)} \min [I(X_1; Y_2), R_f + I(X_1; Y_2 | Z)].$$

The authors in [7] do not consider the secret key rate that can be shared between the two terminals; rather the rate of private communication from the first terminal to the second terminal. Only the term

$R_f + I(X_1; Y_2|Z)$  constitutes an upper bound on the secret key rate that can be shared between the two terminals. To obtain the latter bound from our bound in Corollary 1, choose  $\lambda_{\{1\}} = \lambda_{\{2\}} = 1$ . We also use the auxiliary receiver  $T = Z$ . For the main channel, we can simplify  $V_{\omega, \lambda}(p(t, y_1, y_2, z|x_1, x_2))$  because  $Y_1$  and  $X_2$  are constants:

$$V_{\omega, \lambda}(p(t, y_1, y_2, z|x_1, x_2)) = I(X_1; Y_2|Z).$$

For the parallel channel set  $Y_1 = X_2$  and choose  $X_1$  and  $Z$  as constants. We can use the auxiliary receiver  $T = Z$  to obtain

$$V_{\omega, \lambda}(q_1(t, y_1, y_2, z|x_1, x_2)) = \max_{p(x_2)} I(Y_1; X_2) \leq R_f.$$

These results yield the upper bound  $R_f + I(X_1; Y_2|Z)$ .