

# Quantum Wiretap Channel Coding Assisted by Noisy Correlation

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**Abstract**—We consider the private classical capacity of a quantum wiretap channel, where the users (sender Alice, receiver Bob, and eavesdropper Eve) have access to the resource of a shared quantum state, additionally to their channel inputs and outputs. An extreme case is maximal entanglement or a secret key between Alice and Bob, both of which would allow for one-time padding the message. But here both the wiretap channel and the shared state are general. In the other extreme case that the state is trivial, we recover the wiretap channel and its private capacity [N. Cai, A. Winter and R. W. Yeung, *Probl. Inform. Transm.* 40(4):318-336, 2004]. We show how to use the given resource state to build a code for secret classical communication. Our main result is a lower bound on the assisted private capacity, which asymptotically meets the multi-letter converse and which encompasses all sorts of previous results as special cases.

**Index Terms**—Quantum information; communication via quantum channels; wiretap channels

## I. INTRODUCTION

Entanglement shared between sender and receiver of a transmission is a useful resource that generically increases channel capacity. In this scenario, the two parties may be given some number of quantum bits jointly prepared in a fixed superposition. The advantages of a shared quantum entanglement resource prior to communication over a noisy quantum channel has been considered extensively in [6] and [4], [5], where the entanglement-assisted classical capacity theorem has been derived; see also [29]. These works showed how to increase the classical capacity of quantum channels by the assistance of unlimited pure entanglement. Since then, more recent works have studied extensions of the use of shared entanglement as an assisting quantum resource: entanglement-assisted communication over quantum multiple-access channels [26]; entanglement-assisted communication over compound and arbitrarily varying quantum channels [7]; entanglement-assisted communication over quantum broadcast channels [25]. For these results, it is essential that the shared state be maximally entangled. In a somewhat dual setting, in [8], [9], [19], [20], [33], [22], Alice and Bob are connected by a noiseless quantum channel, but instead share a general mixed quantum state (decoupled from Eve). The case of a noisy quantum channel has been investigated in [3]. These papers showed that sufficiently entangled states, or any entanglement

in conjunction with suitable channels, are useful resources for communication.

Secure communication over a classical channel with an eavesdropper was first introduced by Wyner [34]. The secrecy capacity for quantum wiretap channels has been determined in [16] and [10]. Chen *et al.* [12] demonstrated that correlation between Alice and Bob is able to help secure message transmission over a classical wiretap channel. Inspired by these results, we analyze secure communication when the channel users (Alice, Bob and Eve) share a general correlated state, which is “given by nature”, and assumed to be known to all parties. In this work, we show a way to increase the secure capacity of quantum channels by the assistance of such noisy correlation. Our proof works by considering the shared resource as a component of the channel. Technically, we employ quantum wiretap channel codes, and incorporate the correlated resource via a variant of Gel’fand-Pinsker coding as in [13] and [2]. This has been used similarly in secure “writing on dirty paper” codes [14].

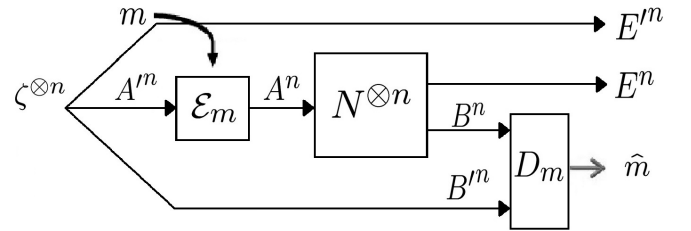


Fig. 1. Communication diagram of the wiretap coding problem over  $\mathcal{N}$  with assistance by a tripartite resource state shared by sender (Alice), receiver (Bob) and eavesdropper (Eve). Unlike the plain wiretap channel, here each message is encoded not into a state, but rather a modulation  $\mathcal{E}_m : A \rightarrow A$ .

We would like to mention another coding protocol, which has appeared previously in the literature, showing a similar result [31]: this paper determined the entanglement generation capacity of a quantum channel with shared quantum states as a resource. The proof of the achievability part uses a similar approach to the dirty paper codes, although our construction cannot be reduced to theirs, in the same way as secret key

generation is more general than entanglement generation [21]. See also [16] for the first description of codes achieving the entanglement generation capacity based on wiretap codes.

In the next section we introduce the communication scenario and the assisted private capacity, followed by a section recalling certain basic notions. After that we state and prove our main result, followed by a discussion of special cases of interest.

## II. COMMUNICATION SCENARIO

Before starting, we review some basic notions from quantum information theory. Let  $\rho_1$  and  $\rho_2$  be Hermitian operators on a finite-dimensional complex Hilbert space  $A$ . We say  $\rho_1 \geq \rho_2$  and  $\rho_2 \leq \rho_1$  if  $\rho_1 - \rho_2$  is positive semidefinite. We denote the set of density operators (states) on  $A$  by  $\mathcal{S}(A) := \{\rho \in \mathcal{L}(A) : \rho \geq 0, \text{Tr}(\rho) = 1\}$ , where  $\mathcal{L}(A)$  is the set of linear operators on  $A$ . For finite-dimensional complex Hilbert spaces  $A$  and  $B$ , a *quantum channel* is a linear, completely positive and trace preserving (cptp) map  $\mathcal{N} : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$ , that acts a  $\mathcal{L}(A) \ni \rho \mapsto \mathcal{N}(\rho) \in \mathcal{L}(B)$ , which accepts input quantum states in  $\mathcal{S}(A)$  and produces output quantum states in  $\mathcal{S}(B)$ . We shall often simplify the notation of a quantum channel as  $\mathcal{N} : A \rightarrow B$ . Likewise, we will often suppress the tensor product sign in  $BE = B \otimes E$  and  $A^n = A^{\otimes n}$ , if there is no danger of confusion.

**Definition 1.** A quantum wiretap channel is a cptp map  $\mathcal{N} : A \rightarrow B \otimes E$ , with finite-dimensional quantum systems  $A$ ,  $B$  and  $E$ , representing Alice's input and Bob's and Eve's outputs, respectively.

Note that in [10], a wiretap channel was more generally a pair of cptp maps, one from  $A$  to  $B$  and one from  $A$  to  $E$ . We can reproduce that notion by letting  $\mathcal{N}_B = \text{Tr}_E \circ \mathcal{N}$  and  $\mathcal{N}_E = \text{Tr}_B \circ \mathcal{N}$ , but not every pair of cptp maps arises in this way. However, the generality offered by [10] comes in handy later on.

**Definition 2.** A quantum wiretap channel assisted by a correlation is a pair  $(\mathcal{N}, \zeta)$  consisting of a wiretap channel  $\mathcal{N} : A \rightarrow B \otimes E$  and a quantum state  $\zeta \in \mathcal{S}(A' \otimes B' \otimes E')$ , with finite-dimensional quantum systems  $A'$ ,  $B'$  and  $E'$  in the possession of Alice, Bob and Eve, respectively, before the transmission.

**Definition 3.** An  $(n, \lambda, \mu)$ -wiretap code for  $(\mathcal{N}, \zeta)$  is a collection  $\{(\mathcal{E}_m, D_m) : m \in [M] = \{1, \dots, M\}\}$  of cptp maps  $\mathcal{E}_m : A'^n \rightarrow A^n$  (the "modulations") and operators  $D_m \geq 0$  on  $B^n B'^n$ ,  $\sum_{m=1}^M D_m \leq \mathbf{1}$ , such that there exists a state  $\sigma$  on  $E^n E'^n$  with

$$\frac{1}{M} \sum_m \text{Tr}((\mathcal{N}^{\otimes n} \circ \mathcal{E}_m) \zeta^{\otimes n} \cdot (D_m \otimes \mathbf{1}_{E^n E'^n})) \geq 1 - \lambda, \quad (1)$$

$$\frac{1}{M} \sum_m \left\| \text{Tr}_{B^n B'^n}(\mathcal{N}^{\otimes n} \circ \mathcal{E}_m) \zeta^{\otimes n} - \sigma^{E^n E'^n} \right\|_1 \leq \mu. \quad (2)$$

See Fig. 1 for the communication diagram.

The rate of the wiretap code is  $\frac{1}{n} \log M$ , and the largest number  $M$  of messages of an  $(n, \lambda, \mu)$ -wiretap code is denoted

$M(n, \lambda, \mu)$ . This allows us to define the (weak) *assisted private capacity* as

$$P(\mathcal{N}, \zeta) := \inf_{\lambda, \mu > 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log M(n, \lambda, \mu), \quad (3)$$

which is the main objective for the rest of the paper.

## III. PRELIMINARIES

For a finite set  $\mathcal{X}$ , we denote the set of probability distributions on  $\mathcal{X}$  by  $\mathbb{P}(\mathcal{X})$ . For a discrete random variable  $X$  on a finite set  $\mathcal{X}$  and a discrete random variable  $Y$  on a finite set  $\mathcal{Y}$  we denote the Shannon entropy of  $X$  by  $H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x)$  and the mutual information between  $X$  and  $Y$  by

$$I(X; Y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log \frac{p(x, y)}{p(x)p(y)},$$

Here  $p(x, y)$  is the joint probability distribution function of  $X$  and  $Y$ , and  $p(x)$  and  $p(y)$  are the marginal probability distribution functions of  $X$  and  $Y$  respectively. Throughout the paper the logarithm base is 2.

For quantum states acting on  $A$  or composite systems  $A \otimes B$ , these concepts generalise. For instance for  $\rho^{AB} \in \mathcal{S}(A \otimes B)$  we have the marginals (reduced states)  $\rho^A = \text{Tr}_B \rho$  and  $\rho^B = \text{Tr}_A \rho$  given by the partial trace. To a finite set  $\mathcal{X}$ , we associate the Hilbert space  $X$  with orthonormal basis  $\{|x\rangle : x \in \mathcal{X}\}$ , such that the possible distributions  $p(x)$  on  $\mathcal{X}$  correspond to diagonal density matrices  $\rho = \sum_x p(x) |x\rangle\langle x|$ .

An ensemble  $\{p(x), \rho_x \in \mathcal{S}(A)\}_{x \in \mathcal{X}}$  of states on  $A$  is faithfully represented by the *classical-quantum (cq-)state*

$$\gamma = \sum_{x \in \mathcal{X}} p(x) |x\rangle\langle x|^X \otimes \rho_x^A \in \mathcal{S}(X \otimes A).$$

The entropy for quantum states  $\rho \in \mathcal{S}(A)$  is the von Neumann entropy,  $S(\rho^A) = S(A)_\rho = -\text{Tr} \rho \log \rho$ , and by analogy with the Shannon entropy we have also the conditional entropy  $S(A|B)_\rho = S(AB)_\rho - S(B)_\rho$  and the mutual information  $I(A : B)_\rho = S(A)_\rho + S(B)_\rho - S(AB)_\rho$ .

A special class of quantum channels are *classical-quantum (cq-)channels*. These are given by  $W : X \rightarrow B$ , with  $W(|x\rangle\langle x'|) = \delta_{xx'} W_x$  for states  $W_x \in \mathcal{S}(B)$ ,  $x \in \mathcal{X}$ . As the channel is entirely described by this family of states, we shall identify a cq-channel with the map  $W : \mathcal{X} \rightarrow \mathcal{S}(B)$ , mapping  $x \mapsto W_x$ .

We shall need two commonly used measures of distance and similarity of quantum states. The fidelity of two quantum states  $\rho$  and  $\sigma$  is defined as  $F(\rho, \sigma) = \|\sqrt{\rho} \sqrt{\sigma}\|_1 = \text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}$ , where  $\|X\|_1 = \text{Tr} \sqrt{X X^\dagger}$  is the trace norm. Then, according to Fuchs and van de Graff,

$$1 - F(\rho, \sigma) \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)^2}.$$

We end this section by recalling the formula for the private capacity of a quantum wiretap channel [10], [16] when no

correlation is present, i.e.  $\zeta^{A'B'E'} = |0\rangle\langle 0|^{A'} \otimes |0\rangle\langle 0|^{E'} \otimes |0\rangle\langle 0|^{E'} =: \emptyset$ :

$$P(\mathcal{N}) = P(\mathcal{N}, \emptyset) = \sup_n \frac{1}{n} \sup_{q(u), \rho_u} (I(U : B^n)_\gamma - I(U : E^n)_\gamma), \quad (4)$$

where the inner maximisation is over ensembles of states  $\rho_u \in \mathcal{S}(A^n)$  with probabilities  $q(u)$ ,  $u \in \mathcal{U}$ , and

$$\gamma^{UB^n E^n} = \sum_{u \in \mathcal{U}} q(u) |u\rangle\langle u|^U \otimes [\mathcal{N}^{\otimes n}(\rho_u)]^{B^n E^n}.$$

#### IV. MAIN RESULT

Looking again at the definitions and the communication diagram in Fig. 1, we note that not just a code but any given modulations  $\mathcal{E}_u$  ( $u \in \mathcal{U}$ ) give rise to a cq-wiretap channel  $\mathcal{M} : \mathcal{U} \rightarrow (BB')^n(EE')^n$ , mapping  $u \in \mathcal{U}$  to  $\mathcal{M}(u) = (\mathcal{N}^{\otimes n} \circ \mathcal{E}_u)\zeta^{\otimes n}$ . We can thus apply the coding theorem and weak converse from [10] to get the following expression for the assisted private capacity:

$$P(\mathcal{N}, \zeta) = \sup_n \frac{1}{n} \sup_{\{q(u), \mathcal{E}_u\}} (I(U : B^n B'^n)_\gamma - I(U : E^n E'^n)_\gamma), \quad (5)$$

with respect to the cq-state

$$\gamma^{U(BB')^n(EE')^n} = \sum_u q(u) |u\rangle\langle u|^U \otimes (\mathcal{N}^{\otimes n} \circ \mathcal{E}_u)\zeta^{\otimes n},$$

where the inner supremum is over all alphabets  $\mathcal{U}$ , probability distribution  $q$  on  $\mathcal{U}$  and modulations  $\mathcal{E}_u$ . Our goal will be to find a better achievability bound, which thus speeds up the convergence of Eq. (5).

For the subsequent analysis, we observe that w.l.o.g. the reduced state  $\zeta^{A'}$  has full rank (i.e. equal to the Hilbert space dimension  $|A'|$ ), because otherwise we can shrink  $A'$  to the support of  $\zeta^{A'}$  and restrict the modulation maps  $\mathcal{E}_u$  accordingly.

**Remark 4.** The key to our coding theorem is the following equivalent description of the code. Consider a Hilbert space  $A'' \simeq A'$  and a pure state vector  $|\phi_0\rangle \in A' \otimes A''$  such that  $\phi_0^{A'} = \phi_0^{A''} = \zeta^{A'}$ . By the generalised Choi theorem, there is a (unique) cptp map  $\mathcal{Z} : A' \rightarrow B' \otimes E'$  such that  $\zeta^{A'B'E'} = (\text{id}_{A'} \otimes \mathcal{Z}^{A'' \rightarrow B'E'})\phi_0$ . Likewise, the generalised Choi states  $\eta_u = (\mathcal{E}_u \otimes \text{id}_{A''})\phi_0^{\otimes n}$  of the modulation maps have the property  $\text{Tr}_{A^n} \eta_u = (\zeta^{A'})^{\otimes n}$  and by these two equations determine  $\mathcal{E}_u$  uniquely as cptp maps.

Now we make the elementary but crucial observation that

$$\begin{aligned} (\mathcal{E}_u \otimes \text{id}_{B'^n E'^n})\zeta^{\otimes n} &= (\text{id}_{A^n} \otimes \mathcal{Z}^{\otimes n})\eta_u, \\ (\mathcal{N}^{\otimes n} \circ \mathcal{E}_u \otimes \text{id}_{B'^n E'^n})\zeta^{\otimes n} &= (\mathcal{N} \otimes \mathcal{Z})^{\otimes n}\eta_u, \end{aligned} \quad (6)$$

which means that our assisted wiretap code for  $(\mathcal{N}, \zeta)$  turns out to be equivalent to a regular wiretap code for  $\mathcal{N} \otimes \mathcal{Z} : AA' \rightarrow BB' \otimes EE'$ , albeit with the restriction  $\text{Tr}_{A^n} \eta_m = (\zeta^{A'})^{\otimes n}$  for all messages  $m$  (see Fig. 2), and similarly for arbitrary modulations  $\mathcal{E}_u$ .

We can now relax the condition  $\text{Tr}_A \eta_u = \zeta^{A'}$  for all  $u \in \mathcal{U}$  to an average one,  $\sum_u q(u) \text{Tr}_A \eta_u = \zeta^{A'}$ , to obtain the

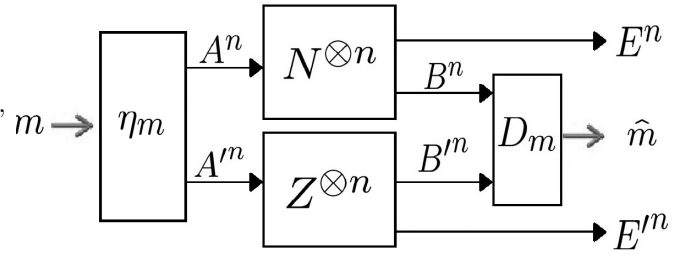


Fig. 2. Reformulation of the assisted wiretap code for  $(\mathcal{N}, \zeta)$  in terms of the tensor product wiretap channel  $\mathcal{N} \otimes \mathcal{Z}$  with a restriction on the marginals of the signal states  $\eta_m$  on  $A'^n$ .

following coding theorem. This is where the Gel'fand-Pinsker and “dirty paper” coding idea is used [17], [14].

**Theorem 5.** Let  $(\mathcal{N}, \zeta)$  be an assisted quantum wiretap channel with the wiretap channel  $\mathcal{Z} : A' \rightarrow B'E'$  defined as above. Assume furthermore a cq-channel  $\mathcal{C} : \mathcal{U} \rightarrow A \otimes A'$  and probabilities  $q(u)$  such that  $\sum_u q(u) \text{Tr}_A \mathcal{C}(u) = \zeta^{A'}$ . Then,

$$P(\mathcal{N}, \zeta) \geq I(U : BB')_\gamma - \max(I(U : EE')_\gamma, I(U : A')_\beta), \quad (7)$$

where

$$\begin{aligned} \beta^{UAA'} &= \sum_u q(u) |u\rangle\langle u|^U \otimes \mathcal{C}(u)^{AA'}, \\ \gamma^{UBB'EE'} &= \sum_u q(u) |u\rangle\langle u|^U \otimes ((\mathcal{N} \otimes \mathcal{Z})\mathcal{C}(u))^{BB'EE'}. \end{aligned} \quad (8)$$

Before proving this, let us remark that it includes the achievability part of Eq. (3), since we may choose  $\mathcal{C}(u) = \eta_u = (\mathcal{E}_u \otimes \text{id}_{A''})\phi_0$  as in Remark 4, which results in  $I(U : A')_\beta = 0$ .

*Proof.* We will construct  $\eta_m$  on block length  $n$  as in Remark 4 by employing the wiretap coding strategy of Devetak [16] for the cq-wiretap channel  $\mathcal{M} = (\mathcal{N} \otimes \mathcal{Z}) \circ \mathcal{C} : \mathcal{U} \rightarrow BB' \otimes EE'$ , and the additional eavesdropper channel  $\mathcal{U} \ni u \mapsto \text{Tr}_A \mathcal{C}(u)$ . This amounts to choosing code words  $u^n \in \mathcal{U}^n$  independently according to the distribution  $q^{\otimes n}$ , and binning them randomly so as to average over bins.

Concretely, let  $\mathcal{U}^n \ni u_{(m,s)} \sim q^{\otimes n}$  be sampled i.i.d. for  $m \in [M]$  and  $s \in [S]$ , where  $S = 2^{n \max(I(U : EE'), I(U : A')) + n\epsilon}$  and  $MS = 2^{nI(U : BB') - n\epsilon}$  for an arbitrary  $\epsilon > 0$ . (Devetak restricts the sampling to the typical sequences of  $q^{\otimes n}$ , but by typicality the difference to the present prescription is small.)

Devetak’s analysis [16] implies that for sufficiently large  $n$ , with high probability Bob’s output states  $\text{Tr}_{E^n E'^n} \mathcal{M}^{\otimes n}(u_{(m,s)})$  are distinguishable reliably by a decoding POVM, say with error probability  $\epsilon$ . Furthermore, the averaged input states

$$\tilde{\eta}_m := \frac{1}{S} \sum_{s=1}^S \mathcal{C}^n(u_{(m,s)}) \in \mathcal{S}(A^n \otimes A'^n)$$

with high probability have the properties

$$\frac{1}{M} \sum_{m=1}^M \left\| \tilde{\eta}_m^{A^n} - (\zeta^{A'})^{\otimes n} \right\|_1 \leq \epsilon, \quad (10)$$

$$\frac{1}{M} \sum_{m=1}^M \left\| (\text{Tr}_{BB'} \circ (\mathcal{N} \otimes \mathcal{Z}))^{\otimes n} \tilde{\eta}_m - (\gamma^{EE'})^{\otimes n} \right\|_1 \leq \epsilon, \quad (11)$$

where  $\gamma$  is as in Eq. (9). The first comes from the Holevo-Schumacher-Westmoreland theorem for classical information transmission over a quantum channel, the second from the matrix tail bounds and the matrix covering lemma in [1].

By the Fuchs-van-de-Graaf relations between trace distance and fidelity, and Uhlmann's theorem (cf. [23], [30]), we can find  $\eta_m \in \mathcal{S}(A^n \otimes A'^n)$  such that  $\eta_m^{A'^n} = (\zeta^{A'})^{\otimes n}$  for all  $m$  and

$$\frac{1}{M} \sum_{m=1}^M \|\tilde{\eta}_m - \eta_m\|_1 \leq 4\sqrt{\epsilon}, \quad (12)$$

and furthermore

$$\frac{1}{M} \sum_{m=1}^M \left\| (\text{Tr}_{BB'} \circ (\mathcal{N} \otimes \mathcal{Z}))^{\otimes n} \eta_m - (\gamma^{EE'})^{\otimes n} \right\|_1 \leq \epsilon + 4\sqrt{\epsilon}. \quad (13)$$

According to Remark 4, this is equivalent to an assisted wiretap code for  $(\mathcal{N}, \zeta)$  with block length  $n$ ,  $\lambda = \mu = \epsilon + 4\sqrt{\epsilon}$ , concluding the proof.  $\square$

## V. DISCUSSION

We have presented a coding strategy for the quantum wiretap channel assisted by a general quantum correlation, which shows a systematic improvement of the private communication rate over the unassisted private capacity of the channel due to the exploitation of the correlation. Our Theorem 5 also directly generalises the main result of [12], since classical channels are a special case of cptp maps and a classical correlation between random variables  $X$ ,  $Y$  and  $Z$  is naturally represented by the diagonal state  $\zeta^{A'B'E'} = \sum_{xyz} P_{XYZ}(xyz) |x\rangle\langle x|^{A'} \otimes |y\rangle\langle y|^{B'} \otimes |z\rangle\langle z|^{E'}$ . As the proposed code uses a method derived from the (quantum) Gel'fand-Pinsker wiretap channel with side information, it is also always at least as good as, or superior to, the "trivial" incorporation of the correlation resource into an augmented wiretap channel. However, due to the general need for regularisation, both our main result (Theorem 5) and the "trivial" achievable rate [Eq. (5)] lead to multi-letter formulas for the same quantity  $P(\mathcal{N}, \zeta)$ .

Of course, there are extreme cases of useless correlation: for instance any state  $\zeta$  such that Alice and Bob are uncorrelated,  $\zeta^{A'B'} = \zeta^{A'} \otimes \zeta^{B'}$ , clearly leads to  $P(\mathcal{N}, \zeta) = P(\mathcal{N})$ , irrespective of the wiretap channel  $\mathcal{N}$ . On the other hand, any  $\zeta^{A'B'} \otimes |0\rangle\langle 0|^{E'}$  that is not a tensor product between Alice and Bob offers an advantage for some channel, indeed we can take the quantum broadcast channel  $\mathcal{B} : A \rightarrow B \otimes E$  with qubits  $A$ ,  $B$  and  $E$ , acting as  $\mathcal{B}(\rho) = B\rho B^\dagger$ ,  $B|x\rangle = |x\rangle^B |x\rangle^E$  for  $x = 0, 1$ . Because  $\mathcal{B}$  is both degradable and anti-degradable,  $P(\mathcal{B}) = 0$ . However,  $P(\mathcal{B}, \zeta) > 0$ , since  $\mathcal{B}$  can be used to communicate classically and thus Alice and Bob can

extract shared randomness that is automatically a secret key from  $\zeta^{A'B'}$  [15], which then can be used to one-time-pad a subsequent classical message.

As indicated in the introduction, various communication problems considered in the past are special cases of our model, and it is interesting to see to which extent our main theorem reproduces known results or sheds new light on them. For instance, the channel without wiretapper,  $\mathcal{N}^{A \rightarrow BE} = \mathcal{N}^{A \rightarrow B} \otimes |0\rangle\langle 0|^{E'}$ , gives rise to all sorts of problems of classical communication assisted by entanglement. For general point-to-point channel  $\mathcal{N} : A \rightarrow B$  but arbitrary pure state  $\zeta^{A'B'} \otimes |0\rangle\langle 0|^{E'}$ , we recover the entanglement-assisted classical capacity  $C_E(\mathcal{N})$  [4], [5], even though some additional work is still required to see how Theorem 5 and Eq. (3) give rise to the familiar maximum quantum mutual information formula from those papers.

If instead our correlation resource is a mixed state, albeit tensor product with Eve, i.e.  $\zeta^{A'B'} \otimes |0\rangle\langle 0|^{E'}$ , we get the classical capacity  $C(\mathcal{N}, \zeta)$  of  $\mathcal{N} : A \rightarrow B$  assisted by  $\zeta^{A'B'}$  considered in [3]. In that paper it was observed that a separable  $\zeta^{A'B'}$  cannot increase the classical capacity of any channel,  $C(\mathcal{N}, \zeta) = C(\mathcal{N})$ , and the hypothesis was advanced that for every entangled  $\zeta$  there might be a channel such that  $C(\mathcal{N}, \zeta) > C(\mathcal{N})$ . Noticing that in our present setting, and due to the triviality of Eve in both the channel and the resource state, the classical capacities (assisted and unassisted) are equal to private capacities, suggests a broader interpretation of the question from [3]: namely, for any state  $\rho^{A'B'}$  and a purification  $\zeta^{A'B'E'}$  of it, is it true that  $\rho^{A'B'}$  is entangled if and only if there exists a wiretap channel  $\mathcal{N} : A \rightarrow BE$  such that  $P(\mathcal{N}, \zeta) > P(\mathcal{N})$ ?

The special case of the ideal channel  $\mathcal{N} = \text{id}_A : A \rightarrow B = A$  merits a separate mention, too. Its assistance by a general product state  $\zeta^{A'B'} \otimes |0\rangle\langle 0|^{E'}$  was considered in multiple works [8], [9], [19], [20], [33], [22], and it was found that  $C(\text{id}_A, \zeta) \geq \log |A| + I(A)B'_\omega$ , where  $\omega^{AB'} = (\Omega \otimes \text{id}_{B'})\zeta$  with a cptp map  $\Omega : A' \rightarrow A$ . The optimisation of the coherent information  $I(A)B'_\omega$  over all systems  $A$  and channels  $\Omega$  leads to the *dense coding advantage*  $\Delta(A')B'_\zeta$  of the resource state:

$$\Delta(A')B'_\zeta := \max_{\Omega \text{ cptp}} I(A)B'_\omega \text{ s.t. } \omega^{AB'} = (\Omega \otimes \text{id}_{B'})\zeta.$$

For sufficiently large  $A$  system, it follows that  $C(\text{id}_A, \zeta) = \log |A| + \Delta^{(\infty)}(A')B'_\zeta$ , where  $\Delta^{(\infty)}$  is the regularised dense coding advantage, cf. [22]:

$$\Delta^{(\infty)}(A')B'_\zeta = \sup_n \frac{1}{n} \Delta(A'^n)B'^n_\zeta.$$

Now this achievability bound and expression for the assisted capacity correspond exactly to Eq. (3), and interestingly it provides an instance where Theorem 5 is better, at least on the single-letter level (naturally, in the regularisation both expression result in the same number). Namely, in [22] a certain duality relation was observed between the dense coding

advantage and the so-called *entanglement of purification*  $E_P$  [28], which extends to the regularisations of the two quantities:

$$E_P(C : D)_\rho := \min_{\mathcal{T}: E \rightarrow F} S(BF)_\omega,$$

for a bipartite state  $\rho^{CD}$  with purification  $\psi^{CDE}$ , where the minimisation is over ctp maps  $\mathcal{T} : E \rightarrow F$  and  $\omega^{BF} = (\text{id}_B \otimes \mathcal{T})\psi^{BE}$ . Namely, for any purification  $\psi^{A'B'C'}$  of  $\zeta^{A'B'}$ ,

$$\Delta(A'B') + E_P(C' : B') = S(B').$$

It remains unknown whether  $\Delta = \Delta^{(\infty)}$  and  $E_P = E_P^{(\infty)}$ , but in [11] numerical evidence to the contrary was provided for the entanglement of purification. Showing  $E_P(\rho) > E_P^{(\infty)}(\rho)$  for a certain state in [11] required calculating the l.h.s. numerically, and finding a new upper bound for the r.h.s. via an asymptotic protocol using the covering lemma. Concretely, it was shown for an ensemble  $\{q(u), \rho_u^{CD}\}$  with  $\rho = \sum_u q(u)\rho_u$  that

$$E_P^{(\infty)}(C : D)_\rho \leq \sum_u q(u)E_P(C : D)_{\rho_u} + I(U : CD)_\gamma,$$

where  $\gamma^{UCD} = \sum_u q(u) |u\rangle\langle u|^U \otimes \rho_u^{CD}$  is the cq-state of the ensemble. Using the duality relation from [22] to translate this protocol to  $\Delta^{(\infty)}$ , it corresponds to the achievable rate from Theorem 5 in the case  $\mathcal{N} = \text{id}_A$  and  $\zeta^{A'B'}$ .

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#### REFERENCES

- [1] R. Ahlswede and A. Winter, Strong converse for identification via quantum channels, *IEEE Trans. Inf. Theory*, vol. 48, no. 3, pp. 569-579, 2002. Addendum: *IEEE Trans. Inf. Theory*, vol. 49, no. 1, p. 346, 2003.
- [2] A. Anshu, M. Hayashi, and N. A. Warsi, Secure communication over fully quantum Gel'fand-Pinsker wiretap channel, *IEEE Trans. Inf. Theory*, vol. 66, no. 9, pp. 5548-5566, 2020.
- [3] S. Bäuml, A. Winter, and D. Yang, Every entangled state provides an advantage in classical communication, *J. Math. Phys.*, vol. 60, no. 7, 072201, 2019.
- [4] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, Entanglement-assisted classical capacity of noisy quantum channels, *Phys. Rev. Lett.*, vol. 83, no. 15, pp. 3081-3084, 1999.
- [5] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem, *IEEE Trans. Inf. Theory*, vol. 48, no. 10, pp. 2637-2655, 2002.
- [6] C. H. Bennett and S. J. Wiesner, Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states, *Phys. Rev. Lett.*, vol. 69, no. 20, pp. 2881-2884, 1992.

- [7] H. Boche, G. Janßen, and S. Kaltenstadler, Entanglement-assisted classical capacities of compound and arbitrarily varying quantum channels, *Quantum Inf. Proc.*, vol. 16, 88, 2017.
- [8] S. Bose, M. B. Plenio, V. Vedral, Mixed state dense coding and its relation to entanglement measures, *J. Mod. Optics*, vol. 47, no. 2-3, pp. 271-310, 2000.
- [9] G. Bowen, Classical information capacity of superdense coding, *Phys. Rev. A*, vol. 63, no. 2, 022302, 2001.
- [10] N. Cai, A. Winter, and R. W. Yeung, Quantum privacy and quantum wiretap channels, *Probl. Inform. Transm.*, vol. 40, no. 4, pp. 318-336, 2004.
- [11] J. Chen and A. Winter, Non-Additivity of the Entanglement of Purification (Beyond Reasonable Doubt), arxiv[quant-ph]:1206.1307, 2012.
- [12] Y. Chen, N. Cai and A. Sezgin, Wiretap Channel with Correlated Sources, *Proc. 2014 IEEE Int. Conf. Cloud Engineering (IC2E)*, Boston MA, pp. 472-477, 2014.
- [13] Y. Chen and A. J. H. Vinck, Wiretap channel with side information, *IEEE Trans. Inf. Theory*, vol. 54, no. 1, pp. 395-402, 2008.
- [14] M. Costa, Writing on dirty paper, *IEEE Trans. Inf. Theory*, vol. 29, no. 3, pp. 439-441, 1983.
- [15] I. Devetak and A. Winter, Distilling common randomness from bipartite quantum states, *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3183-3196, 2004.
- [16] I. Devetak, The private classical information capacity and quantum information capacity of a quantum channel, *IEEE Trans. Inf. Theory*, vol. 51, no. 1, pp. 44-55, 2005.
- [17] S. I. Gel'fand and M. S. Pinsker, Coding for channel with random parameters, *Problemy Pered. Inform. (Probl. Inform. Transm.)*, vol. 9, no. 1, pp. 19-31, 1980.
- [18] M. Hayashi, *Quantum Information Theory*, Springer Verlag, Berlin Heidelberg, 2017.
- [19] T. Hiroshima, Optimal dense coding with mixed state entanglement, *J. Phys. A: Math. Gen.*, vol. 34, no. 35, pp. 6907-6912, 2001.
- [20] M. Horodecki, P. Horodecki, R. Horodecki, D. W. Leung, and B. M. Terhal, Classical capacity of a noiseless quantum channel assisted by noisy entanglement, *Quantum Inf. Comput.*, vol. 1, no. 3, pp. 70-78, 2001.
- [21] K. Horodecki, M. Horodecki, P. Horodecki, and J. Oppenheim, Secure Key from Bound Entanglement, *Phys. Rev. Lett.*, vol. 94, no. 16, 160502, 2005.
- [22] M. Horodecki and M. Piani, On quantum advantage in dense coding. *J. Phys. A: Math. Gen.*, vol. 45, no. 10, 105306, 2012.
- [23] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 2000.
- [24] T. Ogawa and H. Nagaoka, Making good codes for classical-quantum channel coding via quantum hypothesis testing, *IEEE Trans. Inf. Theory*, vol. 53, no. 6, pp. 2261-2266, 2007.
- [25] H. Qi, K. Sharma, and M. Wilde, Entanglement-assisted private communication over quantum broadcast channels, *J. Phys. A: Math. Theor.*, vol. 51, no. 37, 374001, 2018.
- [26] M.-H. Hsieh, I. Devetak, and A. Winter, Entanglement-assisted capacity of quantum multiple-access channels, *IEEE Trans. Inf. Theory*, vol. 54, no. 7, pp. 3078-3090, 2018.
- [27] W. F. Stinespring, Positive functions on  $C^*$ -algebras, *Proc. Amer. Math. Soc.*, vol. 6, no. 2, pp. 211-216, 1955.
- [28] B. M. Terhal, M. Horodecki, D. W. Leung, and D. P. DiVincenzo, The entanglement of purification, *J. Math. Phys.*, vol. 43, no. 9, pp. 4286-4298, 2002.
- [29] K. Wang and M. Hayashi, Permutation enhances classical communication assisted by entangled states, *IEEE Trans. Inf. Theory*, vol. 67, no. 6, pp. 3905-3925, 2021.
- [30] M. M. Wilde, *Quantum Information Theory* (2nd edition), Cambridge University Press, 2017.
- [31] M. M. Wilde and M.-H. Hsieh, Entanglement generation with a quantum channel and a shared state, *Proc. 2010 IEEE Int. Symp. Information Theory (ISIT)*, Austin TX, pp. 2713-2717, 2010.
- [32] A. Winter, Coding theorem and strong converse for quantum channels, *IEEE Trans. Inf. Theory*, vol. 45, no. 7, pp. 2481-2485, 1999.
- [33] A. Winter, Scalable programmable quantum gates and a new aspect of the additivity problem for the classical capacity of quantum channels, *J. Math. Phys.*, vol. 43, no. 9, pp. 4341-4352, 2002.
- [34] A. D. Wyner, The Wire-Tap Channel, *Bell Syst. Tech. J.*, vol. 54, no. 8, pp. 1355-1387, 1975.