Monochromatic Boxes in Colored Grids

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Abstract

A d-dimensional *grid* is a set of the form $R = [a_1] \times \cdots \times [a_d]$. A d-dimensional *box* is a set of the form $\{b_1, c_1\} \times \cdots \times \{b_d, c_d\}$. When a grid is c-colored, must it admit a monochromatic box? If so, we say that R is c-guaranteed. This question is a relaxation of one attack on bounding the van der Waerden numbers, and also arises as a natural hypergraph Ramsey problem (viz. the Ramsey numbers of hyperoctahedra). We give conditions on the a_i for R to be c-guaranteed that are asymptotically tight, and analyze the set of minimally c-guaranteed grids.

1 Introduction

A d-dimensional *grid* is a set $R = [a_1] \times \cdots \times [a_d]$, where $[t] = \{1, \ldots, t\}$. For ease of notation, we write $[a_1, \ldots, a_d]$ for $[a_1] \times \cdots \times [a_d]$. The "volume" of R is $\prod_{i=1}^d a_i$. A d-dimensional *box* is a set of 2^d points of the form

$$\{(x_1 + \epsilon_1 s_1, \ldots, x_d + \epsilon_d s_d) \,|\, \epsilon_i \in \{0,1\} \text{ for } 1 \le i \le d\},$$

with $s_i \neq 0$ for all $1 \le i \le d$. A grid R is (c,t)*-guaranteed*, if for all colorings $f : R \to [c]$, there are at least t distinct monochromatic boxes in R, i.e., boxes $B_j \subseteq R$, $j \in [t]$, so that $|f(B_j)| = 1$. When $t = 1$, we simply say that R is c-guaranteed. If R is not c-guaranteed, we say it is c*-colorable*. Clearly, whether a grid is (c,t)-guaranteed depends only on $a_1, \ldots, a_d$. Furthermore, if $b_i \ge a_i$ for all i such that $1 \le i \le d$, then $[b_1, \ldots, b_d]$ is c-guaranteed if $[a_1, \ldots, a_d]$ is. This ordering on d-tuples is sometimes called the *dominance order*, and we will denote it by $\preceq$. Then one may state the above observation as the fact that the set of c*-guaranteed* grids is an up-set in the $(\mathbb{N}^d, \preceq)$-poset. Hence, we have a full understanding of this family if we know the minimal c-guaranteed grids, an antichain in the $\preceq$ order. (Note that any such antichain is finite, a well-known fact in poset theory.) Call the set of minimal c-guaranteed grids $\mathcal{O}(c,d)$, the *obstruction set* for c colors in dimension d. We will focus our attention on *monotone* obstruction set elements, i.e., those grids for which $a_1 \le \cdots \le a_d$,

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since being c-guaranteed (or (c,t)-guaranteed) is invariant under permutations of the a_j.

The subject of unavoidable configurations in grids has connections with the celebrated Van der Waerden's and Szemerédi's Theorems. (See, for example, [2], [5], and [8].) Our results can be seen as belonging to hypergraph Ramsey theory, as follows. Let G be the complete d-partite d-uniform hypergraph with blocks of size $a_1, \ldots, a_d$. Then an edge of G can be identified with a vertex of $R = [a_1, \ldots, a_d]$ in the natural way. Under this correspondence, a c-coloring of R gives rise to a c-edge coloring of G, and boxes correspond precisely to subgraphs isomorphic to the "generalized octahedron" $K_d(2)$, the complete d-partite d-uniform hypergraph with each block of size 2. The generalized octahedra play an important and closely related role in the work of Kohayakawa, Rödl, and Skokan ([7]) on hypergraph quasirandomness. (Among other interesting results, they show that, asymptotically, a random c-edge coloring of G has the fewest number of monochromatic $K_d(2)$'s possible.) We may translate each of our results into statements about the Ramsey numbers of hyperoctahedra-free d-partite d-uniform graphs. For example, in Section 7, we give a family of upper bounds on the sizes of 3-dimensional grids which have a 2-coloring admitting no monochromatic box; this is equivalent to asking for the extremal tripartite 3-uniform hypergraphs which are $(K_3(2), K_3(2))$-Ramsey.

The present work is even more closely connected to the "Product Ramsey Theorem." Though the proof appears in [6], the statement appearing in [9] best illustrates the connection:

Theorem 1.1 (Product Ramsey Theorem). *Let $k_1, \ldots, k_d$ be nonnegative integers; let c and d be positive integers; and let $m_1, \ldots, m_d$ be integers with $m_i \geq k_i$ for $i \in [d]$. Then there exists an integer $R = R(c, d; k_1, \ldots, k_d; m_1, \ldots, m_d)$ so that if $X_1, \ldots, X_d$ are sets and $|X_i| \geq R$ for $i \in [d]$, then for every function $f : \binom{X_1}{k_1} \times \cdots \times \binom{X_d}{k_d} \to [c]$, there exists an element $\alpha \in [c]$ and subsets $Y_1, \ldots, Y_d$ of $X_1, \ldots, X_d$, respectively, so that $|Y_i| \geq m_i$ for $i \in [d]$ and f maps every element of $\binom{X_1}{k_1} \times \cdots \times \binom{X_d}{k_d}$ to α.*

This result ensures that the quantity $N(c, d) = R(c, d; 1^d; 2^d)$, which corresponds to the least R so that $[R]^d$ is c-guaranteed, is finite. A closer analysis of $N(c, d)$ – in fact, the more general $N(c, d, m) = R(c, d; 1^d; m^d)$ – appears in the manuscript [1] by Agnarsson, Doerr, and Schoen. They obtain asymptotic bounds on $N(c, d, m)$ that are valid for large m. Here, we examine instead the least nontrivial case of $m = 2$, and consider grids which are not necessarily equilateral.

In the next section, we show that any grid of sufficiently small volume (approximately c^{2^d-1}) is c-colorable. The following section shows that the analysis is tight: there are grids of this volume which are c-guaranteed. Not all grids of sufficient volume are c-guaranteed, although Section 4 demonstrates that any grid all of whose lower-dimensional subgrids are sufficiently voluminous is indeed c-guaranteed. The next section gives a tight upper bound on the volume of minimally c-guaranteed grids, i.e., elements of the obstruction set. Section

6 then addresses the question of how many obstructions there are. Finally, as mentioned above, Section 7 considers the case of $c = 2$ and $d = 3$, where some interesting computational questions arise. This extends work of the second two authors ([4]) for $d = 2$ and $2 \le c \le 4$.

Throughout the present manuscript, unless we explicitly say otherwise, we use the notations $x = O(y)$ and $y = \Omega(x)$ to mean that there is a function $F : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $x \le F(d)y$. That is, x is bounded by y times a number that only depends on d. (Naturally, $x = \Theta(y)$ means that $x = O(y)$ and $x = \Omega(y)$, and notation $x = o(y)$ is defined analogously.) In general, x and y will depend on c, d, and perhaps other quantities.

2 All small grids are c-colorable

Define $V(c, d)$ to be the largest integer V so that every d-dimensional grid R with volume at most V is c-colorable. Below, we show that $V(c, d)$ is $\Theta(c^{2^d-1})$.

Theorem 2.1.

$$V(c, d) = \Omega(c^{2^d-1}).$$

In fact, $V(c, d) > c^{2^d-1}/e2^d$, where $e = 2.718\ldots$ is Napier's constant.

Proof. We apply the Lovász Local Lemma (see, e.g., [3]), which states the following. Suppose that $A_1, \ldots, A_t$ are events in some probability space, each of probability at most p. Let G be a "dependency" graph with vertex set $\{A_i\}_{i=1}^t$, i.e., a graph so that, whenever a set S of vertices induces no edges in G, then S is a mutually independent family of events. Then $\mathbf{P}(\bigwedge_{i=1}^t \overline{A_i}) > 0$ if $ep(\Delta+1) \le 1$, where $\Delta = \Delta(G)$ is the maximum degree of G.

Now, suppose $R = [a_1, \ldots, a_d]$ is a grid of volume V, and we color the points of R uniformly at random from $[c]$. Enumerate all boxes in R as $B_1, \ldots, B_t$. Define A_i to be the event that B_t is monochromatic in this random coloring. Clearly, we may take G to have an edge between A_i and A_j whenever $B_i \cap B_j \ne \emptyset$. The degree of a vertex A_i is then the number of boxes B_j, $j \ne i$, which intersect B_i. Since we may specify the list of all such boxes by choosing one of the 2^d points of B_i, and then choosing the d coordinates of its antipodal point, $\deg_G(A_i)$ is at most

$$2^d \prod_{i=1}^{d} (a_i - 1) - 1 < 2^d \prod_{i=1}^{d} a_i - 1 = 2^d V - 1.$$

(The outermost -1 here reflects the fact that B_i may be excluded among these choices.) The probability of each A_i is the same: $p = c^{-2^d+1}$. Therefore,

$$ep(\Delta + 1) < ec^{-2^d+1} 2^d V$$

which is ≤ 1 whenever $V \le c^{2^d-1}/e2^d$. □

3 Some large grids are c-guaranteed

Theorem 3.1. *Fix c, d, define $R = [a_1, \ldots, a_d]$, and let $M = \prod_i \binom{a_i}{2}$ denote the total number of boxes in R. For $\min\{a_1, \ldots, a_d\} \to \infty$, R is $(c, M(1 + o(1))/c^{2^d-1})$-guaranteed.*

Theorem 3.1 follows quickly from the next lemma, whose extra strength we will need later.

Lemma 3.2. *Suppose $c \geq 1$. For $d \geq 1$ and integers $a_1, \ldots, a_d \geq 2$, let $M = \prod_{i=1}^d \binom{a_i}{2}$. The grid $R = [a_1, \ldots, a_d]$ is $(c, M\Delta_d/c^{2^d-1})$-guaranteed provided $\Delta_1, \ldots, \Delta_d > 0$, where Δ_j, $0 \leq j \leq d$, is given by the recurrence*

$$\begin{aligned} \Delta_0 &= 1, \\ \Delta_j &= \Delta_{j-1}^2 \left(1 - \frac{\frac{c^{2^{j-1}}}{\Delta_{j-1}} - 1}{a_j - 1}\right). \end{aligned}$$

Proof. We proceed inductively. Suppose $d = 1$, let $f : [a_1] \to [c]$ be a c-coloring, and define

$$\gamma_i = |f^{-1}(i)|$$

to be the number of points colored i, $1 \leq i \leq c$. Then the number N of monochromatic boxes in f is exactly

$$N = \sum_{i=1}^c \binom{\gamma_i}{2} = \frac{1}{2} \cdot \sum_{i=1}^c \left(\gamma_i^2 - \gamma_i\right) = \frac{1}{2} \cdot \left(\sum_{i=1}^c \gamma_i^2 - a_1\right).$$

Applying Cauchy-Schwarz,

$$\begin{aligned} N &\geq \frac{\left(\sum_{i=1}^c \gamma_i\right)^2}{2c} - \frac{a_1}{2} = \frac{a_1^2}{2c} - \frac{a_1}{2} \\ &= \frac{a_1(a_1 - c)}{2c} = \frac{1}{c}\binom{a_1}{2}\frac{a_1 - c}{a_1 - 1} = \frac{1}{c}\binom{a_1}{2}\Delta_1. \end{aligned}$$

Now, suppose the statement is true for dimensions $< d + 1$, and consider a coloring $f : [a_1, \ldots, a_{d+1}] \to [c]$. Consider the a_{d+1} colorings f_j of the d-dimensional grid $[a_1, \ldots, a_d]$ induced by setting the last coordinate to j, i.e.,

$$f_j(x_1, \ldots, x_d) = f(x_1, \ldots, x_d, j).$$

Let $\gamma_i(B)$, for a box $B \subset [a_1, \ldots, a_d]$ and $i \in [c]$, denote the number of j so that $f_j|_B \equiv i$. Then the number N of monochromatic $(d + 1)$-dimensional boxes in f is

$$N = \sum_i \sum_B \binom{\gamma_i(B)}{2}$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot \sum_i \sum_B (\gamma_i(B)^2 - \gamma_i(B)) \\
&\geq \frac{\left(\sum_B \sum_i \gamma_i(B)\right)^2}{2Mc} - \frac{1}{2} \cdot \sum_B \sum_i \gamma_i(B) \\
&= \frac{\left(\sum_B \sum_i \gamma_i(B)\right)^2 - Mc \sum_B \sum_i \gamma_i(B)}{2Mc}
\end{aligned}
$$

where $M = \prod_{i=1}^{d} \binom{a_i}{2}$. Since, by the inductive hypothesis, f_j induces at least $M\Delta_d / c^{2^d - 1}$ monochromatic boxes,

$$
\sum_i \sum_B \gamma_i(B) \geq \frac{a_{d+1} M \Delta_d}{c^{2^d - 1}},
$$

so that

$$
\begin{aligned}
N &\geq \frac{a_{d+1}^2 M^2 \Delta_d^2 / c^{2^{d+1} - 2} - a_{d+1} M^2 \Delta_d c / c^{2^d - 1}}{2Mc} \\
&= \frac{a_{d+1} M (a_{d+1} \Delta_d^2 - c^{2^d} \Delta_d)}{2c^{2^{d+1} - 1}} \\
&= \frac{M}{c^{2^{d+1} - 1}} \binom{a_{d+1}}{2} \frac{a_{d+1} \Delta_d^2 - c^{2^d} \Delta_d}{a_{d+1} - 1} \\
&= \frac{\prod_{i=1}^{d+1} \binom{a_i}{2}}{c^{2^{d+1} - 1}} \cdot \Delta_d^2 \left(\frac{a_{d+1} - c^{2^d} / \Delta_d}{a_{d+1} - 1} \right) \\
&= \frac{\prod_{i=1}^{d+1} \binom{a_i}{2} \Delta_{d+1}}{c^{2^{d+1} - 1}}.
\end{aligned}
$$

□

Proof of Theorem 3.1. Fix $c, d \geq 1$. It is clear by induction on j that for all $1 \leq j \leq d$, as $\min\{a_1, \ldots, a_d\} \to \infty$, $\Delta_j = 1 + o(1)$, and so in particular, $\Delta_j > 0$ if $\min\{a_1, \ldots, a_d\}$ is large enough. □

Note that, in the notation of Lemma 3.2, if $\Delta_1, \ldots, \Delta_d > 0$, then $[a_1, \ldots, a_d]$ is not c-colorable. Therefore we may conclude the following.

Corollary 3.3. *In the notation of Lemma 3.2, let Γ_j, $0 \leq j \leq d$, be given by the recurrence*

$$
\begin{aligned}
\Gamma_0 &= 1, \\
\Gamma_j &= \Gamma_{j-1}^2 \left(1 - \frac{c^{2^{j-1}} / \Gamma_{j-1}}{a_j - 1} \right) = \Gamma_{j-1} \left(\Gamma_{j-1} - \frac{c^{2^{j-1}}}{a_j - 1} \right).
\end{aligned}
$$

If $\Gamma_1, \ldots, \Gamma_d > 0$, then $[a_1, \ldots, a_d]$ is c-guaranteed.

Proof. Assume $\Gamma_1, \ldots, \Gamma_d > 0$. A routine induction shows that $\Gamma_j \leq \Delta_j$ for $0 \leq j \leq d$. $\square$

Lemma 3.4. *In the notation of Lemma 3.2, let ε_j be given by the recurrence*

$$\begin{aligned} \varepsilon_0 &= 0, \\ \varepsilon_j &= 2\varepsilon_{j-1} + \frac{c^{2^{j-1}}}{a_j - 1}. \end{aligned}$$

If $\varepsilon_d < 1$, then $[a_1, \ldots, a_d]$ is c-guaranteed.

Proof. Clearly, $0 = \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_d$, and so by assumption $\varepsilon_i < 1$ for all $i \in [d]$. An induction on i shows that $\Gamma_i \geq 1 - \varepsilon_i$ for $0 \leq i \leq d$: This is clearly true for $i = 0$. Suppose $i < d$ and $\Gamma_i \geq 1 - \varepsilon_i$. Then setting $\eta := c^{2^i}/(a_{i+1} - 1)$ and noting that $\Gamma_i \geq 0$, we have

$$\Gamma_{i+1} = \Gamma_i(\Gamma_i - \eta) \geq \Gamma_i(1 - \varepsilon_i - \eta). \tag{1}$$

The term in the parentheses is positive:

$$1 - \varepsilon_i - \eta \geq 1 - 2\varepsilon_i - \eta = 1 - \varepsilon_{i+1} > 0$$

by assumption. Thus continuing (1) and using the inductive hypothesis again,

$$\Gamma_i(1 - \varepsilon_i - \eta) \geq (1 - \varepsilon_i)(1 - \varepsilon_i - \eta) \geq 1 - 2\varepsilon_i - \eta = 1 - \varepsilon_{i+1}.$$

$\square$

Motivated by the preceding lemma, for every $c, d \geq 1$ and grid $R = [a_1, \ldots, a_d]$ with $a_i \geq 2$ for all $i \in [d]$, we define

$$\varepsilon_c(R) := \sum_{i=1}^{d} 2^{d-i} \frac{c^{2^{i-1}}}{a_i - 1}.$$

Lemma 3.5. *If $R = [a_1, \ldots, a_d]$ is not c-guaranteed, then $\varepsilon_c(R) \geq 1$.*

Proof. We let $\varepsilon_j := \varepsilon_c([a_1, \ldots, a_j]) = \sum_{i=1}^{j} 2^{j-i} c^{2^{i-1}}/(a_i - 1)$ for all j with $0 \leq j \leq d$, and notice that the ε_j satisfy the recurrence in Lemma 3.4. $\square$

Corollary 3.6. *For any fixed $d \geq 1$ and $c \geq 2$, if n is least such that $[n]^d$ is c-guaranteed, then $n < (d+2)c^{2^{d-1}}$. Furthermore,*

$$2^{-d}e^{-1} < \frac{V(c,d)}{c^{2^d - 1}} < (d+2)^d 2^{d(d-1)/2}.$$

Proof. If we take $a_j = (d+1)2^{d-j}c^{2^{j-1}} + 1$ for all $1 \leq j \leq d$, then $\varepsilon_c(R) = d/(d+1) < 1$. The second result now follows from the fact that

$$\prod_{j=1}^{d} a_j < \prod_{j=1}^{d} (d+2)2^{d-j}c^{2^{j-1}}$$

$$
\begin{aligned}
&= (d+2)^d 2^{\sum_{j=1}^{d}(d-j)} c^{\sum_{j=1}^{d} 2^{j-1}} \\
&= (d+2)^d 2^{\sum_{j=1}^{d-1} j} c^{\sum_{j=0}^{d-1} 2^j} \\
&= (d+2)^d 2^{d(d-1)/2} c^{2^d-1}.
\end{aligned}
$$

The first result follows by taking $n := a_d$. □

4 Hereditarily large grids are c-guaranteed

It is possible for grids of arbitrarily large volume to be c-colorable. Indeed, one need only have one of the dimensions be at most c, and then color the grid with this coordinate. However, if we require that each lower dimensional sub-grid be sufficiently voluminous, then the whole grid is c-colorable. This statement is made precise by the following theorem.

Theorem 4.1. *Fix $d > 0$, and define $C_j = (d2^d)^{\frac{3}{2}(3^{j-1}-1)}$ for $j \geq 1$. For all integers $c \geq 1$ and $1 \leq a_1 \leq a_2 \leq \cdots \leq a_d$, if $\prod_{i=1}^{j} a_i > C_j c^{(3^j-1)/2}$ for all $j \in [d]$, then $[a_1, \ldots, a_d]$ is c-guaranteed.*

We require a lemma and a bit of notation: If $R = [a_1, \ldots, a_d]$ and $1 \leq j < d$, let R_j denote $[a_1, \ldots, a_j]$ and let $\overline{R}_j$ denote $[a_{j+1}, \ldots, a_d]$. Note that, if R is c-guaranteed, then R_j is as well. Indeed, if $f : R_j \to [c]$ is a c-coloring of R_j, then the function $g : R \to [c]$ defined by $g(x_1, \ldots, x_d) = f(x_1, \ldots, x_j)$ is a c-coloring of R. We will also make repeated use of the following easily verified fact: For every integer $j \geq 0$, $j \cdot 2^{j-1} \leq (3^j - 1)/2$ and $j \cdot 2^j + 1 \leq 3^j$.

Lemma 4.2. *Let $c \geq 1$, let $R = [a_1, \ldots, a_d]$ be a grid, and let $j \in [d-1]$. Define*

$$
c' := c \cdot \prod_{i=1}^{j} \binom{a_i}{2} \leq 2^{-j} \cdot c \cdot \prod_{i=1}^{j} a_i^2.
$$

If R_j is c-guaranteed and $\overline{R}_j$ is c'-guaranteed, then R is c-guaranteed.

Proof. Assume that R_j is c-guaranteed and that $\overline{R}_j$ is c'-guaranteed. Suppose that $f : R \to [c]$ is a c-coloring. Consider the coloring $g : \overline{R}_j \to [c']$ that assigns the pair (B, s) to the point $\mathbf{v}$, B being an arbitrary choice of j-dimensional box colored monochromatically by $f_j : R_j \to [c]$, where $f_j(x_1, \ldots, x_j) = f(x_1, \ldots, x_j, \mathbf{v})$, and s being its color. (Note that R_j is c-guaranteed, so such a B always exists.) Then g is a c'-coloring, because there are exactly c' many different (B, s). Since $\overline{R}_j$ is c'-guaranteed, g colors some $(d-j)$-dimensional box B_1 monochromatically, with color (B_2, s). But then $B_2 \times B_1$ is a d-dimensional box monocolored by f with color s. □

Proof of Theorem 4.1. The statement is clearly true when $d = 1$ since $C_1 = 1$. Suppose $d > 1$ and the statement is true for all $d' < d$. Let $R = [a_1, \cdots, a_d]$ be a monotone grid satisfying the hypothesis of the theorem.

Case 1: $\varepsilon_c(R) < 1$. The result follows immediately from Lemma 3.5.

Case 2: $\varepsilon_c(R) \geq 1$. Then there is some $j \in [d]$ such that $2^{d-j}c^{2^{j-1}}/(a_j-1) \geq 1/d$, i.e.,

$$a_j \leq d2^{d-j}c^{2^{j-1}} + 1 < d2^{d-j+1}c^{2^{j-1}}.$$

Since $j2^j \leq 3^j - 1$ for all integers $j \geq 1$,

$$\prod_{i=1}^{j} a_i \leq \prod_{i=1}^{j} a_j < d^j 2^{j(d-j+1)} c^{j2^{j-1}} \leq d^j 2^{j(d-j+1)} c^{(3^j-1)/2},$$

and so for all $k \in [d-j]$,

$$\prod_{i=1}^{k} a_{j+i} > \frac{C_{j+k}}{d^j 2^{j(d-j+1)}} c^{(3^{j+k}-1)/2-(3^j-1)/2} \geq \frac{C_{j+k}}{d^j 2^{j(d-j+1)}} c^{3^j(3^k-1)/2}.$$

Let $c' = d^{2j}2^{2j(d-j+1)}c^{3^j}$. (Note that $c' \geq c \cdot \prod_{i=1}^{j} a_i^2$.) Then for all $k \in [d-j]$,

$$\begin{aligned}
\prod_{i=1}^{k} a_{j+i} &> \frac{C_{j+k}}{d^j 2^{j(d-j+1)}} \left(\frac{c'}{d^{2j}2^{2j(d-j+1)}} \right)^{(3^k-1)/2} \\
&= \frac{(d2^d)^{\frac{3}{2}(3^{j+k-1}-1)}}{(d2^{d-j+1})^{j3^k}} c'^{(3^k-1)/2} \\
&\geq (d2^d)^{\frac{3}{2}(3^{j+k-1}-1)-j3^k} c'^{(3^k-1)/2} \\
&\geq (d2^d)^{\frac{3}{2}(3^{j+k-1}-1)-(3^j-1)3^k/2} c'^{(3^k-1)/2},
\end{aligned}$$

because $j \leq (3^j-1)/2$ for all $j \geq 1$. Continuing the computation,

$$\begin{aligned}
\prod_{i=1}^{k} a_{j+i} &> (d2^d)^{\frac{3}{2}(3^{j+k-1}-1)-(3^j-1)3^k/2} c'^{(3^k-1)/2} \\
&= (d2^d)^{\frac{3}{2}(3^{j+k-1}-1-3^{j+k-1}+3^{k-1})} c'^{(3^k-1)/2} \\
&= (d2^d)^{\frac{3}{2}(3^{k-1}-1)} c'^{(3^k-1)/2} \\
&= C_k c'^{(3^k-1)/2}.
\end{aligned}$$

Therefore $\overline{R}_j = [a_{j+1}, \ldots, a_d]$ is c'-guaranteed by the inductive hypothesis. (It is easy to see that the C_j's are increasing in d, so taking $d' = d - j$ causes no problem here.) Since R_j is also c-guaranteed by the inductive hypothesis, we may apply Lemma 4.2 to conclude that R is c-guaranteed. □

5 Upper bounds on the volume of obstruction grids

Before proceeding, we introduce the following notation. For $d \geq 1$ and any monotone grid $R = [a_1, \ldots, a_d]$ where $a_d > 1$, we let R^- denote the monotone grid obtained from R by subtracting one from a_j, where $j \in [d]$ is least such that $a_j = a_d$. Note that if R is monotone and $R \in \mathcal{O}(c, d)$, then R is c-guaranteed but R^- is not c-guaranteed.

The next theorem gives an asymptotic upper bound on the volume $\prod_{i=1}^d a_i$ of any grid $[a_1, \ldots, a_d] \in \mathcal{O}(c, d)$.

Theorem 5.1. *For every $d \geq 1$ and every grid $R = [a_1, \ldots, a_d] \in \mathcal{O}(c, d)$,*

$$\prod_{i=1}^{d} a_i = O\left(c^{(3^d - 1)/2}\right).$$

The theorem follows immediately from the following lemma:

Lemma 5.2. *For every $d \geq 1$, every $c \geq 2$, and every monotone grid $R = [a_1, \ldots, a_d] \in \mathcal{O}(c, d)$, there is a set $P \subseteq [d]$ such that*

1. *$d \in P$,*
2. *$\prod_{i=1}^{\ell} a_i = O\left(c^{(3^\ell - 1)/2}\right)$ for every $\ell \in P$, and*
3. *For every $k \in [d]$,*
$$a_k = O\left(c^{3^j \cdot 2^{\ell - j - 1}}\right),$$
where ℓ is the least element of P that is $\geq k$, and j is the biggest element of P that is $< k$ ($j = 0$ if there is no such element).

(We call the elements of P pinch points *for R.)*

Proof. Let $d \geq 1$ and $c \geq 2$ be given, and let $R = [a_1, \ldots, a_d] \in \mathcal{O}(c, d)$ be a monotone grid. Then R is c-guaranteed, and thus R_j is also c-guaranteed for all $1 \leq j \leq d$. Since $R \in \mathcal{O}(c, d)$, we have that R^- is not c-guaranteed, and thus $\varepsilon_c(R^-) \geq 1$. This in turn implies that there is some largest $\ell \in [d]$ such that

$$2^{d-\ell} \frac{c^{2^{\ell-1}}}{a_\ell - 2} \geq \frac{1}{d}.$$

(Note that the denominator is positive, because $a_\ell \geq a_1 \geq c + 1 \geq 3$ since R is c-guaranteed.) Thus,

$$a_\ell \leq d 2^{d-\ell} \cdot c^{2^{\ell-1}} + 2 \leq (d+2) 2^{d-\ell} \cdot c^{2^{\ell-1}}, \tag{2}$$

and thus

$$\prod_{i=1}^{\ell} a_i \leq (a_\ell)^\ell \leq \left((d+2)2^{d-\ell}\right)^\ell \cdot c^{\ell \cdot 2^{\ell-1}} \leq \left((d+2)2^{d-1}\right)^d \cdot c^{(3^\ell - 1)/2}, \tag{3}$$

which implies that ℓ satisfies Condition 2 of the lemma. We will make ℓ the least element of P, noticing that Equation (2) and the monotonicity of R imply that a_k satisfies Condition 3 of the lemma for all $k \in [\ell]$ (with $j = 0$).

If $\ell = d$, then we let $P = \{\ell\} = \{d\}$ and we are done.

Otherwise, $\ell < d$. Note that $R^- = R_\ell \times (\overline{R}_\ell)^-$ up to a possible permutation of the coordinates. Recall also that R_ℓ is c-guaranteed, but R^- is not. It follows from Lemma 4.2 that $(\overline{R}_\ell)^-$ is not c'-guaranteed, where

$$c' := c \cdot \prod_{i=1}^{\ell} \binom{a_i}{2} = O\left(c \cdot \left(\prod_{i=1}^{\ell} a_i\right)^2\right).$$

The bound in Equation (3) gives $c' = O\left(c^{3^\ell}\right)$.

We thus have $\varepsilon_{c'}((\overline{R}_\ell)^-) \geq 1$, and so there is some largest m with $\ell < m \leq d$ such that

$$2^{d-m} \frac{(c')^{2^{m-\ell-1}}}{a_m - 2} \geq \frac{1}{d-\ell},$$

which gives

$$\begin{aligned} a_m &\leq (d-\ell)2^{d-m} \cdot (c')^{2^{m-\ell-1}} + 2 && (4)\\ &\leq (d-\ell+2)2^{d-m} \cdot (c')^{2^{m-\ell-1}} && (5)\\ &= O\left(c^{3^\ell \cdot 2^{m-\ell-1}}\right). && (6) \end{aligned}$$

For the volume of R_m, we get

$$\begin{aligned} \prod_{i=1}^{m} a_i &= \prod_{i=1}^{\ell} a_i \cdot \prod_{i=\ell+1}^{m} a_i \\ &\leq \left(\prod_{i=1}^{\ell} a_i\right) \cdot (a_m)^{m-\ell} \\ &= O\left(c^{(3^\ell-1)/2}\right) \cdot O\left(c^{3^\ell \cdot (m-\ell) \cdot 2^{m-\ell-1}}\right) \\ &= O\left(c^{(3^\ell-1)/2} \cdot c^{3^\ell \cdot (3^{m-\ell}-1)/2}\right) \\ &= O\left(c^{(3^m-1)/2}\right). \end{aligned}$$

We make ℓ and m the two least elements of P, and the last calculation shows that $m \in P$ satisfies Condition 2. Further, since $a_k \leq a_m$ for all k such that $\ell < k \leq m$, Condition 3 is also satified for all these a_k by Equations (4)–(6).

If $m = d$, then we let $P = \{\ell, m\}$ and we are done. Otherwise, we repeat the argument above using m instead of ℓ to obtain an n with $m < n \leq d$ such that ℓ, m, and n being the least three elements of P satisfies Conditions 2 and 3 of the lemma, and so on until we arrive at d, whence we set $P := \{\ell, m, n, \ldots, d\}$. □

The next proposition shows that the bounds in Lemma 5.2 are asymptotically tight.

Proposition 5.3. *For $c \geq 2$, there is an infinite sequence $\{\mu_j(c)\}_{j=1}^{\infty}$ of positive integers such that*

1. $\mu_j(c) \geq 1 + 2^{(1-3^{j-1})/2} \cdot c^{3^{j-1}}$ *for all $j \in \mathbb{Z}^+$, and*
2. *for all $d \geq 1$, the grid $[\mu_1(c), \ldots, \mu_d(c)] \in \mathcal{O}(c, d)$ with pinch point set $P = [d]$.*

Proof. For all $c \geq 2$, define

$$\begin{aligned} \mu_1(c) &:= 1 + c, \\ \mu_2(c) &:= 1 + c \cdot \binom{c+1}{2}, \\ &\vdots \\ \mu_{j+1}(c) &:= 1 + c \cdot \prod_{i=1}^{j} \binom{\mu_i(c)}{2}, \\ &\vdots \end{aligned}$$

Fix $c \geq 2$ and let μ_j denote $\mu_j(c)$ for short. A routine induction on j shows (1). For the inductive step, noting that $\sum_{i=0}^{j-1} 3^i = (3^j - 1)/2$, we have

$$\begin{aligned} \mu_{j+1} &= 1 + c \cdot \prod_{i=1}^{j} \binom{\mu_i}{2} \\ &\geq 1 + \frac{c}{2^j} \prod_{i=1}^{j} (\mu_i - 1)^2 \\ &\geq 1 + \frac{c}{2^j} \prod_{i=1}^{j} \frac{c^{2 \cdot 3^{i-1}}}{2^{3^{i-1}-1}} \\ &= 1 + \frac{c^{3^j}}{2^{(3^j-1)/2}}. \end{aligned}$$

For (2), we use induction on $d \geq 1$ to show separately that

1. $[\mu_1, \ldots, \mu_d]$ is c-guaranteed, and
2. $[\mu_1, \ldots, \mu_d]$ is not $(c, 2)$-guaranteed (i.e., there is a coloring $[\mu_1, \ldots, \mu_d] \to [c]$ that monocolors exactly one box).

Clearly $[\mu_1] = [1+c]$ is c-guaranteed by the Pigeonhole Principle. Now let $d \geq 2$ and assume that $[\mu_1, \ldots, \mu_{d-1}]$ is c-guaranteed. Then letting $c' = c \cdot \prod_{i=1}^{d-1} \binom{\mu_i}{2}$, we have $\mu_d = 1 + c'$, and hence $[\mu_d]$ is c'-guaranteed. But then, $[\mu_1, \ldots, \mu_d]$ is c-guaranteed by Lemma 4.2 (letting $j = d - 1$).

Now for claim (2). For $d = 1$, clearly the coloring $[\mu_1] \to [c]$ mapping $j \mapsto (j \bmod c) + 1$ has exactly one monochromatic 1-dimensional box, namely,

$(1;c) = \{1, c+1\}$. Now let $d \geq 2$ and assume claim (2) holds for $d-1$, i.e., there is a coloring $[\mu_1, \ldots, \mu_{d-1}] \to [c]$ that monocolors exactly one box. We will call such a coloring *minimal*. This generates exactly $\prod_{i=1}^{d-1} \binom{\mu_i}{2}$ many boxes in $[\mu_1, \ldots, \mu_{d-1}]$. For each of these boxes B and for each color s, we can find a minimal coloring that monocolors B with s by permuting the order of the hyperplanes along each axis and by permuting the colors. Thus there are exactly $c' = c \cdot \prod_{i=1}^{d-1} \binom{\mu_i}{2}$ many distinct minimal colorings. We overlay these c' many colorings to obtain a coloring of $[\mu_1, \ldots, \mu_{d-1}, c']$ with no monochromatic d-boxes. We then duplicate the first $(d-1)$-dimensional layer to arrive at a c-coloring of $[\mu_1, \ldots, \mu_{d-1}, 1 + c'] = [\mu_1, \ldots, \mu_d]$. This coloring has only one monocolored d-box: the box corresponding to the duplicated layer of unique monocolored $(d-1)$-boxes. This shows Item (2).

It follows from claim (2) that $[\mu_1, \ldots, \mu_{d-1}, \mu_d - 1]$ is not c-guaranteed for any $d \geq 1$, since we can remove a single hyperplane from the only monocolored d-box in some minimal coloring of $[\mu_1, \ldots, \mu_{d-1}, \mu_d]$ to leave a coloring of $[\mu_1, \ldots, \mu_{d-1}, \mu_d - 1]$ without any monochromatic $(d-1)$-boxes. From this it easily follows that $[\mu_1, \ldots, \mu_d] \in \mathcal{O}(c, d)$, because $[\mu_1, \ldots, \mu_{j-1}, \mu_j - 1]$ is not c-guaranteed, and hence $[\mu_1 \ldots, \mu_{j-1}, \mu_j - 1, \mu_{j+1}, \ldots, \mu_d]$ is not c-guaranteed, for any $j \in [d]$.

Finally, it is evident that all $j \in [d]$ are pinch points for $[\mu_1, \ldots, \mu_d]$. (It is interesting to note that $[\mu_1, \ldots, \mu_d]$ is the lexicographically first element of $\mathcal{O}(c, d)$.) □

6 Upper bound on the size of the obstruction set

It was shown in [4] that $|\mathcal{O}(c, 2)| \leq 2c^2$. We give an asymptotic upper bound for $|\mathcal{O}(c, d)|$ for every fixed $d \geq 3$.

Theorem 6.1. *For all $d \geq 3$,*

$$|\mathcal{O}(c, d)| = O\left(c^{(17 \cdot 3^{d-3} - 1)/2}\right).$$

Proof. Fix $d \geq 3$. We give an asymptotic upper bound on the number of monotone grids in $\mathcal{O}(c, d)$. The size of $\mathcal{O}(c, d)$ is at most $d!$ times this bound, and so it is asymptotically equivalent. By Lemma 5.2, every grid $R \in \mathcal{O}(c, d)$ has a set P of pinch points. For each set $P \subseteq [d]$ such that $d \in P$, let $\#_c(P)$ be the number of monotone grids in $\mathcal{O}(c, d)$ having pinch point set P. There are 2^{d-1} many such P, so an asymptotic bound on $\max\{\#_c(P) \mid P \subseteq [d] \wedge d \in P\}$ gives the same asymptotic bound on $|\mathcal{O}(c, d)|$.

Fix a set $P \subseteq [d]$ such that $d \in P$, and let $P = \{\ell_1 < \ell_2 < \cdots < \ell_s = d\}$, where $s = |P|$ and $\ell_1, \ldots, \ell_s$ are the elements of P in increasing order. For convenience, set $\ell_0 := 0$. Lemma 5.2 says that for any monotone grid $R = [a_1, \ldots, a_d] \in \mathcal{O}(c, d)$ having pinch point set P, for any $b \in [s]$, and for any k

such that $\ell_{b-1} < k \le \ell_b$, we have $a_k = O\left(c^{e(b)}\right)$, where

$$e(b) := 3^{\ell_{b-1}} \cdot 2^{\ell_b - \ell_{b-1} - 1}.$$

To bound $\#_c(P)$, we first note that for any choice of $1 \le a_1 \le \cdots \le a_{d-1}$, there can be at most one value of a_d such that $[a_1, \ldots, a_d] \in \mathcal{O}(c, d)$, because any two d-dimensional grids that share the first $d-1$ dimensions are comparable in the dominance order $\preceq$. Thus $\#_c(P)$ is bounded by the number of possible combinations of values of $a_1, \ldots, a_{d-1}$. From the bound on each a_k above, we therefore have

$$\begin{aligned}
\#_c(P) &\le \left(\prod_{b=1}^{s-1} \prod_{k=\ell_{b-1}+1}^{\ell_b} O\left(c^{e(b)}\right) \right) \cdot \prod_{k=\ell_{s-1}+1}^{d-1} O\left(c^{e(s)}\right) \\
&= O\left(\prod_{b=1}^{s-1} \left(c^{e(b)}\right)^{\ell_b - \ell_{b-1}} \right) \cdot O\left(\left(c^{e(s)}\right)^{d-1-\ell_{s-1}} \right) \\
&= O\left(c^{h_1 + h_2}\right)
\end{aligned}$$

where $h_2 = e(s)(d - 1 - \ell_{s-1})$ and

$$\begin{aligned}
h_1 &= \sum_{b=1}^{s-1} e(b)(\ell_b - \ell_{b-1}) \\
&= \sum_{b=1}^{s-1} 3^{\ell_{b-1}} \cdot 2^{\ell_b - \ell_{b-1} - 1} \cdot (\ell_b - \ell_{b-1}) \\
&\le \sum_{b=1}^{s-1} 3^{\ell_{b-1}} \cdot \frac{3^{\ell_b - \ell_{b-1}} - 1}{2} \\
&= \frac{1}{2} \sum_{b=1}^{s-1} \left(3^{\ell_b} - 3^{\ell_{b-1}}\right) \\
&= \frac{3^m - 1}{2},
\end{aligned}$$

where $m = \ell_{s-1}$. We also have

$$\begin{aligned}
h_2 &= 3^{\ell_{s-1}} \cdot 2^{d - \ell_{s-1} - 1} \cdot (d - 1 - \ell_{s-1}) \\
&= 3^m \cdot 2^{d-m-1} \cdot (d - m - 1),
\end{aligned}$$

whence

$$h_1 + h_2 = \frac{3^m - 1}{2} + 3^m \cdot 2^{d-m-1} \cdot (d - m - 1).$$

So our bound on the exponent of c only depends on the value of m, which satisfies $0 \le m < d$. It is more convenient to express $h_1 + h_2$ in terms of $n := d - m$, where $n \in [d]$:

$$h_1 + h_2 = \frac{3^{d-n} - 1}{2} + 3^{d-n} \cdot 2^{n-1} \cdot (n - 1)$$

$$= \frac{3^d}{2} \cdot \frac{1 + 2^n(n-1)}{3^n} - \frac{1}{2}.$$

It is easy to check that $(1 + 2^n(n-1))/3^n$ is greatest (and thus $h_1 + h_2$ is greatest) when $n = 3$. It follows that

$$\begin{aligned} h_1 + h_2 &\leq \frac{3^d}{2} \cdot \frac{1 + 2^3(3-1)}{3^3} - \frac{1}{2} \\ &= \frac{17 \cdot 3^{d-3} - 1}{2}, \end{aligned}$$

which proves the theorem. □

The first few values $(17 \cdot 3^{d-3} - 1)/2$ are given in the Figure 1.

d	$(17 \cdot 3^{d-3} - 1)/2$
3	8
4	25
5	76
6	229

Figure 1: Table of upper bounds on e so that $|\mathcal{O}(c,d)| = O(c^e)$ for small d.

7 Three Dimensions and Two Colors

The following graph (Figure 2, generated using the Jmol module in SAGE) and table (Figure 3) display upper bounds for the smallest a_3 so that $[a_1, a_2, a_3]$ is 2-guaranteed. All three graphical axes run from 3 to 130; the table includes only $3 \leq a_1 \leq 12$ and $3 \leq a_2 \leq 12$. We believe these values to be very close to the truth; indeed, we have matching lower bounds in many cases, and lower bounds that differ from the upper bounds by at most 2 in many more cases.

A few different methods were applied to obtain these bounds. First, the values Δ_j, as in Section 3, were computed, and the least a_3 so that $\Delta_3 > 0$ was recorded. In fact, this idea was improved slightly by applying the observation that, if some grid is $(2,t)$-guaranteed, then it is $(2, \lceil t \rceil)$-guaranteed. In some cases, this increases the value of Δ_j. Second, we used the simple observations that c-colorability is independent of the order of the a_i, and that $R \preceq R'$ when R is c-guaranteed implies that R' is c-guaranteed. Third, we applied the following lemma.

Lemma 7.1. *If the grid $R = [a_1, \ldots, a_d]$ is (c,t)-guaranteed, then $R \times [\lfloor cM/t \rfloor + 1]$ is c-guaranteed, where $M = \prod_{j=1}^{d} \binom{a_j}{2}$*

Proof. Note that $K = \lfloor cM/t \rfloor + 1 > cM/t$ and is integral. If we think of $R \times [K]$ as K copies of R, then any c-coloring of $R \times [K]$ restricts to K c-colorings of R. Since R is (c,t)-guaranteed, each of these c-colorings gives rise to

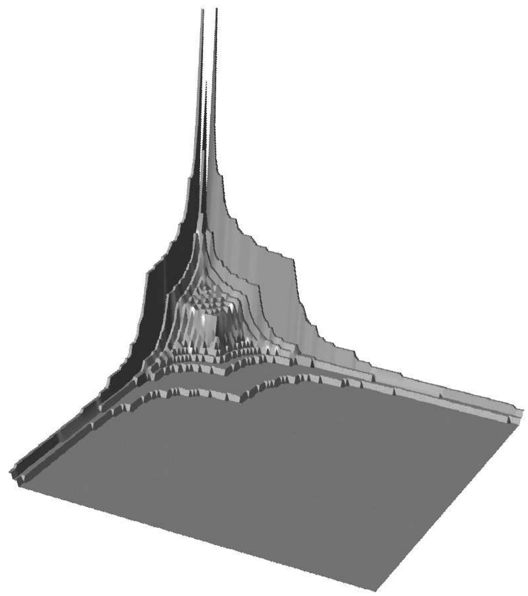

Figure 2: Graph of upper bounds on a_3 so that $[a_1, a_2, a_3]$ is 2-guaranteed.

	3	4	5	6	7	8	9	10	11	12
3					127	85	73	68	67	67
4					127	85	73	68	67	67
5			101	76	53	47	46	46	40	37
6			76	76	53	47	46	46	40	37
7	127	127	53	53	53	46	40	37	34	33
8	85	85	47	47	46	45	40	37	34	33
9	73	73	46	46	40	40	37	34	31	30
10	68	68	46	46	37	37	34	33	31	30
11	67	67	40	40	34	34	31	31	30	28
12	67	67	37	37	33	33	30	30	28	28

Figure 3: Table of bounds on a_3 so that $[a_1, a_2, a_3]$ is 2-guaranteed.

t monochromatic boxes. Hence, in K colorings, there are at least $t(\lfloor cM/t \rfloor + 1) > cM$ monochromatic boxes. Since there are only M total boxes in each copy of R, and any monochromatic box can only be colored in c different ways, there must be two identical boxes (in two different copies of R) which are monochromatic and have the same color. This is precisely a monochromatic $(d+1)$-dimensional box in $R \times [K]$. □

Therefore, in order to obtain upper bounds on $[a_3]$ in the above table, we need to know the greatest t for which $[a_1] \times [a_2]$ is $(2,t)$-guaranteed. To that end, we define the following matrix:

Definition 7.2. Let M_r be the $2^r \times 2^r$ integer matrix whose rows and columns are indexed by all maps $f_j : [r] \to [2]$, $0 \leq j < 2^r$. The (i,j)-entry of M_r is defined to be

$$\binom{|f_i^{-1}(1) \cap f_j^{-1}(1)|}{2} + \binom{|f_i^{-1}(2) \cap f_j^{-1}(2)|}{2}.$$

Then define the quadratic form $Q_r : \mathbb{R}^{2^r} \to \mathbb{R}$ by $Q_r(\mathbf{v}) = \mathbf{v}^* M_r \mathbf{v}$. Let $\delta_r = (M_r(1,1), \ldots, M_r(2^r, 2^r))$, the diagonal of M_r.

Proposition 7.3. *Let t be the least value of $Q_r(\mathbf{v}) - \mathbf{v} \cdot \delta_r$ over all nonnegative integer vectors $\mathbf{v} \in \mathbb{Z}^{2^r}$ with $\mathbf{v} \cdot \mathbb{1} = s$. Then $[r] \times [s]$ is (c,t)-guaranteed, and t is the minimum value so that this is the case.*

Proof. Given a vector $\mathbf{v} = (v_1, \ldots, v_r)$ satisfying the hypotheses, consider the $r \times s$ matrix A with v_j columns of type f_j for each $j \in [r]$. (We may identify f_j with a column vector in $[2]^r$ in the natural way.) It is easy to see that $Q_r(\mathbf{v}) - \delta_r$ exactly counts twice the number of monochromatic rectangles in A, thought of as a 2-coloring of the grid $[r] \times [s]$. □

We applied standard quadratic integer programming tools (XPress-MP) to minimize the appropriate programs. Fortunately, for the cases considered, the matrix M_r was positive semidefinite, meaning that the solver could use polynomial time convex programming techniques during the interior point search. We conjecture that this is always the case.

Conjecture 7.4. *M_r is positive semidefinite for $r \geq 3$.*

In particular, for $r = 3$, the eigenvalues of M_r are 0, 1, and 4, with multiplicities 2, 4, and 2, respectively. For $4 \leq r \leq 9$, the eigenvalues are 0, 2^{r-2}, $2^{r-3}(r-2)$, $2^{r-2}(r-1)$, and $2^{r-4}(r^2 - r + 2)$, with multiplicities $2^r - r(r+1)/2$, $r(r-1)/2 - 1$, $r - 1$, 1, and 1, respectively. We conjecture that this description of the spectrum is valid for all $r \geq 4$.

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