BOUNDS ON THE SIZE OF AN INCLUSION USING THE TRANSLATION METHOD FOR TWO-DIMENSIONAL COMPLEX CONDUCTIVITY*

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Abstract. The size estimation problem in electrical impedance tomography is considered when the conductivity is a complex number and the body is two-dimensional. Upper and lower bounds on the volume fraction of the unknown inclusion embedded in the body are derived in terms of two pairs of voltage and current data measured on the boundary of the body. These bounds are derived using the translation method. We also provide numerical examples to show that these bounds are quite tight and stable under measurement noise.

AMS subject classifications. 65N21, 35J20, 45Q05

Key words. inverse problems, size estimation, electrical impedance tomography, complex conductivity, variational principle, translation method

1. Introduction. The size estimation problem in electrical impedance tomography (EIT) is to estimate the size (area or volume) of unknown inclusions embedded in a conducting body by means of boundary measurements of the voltage and current. The unknown inclusions may represent anomalies in EIT imaging or non-destructive testing or a phase in two phase composite materials. Here we consider the problem where the body is two-dimensional.

To put the problem in a precise way, let Ω be a body in \mathbb{R}^2 occupied by a conducting material and let D be a conducting inclusion inside Ω . Let σ_1 and σ_2 ($\sigma_1 \neq \sigma_2$) be the conductivities (or dielectric constants) of D and $\Omega \setminus D$, respectively, and σ be the conductivity profile of Ω , *i.e.*,

$$\sigma = \sigma_1 \chi(D) + \sigma_2 \chi(\Omega \setminus \overline{D}) \tag{1.1}$$

where $\chi(D)$ is the characteristic function of D. If Ω is a two phase composite, we may write σ as

$$\sigma = \sigma_1 \chi_1 + \sigma_2 \chi_2 \tag{1.2}$$

where $\chi_1 = 1$ in phase 1 and 0 in phase 2, and $\chi_2 = 1 - \chi_1$. We consider the boundary value problem of the conductivity equation assuming that the Dirichlet data ϕ is assigned on $\partial\Omega$. So the problem to be considered is

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial \Omega. \end{cases}$$
 (1.3)

Then the current

$$q := \sigma \nabla u \cdot \mathbf{n} \tag{1.4}$$

is measured on $\partial\Omega$ where **n** is the unit outward normal to $\partial\Omega$. Then the size estimation problem is to estimate the area or volume |D| of the inclusion (or the volume fraction) in terms of a single or finitely many pairs of Cauchy data (ϕ,q) . It is worth mentioning that we may apply a current on the boundary and measure the corresponding voltage, and methods developed in this this paper can be applied to such situation

There has been some significant work on the size estimation problem in the context of the conductivity equation. Upper and lower bounds of |D| were obtained by Kang-Seo-Sheen [11], Alessandrini-Rosset [1], and Alessandrini-Rosset-Seo [2]. These bounds were obtained using estimates of elliptic partial differential equations and expressed by integrals evaluated by a single pair of Cauchy data. A different kind of bound

^{*}This work is supported by the Korean Ministry of Education, Sciences and Technology through NRF grants Nos. 2010-0004091 and 2010-0017532, BK21+ at Inha University, and by the National Science Foundation through grant DMS-1211359

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was obtained by Capdeboscq-Vogelius [4] using variational methods. Their bounds hold asymptotically when |D| is small. They require special Cauchy data. For special Cauchy data, such as affine boundary conditions on the potential, the universal bounds of Nemat-Nasser and Hori [21] may be inverted to bound |D|. Milton [17], generalizing the results of Nemat-Nasser and Hori, showed that bounds on the properties of composites imply bounds on the response of bodies with special Cauchy data, and these too can be inverted to bound |D| and do not require the assumption that |D| is small.

Recently, a completely different method to derive bounds on the volume fraction has been introduced by Kang, Kim and Milton which uses translations of the classical variational principles. The translation method was introduced by Murat-Tartar [20, 23, 24] and Lurie-Cherkaev [12, 13], and has been used in an essential way to derive bounds on the effective properties of two phase composites in terms of the volume fraction. It turns out that this method of translation can be applied effectively to derive bounds of the volume fraction in terms of boundary measurements: see Kang-Kim-Milton [8], and Kang-Milton [9]. Numerical implementations of the bounds presented in [8] show that these bounds work quite well to estimate the volume fraction. These bounds are sharp in the sense that for some geometries and for some boundary data the bounds are attained.

In this paper we deal with the case when the conductivity is a complex number. The subject of EIT imaging using complex conductivity has attracted much attention lately since the imaginary part of the complex conductivity changes depending on frequency and images at different frequencies can be used to generate images of high resolution. We refer to [22] and references therein for this direction of research. Our purpose is to derive bounds for the volume fraction using boundary measurements when the conductivity is a complex number.

The derivation of bounds in this paper is based on the variational principle of Cherkaev and Gibiansky [6] and the translation method. Let u be the solution to (1.3) when σ is complex. Then the corresponding electric and current fields are given by

$$\mathbf{e} = -\nabla u := \mathbf{e}' + i\mathbf{e}'' \tag{1.5}$$

and

$$\mathbf{j} = -\sigma \nabla u := \mathbf{j}' + i \mathbf{j}''. \tag{1.6}$$

In above mentioned paper, a minimizing variational principle is obtained for the field $\begin{bmatrix} \mathbf{j}' \\ \mathbf{e}'' \end{bmatrix}$. We may apply the translation method for this field to derive upper and lower bounds for the volume fraction using two Dirichlet boundary data ϕ_1 and ϕ_2 . But the bounds obtained in this way depends on the choice of boundary data, and it is necessary to consider measurements corresponding to the boundary data $e^{\theta_1}\phi_1$ and $e^{\theta_2}\phi_2$ for all θ_1 and θ_2 . So, we use the parameterized version of the Cherkaev-Gibiansky variational principle which was obtained by Milton-Seppecher-Bouchitte [19]. Using this variational principle (and translation) a set of bounds parameterized by θ_1 and θ_2 is obtained, and by minimizing (or maximizing) them over θ_1 and θ_2 we obtain tighter bounds. We emphasize that only the boundary measurements corresponding to one set of boundary data ϕ_1 and ϕ_2 are used to compute the bounds. We perform numerical experiments using the bounds obtained in this paper. Results show that the bounds are quite tight and stable in presence of measurement noise.

There is already some work on size estimation for complex conductivity, both for two and three dimensional bodies, but only using a single pair of Cauchy data unlike the two pairs we use here. Beretta-Francini-Vessella [3] obtained bounds on the size of the inclusion using elliptic estimates, and Thaler-Milton [25] developed a comprehensive set of sharp bounds on the volume fraction based on the splitting method. The splitting method, like the translation method, uses the fact that certain integrals (null-Lagrangians) are known in terms of boundary values, but unlike the translation method does not use variational principles, but instead uses the positivity of the norm of certain fields. It was first used in [18] in the context of elasticity (see also [10]).

This paper is organized as follows. In section 2 we review the variational principle of Cherkaev and Gibiansky and its parameterized version, and introduce some null Lagrangians. In section 3 we use null Lagrangians with parameters to translate the variational principle and compute the minimum. In section 4 parameters are determined and upper and lower bounds for the volume fraction are derived. Section 5 presents results of numerical experiments which show that bounds can be quite tight. We finish the paper with a short conclusion. The appendix is to prove a lemma used in the text.

2. Variational principle and null-Lagrangian. We suppose the conductivity σ given in (1.1) or (1.2) is a complex constant of the following form:

$$\sigma = \sigma' + i\sigma'', \quad \sigma_1 = \sigma'_1 + i\sigma''_1, \quad \sigma_2 = \sigma'_2 + i\sigma''_2.$$
 (2.1)

Assume that

$$\sigma' > 0, \quad |\sigma_1| \neq |\sigma_2|, \quad \sigma_1/\sigma_2 \notin \mathbb{R}.$$
 (2.2)

The second condition in the above is required to guarantee that all four points $(\sigma'_1, \sigma''_1), (-\sigma'_1, -\sigma''_1), (\sigma'_2, \sigma''_2), (-\sigma'_2, -\sigma''_2)$ are not on the same circle, and the last condition guarantees that all four points are not on the same straight line. (See Section 4.)

Let u be the solution u to (1.3) and let

$$\mathbf{e} = -\nabla u \quad \text{and} \quad \mathbf{j} = -\sigma \nabla u.$$
 (2.3)

Then we have

$$\nabla \cdot \mathbf{j} = 0 \quad \text{and} \quad \nabla \times \mathbf{e} = 0 \quad \text{in } \Omega, \tag{2.4}$$

and

$$\mathbf{j} = \sigma \mathbf{e}.\tag{2.5}$$

Let

$$\mathbf{j} = \mathbf{j}' + i\mathbf{j}''$$
 and $\mathbf{e} = \mathbf{e}' + i\mathbf{e}''$. (2.6)

Then, (2.5) is equivalent to the system of equations

$$\begin{cases} \mathbf{j}' = \sigma' \mathbf{e}' - \sigma'' \mathbf{e}'', \\ \mathbf{j}'' = \sigma'' \mathbf{e}' + \sigma' \mathbf{e}'', \end{cases}$$
(2.7)

which is in turn equivalent to the following matrix equation:

$$\begin{bmatrix} \mathbf{e}' \\ \mathbf{j}'' \end{bmatrix} = \frac{1}{\sigma'} \begin{bmatrix} \mathbf{I} & \sigma'' \mathbf{I} \\ \sigma'' \mathbf{I} & (\sigma'^2 + \sigma''^2) \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{j}' \\ \mathbf{e}'' \end{bmatrix} =: \mathbf{D}_{JE} \begin{bmatrix} \mathbf{j}' \\ \mathbf{e}'' \end{bmatrix}, \tag{2.8}$$

where I is 2×2 identity matrix. We then have the variational principle of Cherkaev and Gibiansky [6]: for a given Cauchy datum (ϕ, q) on $\partial\Omega$,

$$\left\langle \begin{bmatrix} \mathbf{j}' \\ \mathbf{e}'' \end{bmatrix} \cdot \mathbf{D}_{JE} \begin{bmatrix} \mathbf{j}' \\ \mathbf{e}'' \end{bmatrix} \right\rangle = \min_{\substack{\underline{\mathbf{e}''} = -\nabla_{\underline{\mathbf{u}''}} \\ \underline{\mathbf{u}''} = \phi'' \text{ on } \partial \Omega}} \nabla_{\mathbf{j}' \cdot \mathbf{n} = -q' \text{ on } \partial \Omega} \left\langle \begin{bmatrix} \underline{\mathbf{j}'} \\ \underline{\mathbf{e}''} \end{bmatrix} \cdot \mathbf{D}_{JE} \begin{bmatrix} \underline{\mathbf{j}'} \\ \underline{\mathbf{e}''} \end{bmatrix} \right\rangle, \tag{2.9}$$

where $\langle f \rangle$ denotes the average of f over Ω , namely,

$$\langle f \rangle := \frac{1}{|\Omega|} \int_{\Omega} f.$$

From now on, we put $\mathbf{D}_{JE} = \mathbf{D}$ for ease of notation.

We now introduce a parameter θ ($0 \le \theta < 2\pi$) and parameterized variational principle following [19]. Let $\tilde{\mathbf{j}} := e^{i\theta}\mathbf{j}$ and $\tilde{\mathbf{e}} := e^{i\theta}\mathbf{e}$. Then we have

$$\widetilde{\mathbf{e}} = -\nabla(ue^{i\theta})$$
 and $\widetilde{\mathbf{j}} = \sigma\widetilde{\mathbf{e}}$,

and hence

$$\nabla \cdot \widetilde{\mathbf{i}} = 0$$
 and $\nabla \times \widetilde{\mathbf{e}} = 0$ in Ω .

Then we have

$$\begin{bmatrix} \widetilde{\mathbf{e}}' \\ \widetilde{\mathbf{j}}'' \end{bmatrix} = \mathbf{D}_{JE} \begin{bmatrix} \widetilde{\mathbf{j}}' \\ \widetilde{\mathbf{e}}'' \end{bmatrix}, \tag{2.10}$$

and the variational principle:

$$\left\langle \begin{bmatrix} \widetilde{\mathbf{j}}' \\ \widetilde{\mathbf{e}}'' \end{bmatrix} \cdot \mathbf{D}_{JE} \begin{bmatrix} \widetilde{\mathbf{j}}' \\ \widetilde{\mathbf{e}}'' \end{bmatrix} \right\rangle = \min \left\langle \begin{bmatrix} \widetilde{\mathbf{j}}' \\ \widetilde{\underline{\mathbf{e}}}'' \end{bmatrix} \cdot \mathbf{D}_{JE} \begin{bmatrix} \widetilde{\underline{\mathbf{j}}}' \\ \widetilde{\underline{\mathbf{e}}}'' \end{bmatrix} \right\rangle, \tag{2.11}$$

where the minimization is over the trial fields $\underline{\widetilde{\bf e}''}$ and $\underline{\widetilde{\bf j}'}$ such that

$$\begin{cases} \underline{\widetilde{\mathbf{e}}''} = -\nabla(\Im(\underline{u}e^{i\theta})), & \nabla \cdot \underline{\widetilde{\mathbf{j}}'} = 0 & \text{in } \Omega, \\ \underline{u'} = \phi', & \underline{u''} = \phi'', & \underline{\widetilde{\mathbf{j}}'} \cdot \mathbf{n} = -\Re(qe^{i\theta}) & \text{on } \partial\Omega. \end{cases}$$
(2.12)

Here and throughout this paper $\Re(z)$ and $\Im(z)$ stand for the real and imaginary parts of z, respectively.

Let ϕ_j (j=1,2) be given functions (Dirichlet data) defined on $\partial\Omega$ and u_j be the solution to (1.3) when $\phi = \phi_j$. Let $q_j = \sigma \nabla u_j \cdot \mathbf{n}|_{\partial\Omega}$. Then $\mathbf{e}_j = -\nabla u_j$ and $\mathbf{j}_j = -\sigma \nabla u_j$ satisfy (2.4) and (2.5), and $\mathbf{j}_j \cdot \mathbf{n} = -q_j$ on $\partial\Omega$. Set

$$\mathbf{v}_{j} = \begin{bmatrix} \widetilde{\mathbf{j}}_{j}' \\ \widetilde{\mathbf{e}}_{j}'' \end{bmatrix}, \quad j = 1, 2, \tag{2.13}$$

where $\widetilde{\mathbf{j}}_j := e^{i\theta_j}\mathbf{j}$ and $\widetilde{\mathbf{e}}_j := e^{i\theta_j}\mathbf{e}_j$. The measurement (response) matrix is given by $\mathbf{A} = (a_{jk})_{j,k=1,2}$ where

$$a_{jk} = \langle \mathbf{v}_j \cdot \mathbf{D} \mathbf{v}_k \rangle. \tag{2.14}$$

We emphasize that a_{jk} is a null-Lagrangian, i.e., it can be computed from the boundary measurements. In fact, we have from (2.8) that

$$a_{jk} = \frac{1}{|\Omega|} \int_{\Omega} \left[\widetilde{\mathbf{j}}_{j}' \right] \cdot \mathbf{D} \left[\widetilde{\mathbf{j}}_{k}' \right] = \frac{1}{|\Omega|} \int_{\Omega} \left[\widetilde{\mathbf{j}}_{j}' \right] \cdot \left[\widetilde{\mathbf{e}}_{k}' \right]$$

$$= \frac{1}{|\Omega|} \int_{\partial\Omega} \left[\Re(q_{j}e^{i\theta_{j}}) \Re(\phi_{k}e^{i\theta_{k}}) + \Im(q_{k}e^{i\theta_{k}}) \Im(\phi_{j}e^{i\theta_{j}}) \right] ds.$$
(2.15)

It is worth mentioning that the measurement matrix **A** depends on the two independent parameters θ_1 and θ_2 .

Let

$$\mathbf{R}_{\perp} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and define for real numbers t_1 and t_2

$$\mathcal{R} = \mathcal{R}(t_1, t_2) := \begin{bmatrix} t_1 \mathbf{R}_{\perp} & \mathbf{O} \\ \mathbf{O} & t_2 \mathbf{R}_{\perp} \end{bmatrix}. \tag{2.16}$$

Since $\mathcal{R}^T = -\mathcal{R}$, we have $\langle \mathbf{v}_j \cdot \mathcal{R} \mathbf{v}_j \rangle = 0$ for j = 1, 2. Let

$$b = b(t_1, t_2) := \langle \mathbf{v}_1 \cdot \mathcal{R} \mathbf{v}_2 \rangle = -\langle \mathbf{v}_2 \cdot \mathcal{R} \mathbf{v}_1 \rangle. \tag{2.17}$$

Then b can be written as

$$b = \alpha_1 t_1 + \alpha_2 t_2, \tag{2.18}$$

where

$$\alpha_1 := \langle \widetilde{\mathbf{j}}_1' \cdot \mathbf{R}_{\perp} \widetilde{\mathbf{j}}_2' \rangle, \quad \alpha_2 := \langle \widetilde{\mathbf{e}}_1'' \cdot \mathbf{R}_{\perp} \widetilde{\mathbf{e}}_2'' \rangle. \tag{2.19}$$

We emphasize that α_1 and α_2 can be computed using the boundary data. In fact, since $\nabla \times \mathbf{R}_{\perp} \mathbf{j}_2 = -\nabla \cdot \mathbf{j}_2 = 0$, there is a potential ψ_2 such that $\mathbf{R}_{\perp} \mathbf{j}_2 = \nabla \psi_2$ in Ω . Thus, if \mathbf{t} denotes the unit tangent vector on $\partial \Omega$, then we have

$$\mathbf{t} \cdot \nabla \psi_2 = \mathbf{t} \cdot \mathbf{R}_{\perp} \mathbf{j}_2 = -\mathbf{n} \cdot \mathbf{j}_2 = q_2 \quad \text{on } \partial \Omega.$$
 (2.20)

So the boundary value ψ_2^0 of ψ_2 on $\partial\Omega$ is given by

$$\psi_2^0(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} q_2 ds, \quad \mathbf{x} \in \partial\Omega,$$

where the integration is along $\partial\Omega$ in the positive orientation (counterclockwise). Hence

$$\alpha_{1} = \frac{1}{|\Omega|} \int_{\Omega} \widetilde{\mathbf{j}}_{1}' \cdot (\cos \theta_{2} \nabla \psi_{2}' - \sin \theta_{2} \nabla \psi_{2}'') d\mathbf{x}$$

$$= \frac{1}{|\Omega|} \int_{\partial \Omega} \widetilde{\mathbf{j}}_{1}' \cdot \mathbf{n} (\cos \theta_{2} \psi_{2}' - \sin \theta_{2} \psi_{2}'') ds(\mathbf{x})$$

$$= -\frac{1}{|\Omega|} \int_{\partial \Omega} \Re(q_{1} e^{i\theta_{1}}) \Re(\psi_{2}^{0} e^{i\theta_{2}}) ds(\mathbf{x})$$

$$= -\frac{1}{|\Omega|} \int_{\partial \Omega} \Re(q_{1} e^{i\theta_{1}}) \Re(\int_{\mathbf{x}_{0}}^{\mathbf{x}} q_{2} e^{i\theta_{2}} ds) ds(\mathbf{x}). \tag{2.21}$$

Since $\nabla \cdot (\mathbf{R}_{\perp} \nabla u_2) = 0$, we also have

$$\alpha_{2} = \frac{1}{|\Omega|} \int_{\Omega} (\cos \theta_{1} \nabla u_{1}'' + \sin \theta_{1} \nabla u_{1}') \cdot (\cos \theta_{2} \mathbf{R}_{\perp} \nabla u_{2}'' + \sin \theta_{2} \mathbf{R}_{\perp} \nabla u_{2}')$$

$$= \frac{1}{|\Omega|} \int_{\partial \Omega} (u_{1}'' \cos \theta_{1} + u_{1}' \sin \theta_{1}) \frac{\partial}{\partial \mathbf{t}} (u_{2}'' \cos \theta_{2} + u_{2}' \sin \theta_{2}) ds$$

$$= \frac{1}{|\Omega|} \int_{\partial \Omega} \Im(\phi_{1} e^{i\theta_{1}}) \Im\left(\frac{\partial \phi_{2}}{\partial \mathbf{t}} e^{i\theta_{2}}\right) ds. \tag{2.22}$$

3. Translation of the variational principle. We now apply the translation method to derive bounds for f_1 , the volume fraction of the phase 1.

We first note that $\begin{bmatrix} 0 & t_3 \mathbf{I} \\ t_3 \mathbf{I} & 0 \end{bmatrix}$ applied to fields $\begin{bmatrix} \widetilde{\mathbf{j}}' \\ \widetilde{\mathbf{e}}'' \end{bmatrix}$ is a null Lagrangian for any real number t_3 . Define

 $\mathcal{L} = \mathcal{L}(t_1, t_2, t_3)$ to be the translation of $\begin{bmatrix} \mathbf{D} & 0 \\ 0 & \mathbf{D} \end{bmatrix}$ by a null Lagrangian:

$$\mathcal{L} := \begin{bmatrix} \widetilde{\mathbf{D}} & \mathcal{R} \\ -\mathcal{R} & \widetilde{\mathbf{D}} \end{bmatrix}$$
 (3.1)

where

$$\widetilde{\mathbf{D}} := \mathbf{D} + \begin{bmatrix} 0 & t_3 \mathbf{I} \\ t_3 \mathbf{I} & 0 \end{bmatrix}. \tag{3.2}$$

We only consider parameters t_1 , t_2 , t_3 for which \mathcal{L} is positive semi-definite.

$$W := \left\langle \begin{bmatrix} k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 \\ k_3 \mathbf{v}_1 + k_4 \mathbf{v}_2 \end{bmatrix} \cdot \mathcal{L} \begin{bmatrix} k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 \\ k_3 \mathbf{v}_1 + k_4 \mathbf{v}_2 \end{bmatrix} \right\rangle$$
(3.3)

for real numbers k_1, \ldots, k_4 . One can see that

$$W = \begin{bmatrix} k_1 \\ \vdots \\ k_4 \end{bmatrix} \cdot \begin{bmatrix} \widetilde{\mathbf{A}} & b\mathbf{R}_{\perp} \\ -b\mathbf{R}_{\perp} & \widetilde{\mathbf{A}} \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_4 \end{bmatrix}, \tag{3.4}$$

where $\mathbf{\tilde{A}} = (\tilde{a}_{ik})$ with

$$\tilde{a}_{jk} := a_{jk} + t_3 \langle \widetilde{\mathbf{j}}_j' \cdot \widetilde{\mathbf{e}}_k'' + \widetilde{\mathbf{e}}_j'' \cdot \widetilde{\mathbf{j}}_k' \rangle, \tag{3.5}$$

and b is the number defined by (2.17). We emphasize that the new quantity \tilde{a}_{jk} is also determined by the boundary measurements since

$$\begin{split} \langle \widetilde{\mathbf{j}}_{j}' \cdot \widetilde{\mathbf{e}}_{k}'' + \widetilde{\mathbf{e}}_{j}'' \cdot \widetilde{\mathbf{j}}_{k}' \rangle &= \frac{1}{|\Omega|} \int_{\Omega} \left[-\widetilde{\mathbf{j}}_{j}' \cdot \nabla \Im(u_{k}e^{i\theta_{k}}) - \nabla \Im(u_{j}e^{i\theta_{j}}) \cdot \widetilde{\mathbf{j}}_{k}' \right] d\mathbf{x} \\ &= -\frac{1}{|\Omega|} \int_{\partial\Omega} \left[(\widetilde{\mathbf{j}}_{j}' \cdot \mathbf{n}) \Im(u_{k}e^{i\theta_{k}}) + (\widetilde{\mathbf{j}}_{k}' \cdot \mathbf{n}) \Im(u_{j}e^{i\theta_{j}}) \right] ds \\ &= \frac{1}{|\Omega|} \int_{\partial\Omega} \left[\Re(q_{j}e^{i\theta_{j}}) \Im(\phi_{k}e^{i\theta_{k}}) + \Re(q_{k}e^{i\theta_{k}}) \Im(\phi_{j}e^{i\theta_{j}}) \right] ds. \end{split}$$
(3.6)

Let

$$\mathcal{D} = \begin{bmatrix} \widetilde{\mathbf{A}} & b\mathbf{R}_{\perp} \\ -b\mathbf{R}_{\perp} & \widetilde{\mathbf{A}} \end{bmatrix}. \tag{3.7}$$

Since null Lagrangians are determined by boundary values, one can see from (2.9) that the following variational principle holds:

$$W = \min \left\langle \begin{bmatrix} k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 \\ k_3 \mathbf{v}_1 + k_4 \mathbf{v}_2 \end{bmatrix} \cdot \mathcal{L} \begin{bmatrix} k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 \\ k_3 \mathbf{v}_1 + k_4 \mathbf{v}_2 \end{bmatrix} \right\rangle, \tag{3.8}$$

where the minimum is taken over all $\underline{\mathbf{v}_j} = \begin{bmatrix} \widetilde{\mathbf{j}}_j' \\ \widetilde{\mathbf{e}}_j'' \end{bmatrix}$, j = 1, 2, satisfying

$$\begin{cases} \widetilde{\mathbf{e}}_{\underline{j}}^{"} = -\nabla \Im(\underline{u}_{\underline{j}}e^{i\theta_{\underline{j}}}), & \nabla \cdot \widetilde{\mathbf{j}}_{\underline{j}}^{'} = 0 & \text{in } \Omega, \\ \overline{u}_{\underline{j}}^{'} = \phi_{\underline{j}}^{'}, & u_{\underline{j}}^{"} = \phi_{\underline{j}}^{"}, & \widetilde{\mathbf{j}}_{\underline{j}}^{'} \cdot \mathbf{n} = -\Re(q_{\underline{j}}e^{i\theta_{\underline{j}}}) & \text{on } \partial\Omega. \end{cases}$$
(3.9)

If $\tilde{\mathbf{e}}_{i}^{"}$ and $\tilde{\mathbf{j}}_{i}^{'}$ satisfy (3.9), one can see that

$$\begin{split} &\widetilde{(\underline{\mathbf{j}}_j')} = \frac{1}{|\Omega|} \int_{\partial\Omega} \mathbf{x} \widetilde{\underline{\mathbf{j}}_j'} \cdot \mathbf{n} ds = -\frac{1}{|\Omega|} \int_{\partial\Omega} \mathbf{x} \Re(q_j e^{i\theta_j}) = \widetilde{(\underline{\mathbf{j}}_j')}, \\ &\langle \widetilde{\underline{\mathbf{e}}_j''} \rangle = -\frac{1}{|\Omega|} \int_{\partial\Omega} \Im(\underline{u_j} e^{i\theta_j}) \mathbf{n} = -\frac{1}{|\Omega|} \int_{\partial\Omega} \Im(\phi_j e^{i\theta_j}) \mathbf{n} = \langle \widetilde{\mathbf{e}}_j'' \rangle. \end{split}$$

Hence by relaxing the constraints (3.9) for minimization we have

$$W \ge \min_{\langle \mathbf{v}_j \rangle = \langle \mathbf{v}_j \rangle} \left\langle \begin{bmatrix} k_1 \underline{\mathbf{v}_1} + k_2 \underline{\mathbf{v}_2} \\ k_3 \underline{\mathbf{v}_1} + k_4 \underline{\mathbf{v}_2} \end{bmatrix} \cdot \mathcal{L} \begin{bmatrix} k_1 \underline{\mathbf{v}_1} + k_2 \underline{\mathbf{v}_2} \\ k_3 \underline{\mathbf{v}_1} + k_4 \underline{\mathbf{v}_2} \end{bmatrix} \right\rangle. \tag{3.10}$$

Here the existence of minimum is guaranteed by the positive semi-definiteness of \mathcal{L} . If the pair $(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2)$ is a minimizer of the righthand side of (3.10), then we have

$$\begin{split} 0 &= \frac{d}{dt}\Big|_{t=0} \Big\langle \begin{bmatrix} k_1(\hat{\mathbf{v}}_1 + t\psi_1) + k_2(\hat{\mathbf{v}}_2 + t\psi_2) \\ k_3(\hat{\mathbf{v}}_1 + t\psi_1) + k_4(\hat{\mathbf{v}}_2 + t\psi_2) \end{bmatrix} \cdot \mathcal{L} \begin{bmatrix} k_1(\hat{\mathbf{v}}_1 + t\psi_1) + k_2(\hat{\mathbf{v}}_2 + t\psi_2) \\ k_3(\hat{\mathbf{v}}_1 + t\psi_1) + k_4(\hat{\mathbf{v}}_2 + t\psi_2) \end{bmatrix} \Big\rangle \\ &= 2 \Big\langle \begin{bmatrix} k_1\psi_1 + k_2\psi_2 \\ k_3\psi_1 + k_4\psi_2 \end{bmatrix} \cdot \mathcal{L} \begin{bmatrix} k_1\hat{\mathbf{v}}_1 + k_2\hat{\mathbf{v}}_2 \\ k_3\hat{\mathbf{v}}_1 + k_4\hat{\mathbf{v}}_2 \end{bmatrix} \Big\rangle \end{split}$$

for any pair (ψ_1, ψ_2) satisfying $\langle \psi_1 \rangle = \langle \psi_2 \rangle = 0$. Thus we have

$$\mathcal{L}\begin{bmatrix}k_1\hat{\mathbf{v}}_1 + k_2\hat{\mathbf{v}}_2\\k_3\hat{\mathbf{v}}_1 + k_4\hat{\mathbf{v}}_2\end{bmatrix} = \mu \tag{3.11}$$

for some constant vector μ .

Let \mathcal{L}_1 and \mathcal{L}_2 be restrictions of \mathcal{L} to phase 1 and phase 2, respectively, *i.e.*,

$$\mathcal{L} = \mathcal{L}_1 \chi_1 + \mathcal{L}_2 \chi_2.$$

Note that \mathcal{L}_1 and \mathcal{L}_2 are 8×8 constant matrices. The relation (3.11) says that the component of $\begin{bmatrix} k_1\hat{\mathbf{v}}_1 + k_2\hat{\mathbf{v}}_2 \\ k_3\hat{\mathbf{v}}_1 + k_4\hat{\mathbf{v}}_2 \end{bmatrix}\chi_1$ which is orthogonal to $\ker \mathcal{L}_1$ is constant. Likewise, the component of $\begin{bmatrix} k_1\hat{\mathbf{v}}_1 + k_2\hat{\mathbf{v}}_2 \\ k_3\hat{\mathbf{v}}_1 + k_4\hat{\mathbf{v}}_2 \end{bmatrix}\chi_2$ orthogonal to $\ker \mathcal{L}_2$ is constant. Since components in $\ker \mathcal{L}_1$ and $\ker \mathcal{L}_2$ do not contribute to minimum value in (3.10), we obtain

$$W \ge \min_{f_1 V_1 + f_2 V_2 = V} \left(f_1 V_1 \cdot \mathcal{L}_1 V_1 + f_2 V_2 \cdot \mathcal{L}_2 V_2 \right), \tag{3.12}$$

where V_1 and V_2 are constant vectors and

$$V := \begin{bmatrix} k_1 \langle \mathbf{v}_1 \rangle + k_2 \langle \mathbf{v}_2 \rangle \\ k_3 \langle \mathbf{v}_1 \rangle + k_4 \langle \mathbf{v}_2 \rangle \end{bmatrix}. \tag{3.13}$$

We use the following lemma whose proof will be given in Appendix.

LEMMA 3.1. Let V be a finite dimensional vector space, \mathcal{L}_1 , $\mathcal{L}_2: V \to V$ self-adjoint linear operators, f_1 and f_2 positive numbers, and $E_0 \in V$. Then

$$\min_{f_1 E_1 + f_2 E_2 = E_0} \left(f_1 E_1 \cdot \mathcal{L}_1 E_1 + f_2 E_2 \cdot \mathcal{L}_2 E_2 \right) = (\pi E_0) \cdot \left[\pi \left(f_1 \mathcal{L}_1^{-1} + f_2 \mathcal{L}_2^{-1} \right) \pi \right]^{-1} \pi E_0, \tag{3.14}$$

where π is the orthogonal projection onto Range $\mathcal{L}_1 \cap \text{Range } \mathcal{L}_2$ and all the inverses are pseudo-inverses.

Let π be the orthogonal projection onto Range $\mathcal{L}_1 \cap \text{Range } \mathcal{L}_2$. Using Lemma 3.1, we know that the minimum on the righthand side of (3.12) is $V \cdot \mathcal{L}_*V$ where

$$\mathcal{L}_* := \pi \left(\pi (f_1 \mathcal{L}_1^{-1} + f_2 \mathcal{L}_2^{-1}) \pi \right)^{-1} \pi. \tag{3.15}$$

So, we have

$$W \ge V \cdot \mathcal{L}_* V$$
.

We finally obtain from (3.4) and (3.7) that

$$\begin{bmatrix} k_1 \\ \vdots \\ k_4 \end{bmatrix} \cdot \mathcal{D} \begin{bmatrix} k_1 \\ \vdots \\ k_4 \end{bmatrix} \ge \begin{bmatrix} k_1 \langle \mathbf{v}_1 \rangle + k_2 \langle \mathbf{v}_2 \rangle \\ k_3 \langle \mathbf{v}_1 \rangle + k_4 \langle \mathbf{v}_2 \rangle \end{bmatrix} \cdot \mathcal{L}_* \begin{bmatrix} k_1 \langle \mathbf{v}_1 \rangle + k_2 \langle \mathbf{v}_2 \rangle \\ k_3 \langle \mathbf{v}_1 \rangle + k_4 \langle \mathbf{v}_2 \rangle \end{bmatrix}. \tag{3.16}$$

We emphasize that \mathcal{L}_* depends on the parameters t_1, t_2, t_3 . We will choose these parameters in a special way and calculate the corresponding \mathcal{L}_* in the next section. In doing so, the following observation plays a crucial role. Let

$$\mathbf{J} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$
(3.17)

The matrix \mathbf{J} has very special properties: it is an orthogonal matrix, namely, $\mathbf{J}\mathbf{J}^T = \mathbf{I}$, and the following holds:

$$\mathcal{L} = \mathbf{J} \begin{bmatrix} \mathbf{D}_{t_3} + \mathbf{T} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{D}_{t_3} - \mathbf{T} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{D}_{t_3} + \mathbf{T} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{D}_{t_3} - \mathbf{T} \end{bmatrix} \mathbf{J}^T, \tag{3.18}$$

where

$$\mathbf{D}_{t_3} := \begin{bmatrix} \frac{1}{\sigma'} & \frac{\sigma''}{\sigma'} + t_3 \\ \frac{\sigma''}{\sigma'} + t_3 & \frac{\sigma'^2 + \sigma''^2}{\sigma'} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}. \tag{3.19}$$

4. Translation bounds. One can see from (3.18) that $\mathcal{L} \geq 0$ if and only if

$$\mathbf{D}_{t_3} + \mathbf{T} \ge 0, \quad \mathbf{D}_{t_3} - \mathbf{T} \ge 0. \tag{4.1}$$

We choose parameters (t_1, t_2, t_3) so that \mathcal{L} is positive semi-definite, more precisely the sum of the ranks of matrices \mathcal{L}_1 and \mathcal{L}_2 is minimized. Let

$$\mathbf{P}_1^{\pm} := \mathbf{D}_{t_3}|_{\mathrm{phase1}} \pm \mathbf{T}, \quad \mathbf{P}_2^{\pm} := \mathbf{D}_{t_3}|_{\mathrm{phase2}} \pm \mathbf{T}.$$

Then, (t_1, t_2, t_3) are chosen to be minimizers of

$$\min_{t_1, t_2, t_3} \left[\operatorname{rank} \mathbf{P}_1^+ + \operatorname{rank} \mathbf{P}_1^- + \operatorname{rank} \mathbf{P}_2^+ + \operatorname{rank} \mathbf{P}_2^- \right]. \tag{4.2}$$

Such a rank minimizing condition has been used in [14, 15].

Since rank $\mathbf{P}_{i}^{\pm} \geq 1$, we have

$$\min \left[\operatorname{rank} \mathbf{P}_{1}^{+} + \operatorname{rank} \mathbf{P}_{1}^{-} + \operatorname{rank} \mathbf{P}_{2}^{+} + \operatorname{rank} \mathbf{P}_{2}^{-} \right] = 5, \tag{4.3}$$

and hence there are four possibilities:

$$\det \mathbf{P}_1^+ > 0$$
, $\det \mathbf{P}_1^- = 0$, $\det \mathbf{P}_2^{\pm} = 0$, (4.4)

$$\det \mathbf{P}_{2}^{+} > 0$$
, $\det \mathbf{P}_{1}^{\pm} = 0$, $\det \mathbf{P}_{2}^{-} = 0$, (4.5)

$$\det \mathbf{P}_{1}^{-} > 0, \quad \det \mathbf{P}_{1}^{+} = 0, \quad \det \mathbf{P}_{2}^{\pm} = 0, \tag{4.6}$$

$$\det \mathbf{P}_{2}^{-} > 0, \quad \det \mathbf{P}_{1}^{\pm} = 0, \quad \det \mathbf{P}_{2}^{+} = 0.$$
 (4.7)

The possibilities (4.4) and (4.5) yield upper and lower bounds for f_1 as we shall see shortly. But, (4.6) and (4.7) are equivalent to (4.4) and (4.5), respectively, changing signs of t_1 and t_2 , and hence they yield the same bounds.

Suppose that (t_1, t_2, t_3) satisfies (4.4). Following [7] (see also [16, Section 23.7]), we interpret this condition in terms of circles. By explicit calculations, one can see that the last three conditions in (4.4) are equivalent to the fact that $(-\sigma'_1, -\sigma''_1)$, (σ'_2, σ''_2) , $(-\sigma'_2, -\sigma''_2)$ pass through the circle

$$t_1(x^2 + y^2) + (1 + t_1t_2 - t_3^2)x - 2t_3y + t_2 = 0. (4.8)$$

Under the last condition in (2.2), the circle is determined uniquely and t_1 , t_2 , t_3 are given as follows:

$$1/t_1 = r\sigma_2'' \pm \sqrt{(r^2 + 1)|\sigma_2|^2}, \quad t_2 = -|\sigma_2|^2 t_1, \quad t_3 = r\sigma_2' t_1$$
(4.9)

where

$$r := \frac{|\sigma_1|^2 - |\sigma_2|^2}{2(\sigma_1'\sigma_2'' - \sigma_2'\sigma_1'')}. (4.10)$$

Moreover, the second condition in (2.2) guarantees the first condition in (4.4).

There are additional conditions for t_1 , t_2 , t_3 to fulfill. To ensure (4.1), they should satisfy

$$|t_1| \le 1/\sigma_1', \quad |t_1| \le 1/\sigma_2', \quad \det \mathbf{P}_1^+ > 0.$$
 (4.11)

We show that these conditions can be fulfilled by choosing t_1 properly in (4.9).

Since $\det \mathbf{P}_2^+ = 0$, we have

$$|\sigma_2|^2 (1/(\sigma_2')^2 - t_1^2) = (1/\sigma_2' + t_1)(|\sigma_2|^2/\sigma_2' + t_2) = (\sigma_2''/\sigma_2' + t_3)^2 > 0,$$

and hence $|t_1| \leq 1/\sigma_2'$.

On the other hand, since $\det \mathbf{P}_1^- = 0$, we have

$$|\sigma_{1}|^{2} \left(\frac{1}{(\sigma'_{1})^{2}} - t_{1}^{2} \right) = \left(\frac{1}{\sigma'_{1}} - t_{1} \right) t_{1} (|\sigma_{1}|^{2} - |\sigma_{2}|^{2}) + \left(\frac{1}{\sigma'_{1}} - t_{1} \right) \left(\frac{|\sigma_{1}|^{2}}{\sigma'_{1}} - t_{2} \right)$$

$$= \left(\frac{1}{\sigma'_{1}} - t_{1} \right) t_{1} (|\sigma_{1}|^{2} - |\sigma_{2}|^{2}) + \left(\frac{\sigma''_{1}}{\sigma'_{1}} + t_{3} \right)^{2}. \tag{4.12}$$

Let f be a quadratic function whose roots are $1/t_1 = r\sigma_2'' \pm \sqrt{(r^2+1)|\sigma_2|^2}$. In fact, it is given by

$$f(x) := x^2 - 2r\sigma_2''x - r^2(\sigma_2')^2 - |\sigma_2|^2.$$

Then one can see that

$$f(\sigma_1') = -\frac{\left[((\sigma_1')^2 - (\sigma_2')^2) |\sigma_2|^2 - (\sigma_1'\sigma_2'' - \sigma_1''\sigma_2')^2 \right]^2}{4(\sigma_1'\sigma_2'' - \sigma_1''\sigma_2')^2 (\sigma_2')^2} \le 0. \tag{4.13}$$

Therefore we have

$$r\sigma_2'' - \sqrt{(r^2+1)|\sigma_2|^2} \le \sigma_1' \le r\sigma_2'' + \sqrt{(r^2+1)|\sigma_2|^2},$$

and hence we can choose t_1 (among $1/t_1 = r\sigma_2'' \pm \sqrt{(r^2+1)|\sigma_2|^2}$) so that

$$\left(\frac{1}{\sigma_1'} - t_1\right) t_1(|\sigma_1|^2 - |\sigma_2|^2) \ge 0. \tag{4.14}$$

Then (4.12) implies $|t_1| \leq 1/\sigma_1'$. Here we know $t_1 \neq 1/\sigma_1'$ because $t_1 = 1/\sigma_1'$ would imply $\sigma_2' > \sigma_1'$ so that $0 = f(1/t_1) = f(\sigma_1') < 0$ by (4.13). Thus the condition det $\mathbf{P}_1^+ > 0$ (equivalently, $t_1(|\sigma_1|^2 - |\sigma_2|^2) > 0$) is satisfied automatically with the choice of t_1 satisfying (4.14).

Now we calculate \mathcal{L}_* . First we observe that

$$\det\left(\mathbf{P}_{1}^{\pm} - \mathbf{P}_{2}^{\pm}\right) = \det\left(\mathbf{D}_{t_{3}}|_{\text{phase}1} - \mathbf{D}_{t_{3}}|_{\text{phase}2}\right) = -\frac{(\sigma'_{1} - \sigma'_{2})^{2} + (\sigma''_{1} - \sigma''_{2})^{2}}{\sigma'_{1}\sigma'_{2}} < 0. \tag{4.15}$$

Since $\det \mathbf{P}_1^- = \det \mathbf{P}_2^- = 0$ while $\mathbf{P}_1^- - \mathbf{P}_2^-$ has rank 2, we have

range
$$\mathbf{P}_1^- \cap \text{range } \mathbf{P}_2^- = 0.$$
 (4.16)

Since $\det \mathbf{P}_1^+ \neq 0$, we have

range
$$\mathbf{P}_1^+ \cap \text{range } \mathbf{P}_2^+ = \text{range } \mathbf{P}_2^+.$$
 (4.17)

Recalling from (3.18) that

$$\mathcal{L}_{j} = \mathbf{J} egin{bmatrix} \mathbf{P}_{j}^{+} & \mathbf{O} & \mathbf{O} & \mathbf{O} \ \mathbf{O} & \mathbf{P}_{j}^{-} & \mathbf{O} & \mathbf{O} \ \mathbf{O} & \mathbf{O} & \mathbf{P}_{j}^{+} & \mathbf{O} \ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{P}_{j}^{-} \end{bmatrix} \mathbf{J}^{T} \,, \quad j = 1, 2,$$

it follows that

range
$$\mathcal{L}_1 \cap \text{range } \mathcal{L}_2 = \text{range } \mathbf{J} \begin{bmatrix} \mathbf{P}_2^+ & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{P}_2^+ & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix} \mathbf{J}^T$$
. (4.18)

Let \mathbf{p} be an unit vector generating range \mathbf{P}_2^+ , and let \mathbf{P} be the orthogonal projection onto range \mathbf{P}_2^+ , namely,

$$\mathbf{P} = \mathbf{p}\mathbf{p}^T$$
.

Then the orthogonal projection π onto range $\mathcal{L}_1 \cap \mathrm{range}\ \mathcal{L}_2$ is given by

$$\pi = \mathbf{J} \begin{bmatrix} \mathbf{P} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{P} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix} \mathbf{J}^{T}.$$
 (4.19)

Thus, we have

$$\pi(f_1\mathcal{L}_1^{-1} + f_2\mathcal{L}_2^{-1})\pi = f_1\mathbf{J} \begin{bmatrix} \mathbf{P}(\mathbf{P}_1^+)^{-1}\mathbf{P} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{P}(\mathbf{P}_1^+)^{-1}\mathbf{P} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix} \mathbf{J}^T$$

$$+ f_2\mathbf{J} \begin{bmatrix} \mathbf{P}(\mathbf{P}_2^+)^{-1}\mathbf{P} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{P}(\mathbf{P}_2^+)^{-1}\mathbf{P} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix} \mathbf{J}^T.$$

Here $(\mathbf{P}_{j}^{+})^{-1}$ is the pseudo-inverse. Since \mathbf{P}_{2}^{+} is symmetric, we have

$$\mathbf{P}_2^+ = (\operatorname{tr}\mathbf{P}_2^+)\mathbf{P}.\tag{4.20}$$

One can also see that

$$\mathbf{P}(\mathbf{P}_1^+)^{-1}\mathbf{P} = \left(\mathbf{p} \cdot (\mathbf{P}_1^+)^{-1}\mathbf{p}\right)\mathbf{P}.\tag{4.21}$$

Therefore, we have

$$\pi(f_1\mathcal{L}_1^{-1} + f_2\mathcal{L}_2^{-1})\pi = \left[f_1\left(\mathbf{p}\cdot(\mathbf{P}_1^+)^{-1}\mathbf{p}\right) + \frac{f_2}{\operatorname{tr}\mathbf{P}_2^+}\right]\pi,$$

and hence

$$\mathcal{L}_* := \pi \left(\pi (f_1 \mathcal{L}_1^{-1} + f_2 \mathcal{L}_2^{-1}) \pi \right)^{-1} \pi = \left[f_1 \left(\mathbf{p} \cdot (\mathbf{P}_1^+)^{-1} \mathbf{p} \right) + \frac{f_2}{\operatorname{tr} \mathbf{P}_2^+} \right]^{-1} \pi.$$
 (4.22)

By positive semi-definiteness of \mathbf{P}_1^+ and $\mathbf{P}_2^+,$ and by (4.4) and (4.15), we have

$$\mathbf{p} \cdot (\mathbf{P}_1^+)^{-1} \mathbf{p} > 0, \quad \text{tr} \mathbf{P}_2^+ > 0.$$
 (4.23)

Moreover we have

$$f_{1}\left(\mathbf{p}\cdot(\mathbf{P}_{1}^{+})^{-1}\mathbf{p}\right) + \frac{f_{2}}{\operatorname{tr}\mathbf{P}_{2}^{+}} = f_{1}\left(\mathbf{p}\cdot(\mathbf{P}_{1}^{+})^{-1}\mathbf{p} - \frac{1}{\operatorname{tr}\mathbf{P}_{2}^{+}}\right) + \frac{1}{\operatorname{tr}\mathbf{P}_{2}^{+}}$$
$$= -f_{1}\frac{\det(\mathbf{P}_{1}^{+} - \mathbf{P}_{2}^{+})}{\operatorname{tr}\mathbf{P}_{2}^{+}\det\mathbf{P}_{1}^{+}} + \frac{1}{\operatorname{tr}\mathbf{P}_{2}^{+}}.$$

We emphasize that

$$-\frac{\det(\mathbf{P}_{1}^{+} - \mathbf{P}_{2}^{+})}{\operatorname{tr}\mathbf{P}_{2}^{+}\det\mathbf{P}_{1}^{+}} > 0 \tag{4.24}$$

which is a consequence of (4.4), (4.15) and (4.23).

Let

$$F(f_1) := \left(-f_1 \frac{\det(\mathbf{P}_1^+ - \mathbf{P}_2^+)}{\operatorname{tr} \mathbf{P}_2^+ \det \mathbf{P}_1^+} + \frac{1}{\operatorname{tr} \mathbf{P}_2^+}\right)^{-1}.$$
 (4.25)

By (3.16) we obtain

$$\mathcal{D} = \begin{bmatrix} \widetilde{\mathbf{A}} & b\mathbf{R}_{\perp} \\ -b\mathbf{R}_{\perp} & \widetilde{\mathbf{A}} \end{bmatrix} \ge F(f_1)\mathcal{C}^T \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{P} \end{bmatrix} \mathcal{C}$$
(4.26)

where

$$C = \frac{1}{\sqrt{2}} \begin{bmatrix} \widetilde{j}'_{1,1} \rangle & \widetilde{j}'_{2,1} \rangle & \widetilde{j}'_{1,2} \rangle & \widetilde{j}'_{2,2} \rangle \\ \langle \widetilde{e}''_{1,1} \rangle & \langle \widetilde{e}''_{2,1} \rangle & \langle \widetilde{e}''_{1,2} \rangle & \langle \widetilde{e}''_{2,2} \rangle \\ -\langle \widetilde{j}'_{1,2} \rangle & -\langle \widetilde{j}'_{2,2} \rangle & \langle \widetilde{j}'_{1,1} \rangle & \langle \widetilde{j}'_{2,1} \rangle \\ -\langle \widetilde{e}''_{1,2} \rangle & -\langle \widetilde{e}''_{2,2} \rangle & \langle \widetilde{e}''_{1,1} \rangle & \langle \widetilde{e}''_{2,1} \rangle \end{bmatrix}.$$

$$(4.27)$$

Here $\tilde{j}'_{k,l}$ and $\tilde{e}''_{k,l}$ (k=1,2) are defined by

$$\widetilde{\mathbf{j}}_k' = \begin{bmatrix} \widetilde{j}_{k,1}' \\ \widetilde{j}_{k,2}' \end{bmatrix}, \quad \widetilde{\mathbf{e}}_k'' = \begin{bmatrix} \widetilde{e}_{k,1}'' \\ \widetilde{e}_{k,2}'' \end{bmatrix}.$$

We emphasize that \mathcal{C} can be computed using boundary data

Straightforward calculations show that $C \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{P} \end{bmatrix} C^T$ takes the form

$$C^{T} \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{P} \end{bmatrix} C := \begin{bmatrix} \mathbf{M} & m\mathbf{R}_{\perp} \\ -m\mathbf{R}_{\perp} & \mathbf{M} \end{bmatrix}$$
(4.28)

where \mathbf{M} is a 2 × 2 symmetric matrix and m is a real number, which can be computed from boundary values since so does \mathcal{C} . Since \mathbf{P} is singular, we know that $m^2 = \det \mathbf{M}$. Calculating the eigenvalues of the matrix appearing above, one can see that the inequality (4.26) is equivalent to the following two inequalities:

$$\operatorname{tr}\tilde{\mathbf{A}} \ge F(f_1)\operatorname{tr}\mathbf{M}$$
 (4.29)

and

$$\det\left(\tilde{\mathbf{A}} - F(f_1)\mathbf{M}\right) \ge (b - F(f_1)m)^2. \tag{4.30}$$

Inequality (4.29) yields a lower bound:

$$f_1 \ge -\frac{\operatorname{tr} \mathbf{P}_2^+ \det \mathbf{P}_1^+}{\det(\mathbf{P}_1^+ - \mathbf{P}_2^+)} \left(\frac{\operatorname{tr} \mathbf{M}}{\operatorname{tr} \tilde{\mathbf{A}}} - \frac{1}{\operatorname{tr} \mathbf{P}_2^+} \right). \tag{4.31}$$

Note that

$$\det\left(\tilde{\mathbf{A}} - F(f_1)\mathbf{M}\right) = \det\tilde{\mathbf{A}} - F(f_1)\operatorname{tr}(\tilde{\mathbf{A}}\mathbf{M}^*) + F(f_1)^2 \det\mathbf{M},\tag{4.32}$$

where \mathbf{M}^* is the adjugate matrix of \mathbf{M} . So, we obtain from (4.30) another lower bound:

$$f_1 \ge -\frac{\operatorname{tr}\mathbf{P}_2^+ \det \mathbf{P}_1^+}{\det(\mathbf{P}_1^+ - \mathbf{P}_2^+)} \left(\frac{\operatorname{tr}(\tilde{\mathbf{A}}\mathbf{M}^*) - 2bm}{\det \tilde{\mathbf{A}} - b^2} - \frac{1}{\operatorname{tr}\mathbf{P}_2^+} \right). \tag{4.33}$$

Observe that $\widetilde{\mathbf{A}}, b$ and thus \mathbf{M}, m depend on θ_1 and θ_2 while \mathbf{P}_1 and \mathbf{P}_2 do not. Denoting the quantities on the righthand sides of inequalities in (4.31) and (4.33) by $L_1(\theta_1, \theta_2)$ and $L_2(\theta_1, \theta_2)$, we have

$$f_1 \ge \max_{\theta_1, \theta_2} L_1(\theta_1, \theta_2) \vee \max_{\theta_1, \theta_2} L_2(\theta_1, \theta_2).$$
 (4.34)

Table 5.1

Two concentric disks. σ_1 : the conductivity of the inclusion D, σ_2 : conductivity of $\Omega \setminus \overline{D}$, f_1 : the exact area fraction of the inclusion, max(L): the lower bound, min(U): the upper bound.

σ_1	σ_2	f_1	max(L)	$max(L)/f_1$	min(U)	$min(U)/f_1$
1+i	1	0.16	0.159919	0.999492	0.160044	1.000274
2 + 0.5i	1	0.16	0.159944	0.999652	0.160015	1.000097
2+5i	1	0.16	0.159937	0.999608	0.160008	1.000049
4 + 100i	1	0.16	0.159839	0.998992	0.160026	1.000165

Here $a \vee b$ is the maximum of a and b. It is worth mentioning that the bound $L_j(\theta_1, \theta_2)$ is the same as the bound $L_j(0,0)$ when the boundary data are $e^{\theta_1}\phi_1$ and $e^{\theta_2}\phi_2$.

Now suppose that (t_1, t_2, t_3) satisfies (4.5). By interchanging the role of phase 1 and phase 2, we obtain

$$1 - f_1 = f_2 \ge -\frac{\operatorname{tr} \mathbf{P}_1^+ \det \mathbf{P}_2^+}{\det(\mathbf{P}_1^+ - \mathbf{P}_2^+)} \left(\frac{\operatorname{tr} \mathbf{M}}{\operatorname{tr} \tilde{\mathbf{A}}} - \frac{1}{\operatorname{tr} \mathbf{P}_1^+} \right), \tag{4.35}$$

$$1 - f_1 = f_2 \ge -\frac{\operatorname{tr} \mathbf{P}_1^+ \det \mathbf{P}_2^+}{\det(\mathbf{P}_1^+ - \mathbf{P}_2^+)} \left(\frac{\operatorname{tr}(\tilde{\mathbf{A}} \mathbf{M}^*) - 2bm}{\det \tilde{\mathbf{A}} - b^2} - \frac{1}{\operatorname{tr} \mathbf{P}_1^+} \right). \tag{4.36}$$

Here the matrix \mathbf{M} and the constant m are defined by (4.28), but \mathbf{P} here is the orthogonal projection onto range \mathbf{P}_1^+ , not range \mathbf{P}_2^+ .

Let

$$U_1(\theta_1, \theta_2) = 1 + \frac{\operatorname{tr} \mathbf{P}_1^+ \det \mathbf{P}_2^+}{\det(\mathbf{P}_1^+ - \mathbf{P}_2^+)} \left(\frac{\operatorname{tr} \mathbf{M}}{\operatorname{tr} \tilde{\mathbf{A}}} - \frac{1}{\operatorname{tr} \mathbf{P}_1^+} \right),$$

$$U_2(\theta_1, \theta_2) = 1 + \frac{\operatorname{tr} \mathbf{P}_1^+ \det \mathbf{P}_2^+}{\det(\mathbf{P}_1^+ - \mathbf{P}_2^+)} \left(\frac{\operatorname{tr} (\tilde{\mathbf{A}} \mathbf{M}^*) - 2bm}{\det \tilde{\mathbf{A}} - b^2} - \frac{1}{\operatorname{tr} \mathbf{P}_1^+} \right).$$

Then we have

$$f_1 \le \min_{\theta_1, \theta_2} U_1(\theta_1, \theta_2) \land \min_{\theta_1, \theta_2} U_2(\theta_1, \theta_2). \tag{4.37}$$

Here $a \wedge b$ is the minimum of a and b.

5. Numerical experiments. This section presents results of some numerical experiments. We compute the bounds for various configurations: (1) the domain is a disk and the inclusion is a concentric disk (Fig. 5.1, Table 5.1), (2) domain: a disk, inclusion: an ellipse (Fig. 5.2, Table 5.2), (3) multiple inclusions (Fig. 5.3, Table 5.3), (4) the domain of general shape (Fig. 5.4, Table 5.4). The results clearly show that bounds obtained in this paper are quite tight, very close to the actual volume fraction. In all computations, we use the Dirichlet boundary data $\phi_1 = x$ and $\phi_2 = y$, and acquire the corresponding Neumann data by solving (1.3) numerically using FEM. We then discretize $[0, 2\pi)$ into 200 points, which means 200×200 pairs of (θ_1, θ_2) are used to optimize the bounds.

We also consider stability of the bounds under measurement noise. In the example of multiple inclusions we add 5, 10, 15, 20% noise to the Neumann data. We first compute ∇u by solving (1.3) corresponding to the Dirichlet data $\phi_1 = x$ and $\phi_2 = y$, and then compute

$$\nabla u^* = [1 + p * \text{rand}] \nabla u$$

for p = 0.05, 0.1, 0.15, 0.2 where rand is a generator of Gaussian white noise. So the measured data (with noise) is $q = \nabla u^* \cdot \mathbf{n}$. As Table 5.3 shows, the bounds are stable under measurement noise.

Finally we took a configuration (Fig. 5.5) which was considered in [25] for the purpose of comparing bounds by the splitting method and those of this paper (translation method). The results presented in Table 5.5 show that the method of this paper yields better bounds than the slitting method. It is worth emphasizing that the splitting method in [25] uses a single measurement while the translation method uses two measurements.

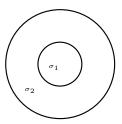


Fig. 5.1. Concentric disks. Inclusion D is the disk of radius $r_1=0.4$ and of conductivity σ_1 . Ω is the unit disk and the conductivity of $\Omega\setminus\overline{D}$ is $\sigma_2=1$.

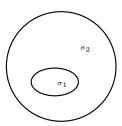


Fig. 5.2. Ω is the unit disk, D is an ellipse with center point lying at (-0.1, -0.3) with the major axis a=0.4 and the minor axis b=0.3. The conductivity of inclusion is $\sigma_1=2+i$ and the background conductivity is $\sigma_2=1$.



FIG. 5.3. Ω is the unit disk, D is composed of three parts: two circles with radius 0.25 centered at (-0.4, 0.3) and (0.4, 0.3), and one crescent inclusion with area 0.0225. The conductivity of the three inclusions is $\sigma_1 = 2 + i$ and the background conductivity is $\sigma_2 = 1$. The exact area is $f_1 = 0.1475$.

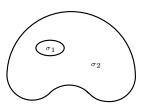


Fig. 5.4. Ω is of general shape. D is with conductivity $\sigma_1=2+i$ and the background conductivity is $\sigma_2=1$. The exact area fraction is $f_1=0.029281$.

Table 5.2
Elliptic inclusion

σ_1	σ_2	f_1	max(L)	$max(L)/f_1$	min(U)	$min(U)/f_1$
2+i	1	0.12	0.119559	0.996324	0.120800	1.00667

Table 5.3 Multiple inclusions. $\sigma_1 = 2 + i, \ \sigma_2 = 1, \ f_1 = 0.1475.$

noise level	max(L)	$max(L)/f_1$	min(U)	$min(U)/f_1$
0	0.146614	0.993991	0.148187	1.004655
5%	0.143527	0.973066	0.151170	1.024883
10%	0.134217	0.909943	0.159726	1.082891
15%	0.119537	0.810420	0.174828	1.185274
20%	0.098495	0.667764	0.194967	1.321812

Conclusion. We have derived upper and lower bounds of the volume fraction of an unknown inclusion (or two phase composites) using boundary measurements when the conductivity is complex. We use the minimizing variational principles with parameters for the fields $\mathbf{e} = \mathbf{e}' + i\mathbf{e}''$ and $\mathbf{j} = \mathbf{j}' + i\mathbf{j}''$. The bounds are given in a nonlinear way in terms of the determinant and the trace of the measurement matrix, and some other null Lagrangians which can be computed using boundary measurements. We perform numerical experiments to validate the effectiveness of the bounds obtained in this paper and to compare them with those in [25]. Results show that the bounds obtained in this paper are quite tight and stable under measurement (white) noise. They also show that these bounds are better than those obtained in [25] using less boundary measurement data.

Appendix A. Proof of Lemma 3.1.

This appendix is to prove Lemma 3.1.

We consider the following minimization problem:

$$\min_{E_1, E_2 \in V} \left(f_1 E_1 \cdot \mathcal{L}_1 E_1 + f_2 E_2 \cdot \mathcal{L}_2 E_2 - 2A \cdot \left(f_1 E_1 + f_2 E_2 \right) \right), \tag{A.1}$$

where the Lagrange multiplier A is a vector in Range $\mathcal{L}_1 \cap \text{Range } \mathcal{L}_2$ (otherwise there is no minimum). If E_1 and E_2 are minimizers, they should satisfy

$$2f_1 \delta E_1 \cdot (\mathcal{L}_1 E_1 - A) = 0, \quad 2f_2 \delta E_2 \cdot (\mathcal{L}_2 E_2 - A) = 0$$
 (A.2)

for all the increments δE_1 and δE_2 . Then

$$E_1 = \mathcal{L}_1^{-1} A + E_1^0, \quad E_2 = \mathcal{L}_2^{-1} A + E_2^0$$
 (A.3)

for some $E_1^0 \in \ker \mathcal{L}_1$ and $E_2^0 \in \ker \mathcal{L}_2$. Since Range \mathcal{L}_j is orthogonal to $\ker \mathcal{L}_j$, if we impose the constraint $f_1E_1 + f_2E_2 = E_0$, then we have

$$\pi E_0 = \pi (f_1 \mathcal{L}_1^{-1} + f_2 \mathcal{L}_2^{-1}) \pi A,$$

and hence

$$A = \left[\pi (f_1 \mathcal{L}_1^{-1} + f_2 \mathcal{L}_2^{-1}) \pi \right]^{-1} \pi E_0.$$
 (A.4)

And we have

$$f_1 E_1 \cdot \mathcal{L}_1 E_1 + f_2 E_2 \cdot \mathcal{L} E_2 = f_1 A \cdot \mathcal{L}_1^{-1} A + f_2 A \cdot \mathcal{L}_2^{-1} A$$
$$= (\pi E_0) \cdot \left[\pi (f_1 \mathcal{L}_1^{-1} + f_2 \mathcal{L}_2^{-1}) \pi \right]^{-1} \pi E_0.$$

This completes the proof.

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 $\begin{array}{c} \text{Table 5.4} \\ \text{General shape domain} \end{array}$

σ_1	σ_2	f_1	max(L)	$max(L)/f_1$	min(U)	$min(U)/f_1$
1+2i	1	0.029281	0.029172	0.996278	0.029631	1.011941

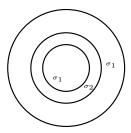


Fig. 5.5. The configuration from [25]. Phase 1 consist of the core and the outer annulus, and its area fraction is $f_1=0.8$. The conductivity of phase 1 is $\sigma_1=3+8i$, and that of phase 2 is $\sigma_2=8+6i$. The radii of three circles are $R_1=2$, $R_2=3$, $R_3=5$.

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Table 5.5 Comparison with the splitting method results. $\sigma_1 = 3 + 8i, \ \sigma_2 = 8 + 6i, \ f_1 = 0.8$

Method	max(L)	$max(L)/f_1$	min(U)	$min(U)/f_1$
splitting method	0.794	0.9925	0.808	1.01
translation method	0.799485	0.999356	0.800064	1.000080

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