Article:

Miller, Craig Andrew (2023) The R-height of semigroups and their bi-ideals. International Journal of Algebra and Computation. pp. 47-66. ISSN 1793-6500

https://doi.org/10.1142/S0218196723500054

THE $\mathcal{R}$-HEIGHT OF SEMIGROUPS AND THEIR BI-IDEALS

CRAIG MILLER

Abstract. The $\mathcal{R}$-height of a semigroup S is the height of the poset of $\mathcal{R}$-classes of S. Given a semigroup S with finite $\mathcal{R}$-height, we establish bounds on the $\mathcal{R}$-height of bi-ideals, one-sided ideals and two-sided ideals; in particular, these substructures inherit the property of having finite $\mathcal{R}$-height. We then investigate whether these bounds can be attained.

Keywords: Semigroup, bi-ideal, poset of $\mathcal{R}$-classes, $\mathcal{R}$-height.
Mathematics Subject Classification 2010: 20M10, 20M12.

1. Introduction

Green's relations, five equivalence relations based on mutual divisibility, are arguably the most important tools for analysing the structure of semigroups. It is natural to consider how each of Green's relations on a semigroup relates to the corresponding relation on a subsemigroup. In general, they bear little resemblance to each other. However, Green's relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$ on a regular subsemigroup T of a semigroup S are the restrictions of the corresponding relations on S. If T is an ideal of S, then the same is also true for Green's relations $\mathcal{D}$ and $\mathcal{J}$ [6, Lemma 2.1]. In [4], East and Higgins investigated the inheritance of Green's relations by subsemigroups in the presence of stability of elements. East also, in [5], characterised Green's relations on principal one-sided ideals of an arbitrary semigroup, and then applied this theory to full transformation monoids and symmetric inverse monoids.

This article is concerned with the poset of $\mathcal{R}$-classes of a semigroup. (Two elements of a semigroup S are $\mathcal{R}$-related if they generate the same principal right ideal, and the set of $\mathcal{R}$-classes of S is a poset under the natural partial order associated with $\mathcal{R}$.) The height of this poset is called the $\mathcal{R}$-height. The term *$\mathcal{R}$-height* first appeared in [8], in which the $\mathcal{R}$-height of certain finite transformation semigroups was considered. However, as alluded to in [8], this parameter plays an implicit role in the (right) Rhodes expansion of a semigroup, a powerful tool in complexity theory, as well as in similar constructions such as the cover expansion; see [1] and [7, Chapter XII].

The purpose of this article is to compare the $\mathcal{R}$-height of a semigroup with that of its bi-ideals, one-sided ideals and two-sided ideals. A *bi-ideal* of a semigroup S is a subset B of S such that $BS^1B \subseteq B$. This notion generalises that of one-sided (and hence two-sided) ideals. Bi-ideals were introduced by Good and Hughes in [9], and were then studied systematically by Lajos in [12, 13].

The paper is structured as follows. In Section 2, we first present the necessary preliminary material, and then provide some basic results regarding chains of $\mathcal{R}$-classes. In Section 3, given a semigroup S with finite $\mathcal{R}$-height, we establish bounds on the $\mathcal{R}$-height of arbitrary bi-ideals, one-sided ideals and two-sided ideals of S. We then investigate in Section 4 whether these bounds can be attained. We conclude with some open questions and potential directions for future research in Section 5.

2. Preliminaries

2.1. **Definitions.** Throughout this section, S will denote a semigroup. We denote by S^1 the monoid obtained from S by adjoining an identity if necessary (if S is already a monoid, then $S^1 = S$).

A subset $A \subseteq S$ is said to be a *right ideal* of S if $AS \subseteq A$. Left ideals are defined dually, and an *ideal* of S is a subset that is both a right ideal and a left ideal.

A right (resp. left) ideal A of S is said to be *generated by* $X \subseteq A$ if $A = XS^1$ (resp. $A = S^1X$). A right (resp. left) ideal is said to be *finitely generated* if it can be generated by a finite set, and *principal* if it can be generated by a single element.

Principal (one-sided) ideals lead to the well-known Green's relations. For this article, we only require Green's relation $\mathcal{R}$ and its associated pre-order. Green's preorder $\leq_{\mathcal{R}}$ is defined by

$$a \leq_{\mathcal{R}} b \Leftrightarrow aS^1 \subseteq bS^1,$$

and this yields the relation $\mathcal{R}$:

$$a \,\mathcal{R}\, b \Leftrightarrow a \leq_{\mathcal{R}} b \text{ and } b \leq_{\mathcal{R}} a.$$

It is easy to see that $\mathcal{R}$ is an equivalence relation on S that is compatible with left multiplication (i.e. it is a *left congruence*).

When we need to distinguish between Green's relation $\mathcal{R}$ on different semigroups, we will write the semigroup as a subscript, i.e. $\mathcal{R}_S$ for $\mathcal{R}$ on S. For convenience, we will write $\leq_S$ rather than $\leq_{\mathcal{R}_S}$, and $a <_S b$ if $a \leq_S b$ but $(a, b) \notin \mathcal{R}_S$.

Following standard convention, we write R_a (or R_a^S) to denote the $\mathcal{R}$-class of an element $a \in S$.

Green's pre-order $\leq_{\mathcal{R}}$ induces a partial order on the set of $\mathcal{R}$-classes of S, given by

$$R_a \leq R_b \Leftrightarrow a \leq_{\mathcal{R}} b.$$

We note that the poset $S/\mathcal{R}$ is isomorphic to the poset of principal right ideals of S (under $\subseteq$). The *$\mathcal{R}$-height* of S is the height of the poset $S/\mathcal{R}$; i.e. the supremum of the lengths of chains of $\mathcal{R}$-classes (where the *length* of a chain is its cardinality). We denote the $\mathcal{R}$-height of S by $H_{\mathcal{R}}(S)$.

The semigroup S is said to be *right simple* if it has no proper right ideals, and S is said to be *simple* if it has no proper ideals. Certainly right simple semigroups are simple.

A right ideal A of S is said to be *minimal* if there is no right ideal of S properly contained in A. Minimal left ideals are defined dually. Similarly, an ideal A is called *minimal* if it does not contain any other ideal of S. It turns out that, considered as semigroups, minimal right ideals are right simple [3, Theorem 2.4], and minimal ideals are simple [3, Theorem 1.1]. There is at most one minimal ideal of S; if it exists, we call it the *kernel* of S and denote it by $K(S)$. On the other hand, S may possess multiple minimal right ideals. If S has a minimal right ideal, then $K(S)$ is equal to the union of all the minimal right ideals [3, Theorem 2.1]. A *completely simple* semigroup is a simple semigroup that possesses both minimal right ideals and minimal left ideals.

An element $a \in S$ is said to be *regular* if there exists $b \in S$ such that $a = aba$. We denote the set of regular elements of S by $\mathrm{Reg}(S)$. The semigroup S is said to be *regular* if $S = \mathrm{Reg}(S)$. It turns that for every regular element $a \in S$ there exists $b \in S$ such that $a = aba$ and $b = bab$; in this case, the element b is said to be an *inverse* of a, and vice versa. If S is regular and each of its elements has a unique inverse, then S is called *inverse*.

One of the most useful means of constructing semigroups is via a presentation. We briefly discuss presentations here; we refer the reader to [10] for more information.

The *free semigroup* on a non-empty set X, denoted by X^+, is the set of all words over X under the operation of concatenation. A *presentation* is a pair $\langle X \mid R \rangle$, where X is a non-empty set and R is a binary relation on X^+. We call R a set of *defining relations*, and we write $u = v$ for $(u, v) \in R$. A semigroup S is *defined by the presentation* $\langle X \mid R \rangle$ if it is isomorphic to $X^+/R^\sharp$, where $R^\sharp$ denotes the congruence generated by R (that is, the smallest congruence on X^+ containing R).

Adjoining an empty word ϵ to X^+ yields the *free monoid* on X, denoted by X^*. Similarly, one can adjoin a zero to X^+ to obtain the *free semigroup with zero* on X, denoted by X_0^+. By replacing X^+ with X^* or X_0^+ in the above definition of a presentation, one obtains the notion of a monoid presentation or a presentation of a semigroup with zero, respectively.

A presentation $\langle X \mid R \rangle$ can be viewed as a rewriting system, where each defining relation $u = v$ corresponds to a rewriting rule $u \to v$. We define a binary relation $\to$ on X^* by $w \to w'$ if and only if $w = w_1uw_2, w' = w_1vw_2$ for some $(u, v) \in R$ and $w_1, w_2 \in X^*$. We denote by $\xrightarrow{*}$ the reflexive and transitive closure of $\to$. The rewriting system $\langle X \mid R \rangle$ is *noetherian* if it is well-founded, i.e. if there is no infinite chain $w_1 \to w_2 \to \cdots$ of words from X^*. The rewriting system is *confluent* if for any $w, w_1, w_2 \in X^*$ with $w \xrightarrow{*} w_1$ and $w \xrightarrow{*} w_2$, there exists $z \in X^*$ such that $w_1 \xrightarrow{*} z$ and $w_2 \xrightarrow{*} z$. If a rewriting system is both noetherian and confluent, it is said to be *complete*. For a noetherian rewriting system, to determine confluence it suffices to consider the critical pairs. A *critical pair* of $\langle X \mid R \rangle$ is a pair $(w_1, w_2) \in X^* \times X^*$ with $w_1 \neq w_2$ for which there exists $w \in X^*$ such that $w \to w_1$ and $w \to w_2$. A critical pair (w_1, w_2) is said to *resolve* if there exists $z \in X^*$ such that $w_1 \xrightarrow{*} z$ and $w_2 \xrightarrow{*} z$. A noetherian rewriting system is complete if and only if all critical pairs resolve [11, Lemma 2.4].

A word in X^* is called *irreducible* if it is does not contain a subword that forms the left-hand side of a rewriting rule. If the rewriting system $\langle X \mid R \rangle$ is complete, then for any word $w \in X^*$ there is a unique irreducible word $z \in X^*$ with $w \xrightarrow{*} z$ [2, Theorem 1.1.12]. In this case, the semigroup defined by $\langle X \mid R \rangle$ has a normal form consisting of all the irreducible words over X.

We refer the reader to [2] for more information about rewriting systems.

2.2. **Elementary results.** We now provide a few basic results that will be useful in the next section.

Lemma 2.1. *Let S be a semigroup with finite $\mathcal{R}$-height. Then S has maximal and minimal $\mathcal{R}$-classes. In particular, S has a kernel.*

Proof. Consider a chain of $\mathcal{R}$-classes of S of maximal length. Since S has finite $\mathcal{R}$-height, this chain is finite and hence has both a maximal element and a minimal element. By the maximality of the length of the chain, it follows that S has a maximal $\mathcal{R}$-class and a minimal $\mathcal{R}$-class. □

Lemma 2.2. *Let S be a semigroup. Then $H_{\mathcal{R}}(S) = 1$ if and only if S is a union of minimal right ideals.*

Proof. Clearly $H_{\mathcal{R}}(S) = 1$ if and only if every $\mathcal{R}$-class of S is minimal, which is equivalent to S being a union of minimal right ideals. □

Lemma 2.3. *Let S be a semigroup with finite $\mathcal{R}$-height, and let B be a bi-ideal of S. If $H_{\mathcal{R}}(B) \geq n$, then there exists a chain*

$$b_1 <_B b_2 <_B \cdots <_B b_n$$

where $b_1 \in K(S)$.

Proof. Since $H_{\mathcal{R}}(B) \geq n$, there exists a chain

$$a_1 <_B a_2 <_B \cdots <_B a_n.$$

Since S has finite $\mathcal{R}$-height, the kernel $K = K(S)$ exists by Lemma 2.1. If $a_1 \in K$, we just set $b_i = a_i$ for all $i \in \{1, \dots, n\}$. Suppose then that $a_1 \notin K$. Since $BKB \subseteq B \cap K$, the intersection $B \cap K$ is non-empty. Choose $u \in B \cap K$ and let $b_1 = a_1 u$. Since $b_1 \in B^2$ and B is a subsemigroup of S, we have that $b_1 \in B$. Also, we have that $b_1 \in a_1 K \subseteq K$, since K is an ideal of S. Thus $b_1 \in B \cap K$. Clearly $b_1 \leq_B a_1$. In fact, we have that $b_1 <_B a_1$, for otherwise we would have $a_1 \in b_1 B \in K$. Thus, setting $b_{i+1} = a_i$ for all $i \in \{1, \dots, n-1\}$ yields the desired chain. □

An element $a \in S$ is said to have a *local right identity* (in S) if $a \in aS$ (that is, $a = ab$ for some $b \in S$). If S is a monoid, a regular semigroup or a right simple semigroup, then every element has a local right identity.

Lemma 2.4. *Let S be a semigroup and let B be a bi-ideal of S. If $b, c \in B$ have local right identites in B, then $b \leq_B c$ if and only if $b \leq_S c$, and $b <_B c$ if and only if $b <_S c$.*

Proof. If $b \leq_B c$ then clearly $b \leq_S c$. Suppose that $b \leq_S c$. Then $b = cs$ for some $s \in S^1$. Now, by assumption, there exist $u, v \in B$ such that $b = bu$ and $c = cv$. Then we have

$$b = bu = csu = c(vsu) \in cB,$$

using the fact that B is a bi-ideal of S. Thus $b \leq_B c$.

Now, using the first part of the lemma, we have

$$b <_B c \;\Leftrightarrow\; [b \leq_B c \text{ and } c \nleq_B b] \;\Leftrightarrow\; [b \leq_S c \text{ and } c \nleq_S b] \;\Leftrightarrow\; b <_S c,$$

as required. □

Corollary 2.5. *Let S be a semigroup with a completely simple kernel $K = K(S)$, and let B be a bi-ideal of S. For any $b, c \in B \cap K$, we have $b \leq_B c$ if and only if $b \leq_S c$, and $b <_B c$ if and only if $b <_S c$.*

Proof. We show that every element of $B \cap K$ has a local right identity in B, and the result then follows from Lemma 2.4. Consider $b \in B \cap K$. Since $b \in K$, we have that $b \,\mathcal{R}_S\, b^2$, so $b = b^2 s$ for some $s \in S^1$. Since K is regular, there exists $x \in K$ such that $b = bxb$. Thus

$$b = b^2 sxb = b(bsxb) \in bB,$$

so b has a local right identity in B. □

3. Bounds on the $\mathcal{R}$-height of Bi-ideals

In general, the property of having finite $\mathcal{R}$-height is not inherited by subsemigroups. For example, the group of integers $\mathbb{Z}$ has $\mathcal{R}$-height 1 but its subsemigroup $\mathbb{N}$ has infinite $\mathcal{R}$-height. Perhaps surprisingly, however, bi-ideals *do* inherit the property of having finite $\mathcal{R}$-height. In fact, given a semigroup S with finite $\mathcal{R}$-height, the following result establishes a bound on the $\mathcal{R}$-height of an arbitrary bi-ideal of S.

Theorem 3.1. *Let S be a semigroup with finite $\mathcal{R}$-height, and let B be a bi-ideal of S. Then*

$$H_{\mathcal{R}}(B) \leq 3n - 1,$$

where n is the maximum length of a chain of $\mathcal{R}$-classes of S that intersect B.

Proof. Suppose for a contradiction that $H_{\mathcal{R}}(B) \geq 3n$. Then, by Lemma 2.3, there exists a chain

$$b_1 <_B b_2 <_B \cdots <_B b_{3n}$$

where $b_1 \in K(S)$. For each $i \in \{1, \dots, 3n-1\}$ we have $b_i \in b_{i+1}B \subseteq b_{i+1}S$, so $b_i \leq_S b_{i+1}$. Thus we have a chain

$$b_1 \leq_S b_2 \leq_S \cdots \leq_S b_{3n}.$$

Since each b_i belongs to B, by assumption the chain

$$R^S_{b_1} \leq_S R^S_{b_2} \leq_S \cdots \leq_S R^S_{b_{3n}}$$

of $\mathcal{R}_S$-classes has size at most n.

We claim that $b_1 <_S b_3$. Indeed, suppose that $b_1 \mathrel{\mathcal{R}_S} b_3$. Since the $\mathcal{R}_S$-classes in K are minimal and $b_1^2 \leq_S b_1$, it follows that $b_1 \mathrel{\mathcal{R}_S} b_1^2$. Thus $b_3 \mathrel{\mathcal{R}_S} b_1^2$. We then have

$$b_2 \in b_3B \subseteq (b_1^2S^1)B \subseteq b_1(BS^1B) \subseteq b_1B,$$

where for the final containment we use the fact that B is a bi-ideal of S. But then $b_2 \leq_B b_1$, contradicting the fact that $b_1 <_B b_2$, so we have established the claim.

It follows from the above claim that $n > 1$. Since $b_1 <_S b_3$, the chain

$$R^S_{b_2} \leq_S R^S_{b_2} \leq_S \cdots \leq_S R^S_{b_{3n}}$$

has size at most $n-1$. Since $\frac{3n-2}{n-1} = 3 + \frac{1}{n-1} > 3$, by the generalised pigeonhole principle there exist $i, j, k, l \in \{3, \dots, 3n\}$ with $i < j < k < l$ such that $b_i \mathrel{\mathcal{R}_S} b_j \mathrel{\mathcal{R}_S} b_k \mathrel{\mathcal{R}_S} b_l$. Since $b_i \leq_S b_{i+1} \leq_S \cdots \leq_S b_l$, we deduce that $b_i \mathrel{\mathcal{R}_S} b_m$ for all $m \in \{i+1, \dots, l\}$. In particular, we have $b_i \mathrel{\mathcal{R}_S} b_{i+3}$, and hence $b_{i+3} \in b_iS$. Therefore, we have that

$$b_{i+2} \in b_{i+3}B \subseteq b_iSB \subseteq b_{i+1}BSB \subseteq b_{i+1}B,$$

using the fact that B is a bi-ideal of S. But this contradicts the fact that $b_{i+1} <_B b_{i+2}$. This completes the proof. □

Corollary 3.2. *Let S be a semigroup with finite $\mathcal{R}$-height, and let B be a bi-ideal of S. Then $H_{\mathcal{R}}(B) \leq 3H_{\mathcal{R}}(S) - 1$.*

In the case that the kernel $K(S)$ is completely simple, we obtain a slightly shorter bound for the $\mathcal{R}$-height of a bi-ideal than that given in Theorem 3.1.

Theorem 3.3. *Let S be a semigroup with finite $\mathcal{R}$-height whose kernel is completely simple, and let B be a bi-ideal of S. Then*

$$H_{\mathcal{R}}(B) \leq 3n - 2,$$

where n is the maximum length of a chain of $\mathcal{R}$-classes of S that intersect B.

Proof. Suppose for a contradiction that $H_{\mathcal{R}}(B) \geq 3n - 1$. Then, by Lemma 2.3, there exists a chain

$$b_1 <_B b_2 <_B \cdots <_B b_{3n-1}$$

where $b_1 \in K(S)$. Then we have a chain

$$b_1 \leq_S b_2 \leq_S \cdots \leq_S b_{3n-1}.$$

By assumption, the chain

$$R^S_{b_1} \leq_S R^S_{b_2} \leq_S \cdots \leq_S R^S_{b_{3n-1}}$$

has size at most n. We cannot have $b_1 \mathrel{\mathcal{R}_S} b_2$, since that would imply that $b_1 \mathrel{\mathcal{R}_B} b_2$ by Corollary 2.5, so $b_1 <_S b_2$. Therefore, the chain

$$R^S_{b_2} \leq_S R^S_{b_3} \leq_S \cdots \leq_S R^S_{b_{3n-1}}$$

has size at most $n-1$. Since $\frac{3n-2}{n-1} > 3$, by the generalised pigeonhole principle we obtain $b_i \, \mathcal{R}_S \, b_{i+3}$ for some $i \in \{2, \dots, 3n-4\}$. But then the same argument as that of Theorem 3.1 yields a contradiction. □

Corollary 3.4. *Let S be a semigroup with finite $\mathcal{R}$-height whose kernel is completely simple, and let B be a bi-ideal of S. Then $H_{\mathcal{R}}(B) \leq 3H_{\mathcal{R}}(S) - 2$.*

If B is a bi-ideal in which every element has a local right identity, then it follows from Lemma 2.4 that there exists a chain of $\mathcal{R}_B$-classes of length i if and only if there exists a chain of $\mathcal{R}_S$ classes that intersect B of length i. Thus we deduce:

Proposition 3.5. *Let S be a semigroup with finite $\mathcal{R}$-height, and let B be a bi-ideal of S in which every element has a local right identity in B. Then*

$$H_{\mathcal{R}}(B) = n,$$

where n is the maximal length of a chain of $\mathcal{R}$-classes of S that intersect B.

The next result provides a bound on the $\mathcal{R}$-height of a right ideal of a semigroup with finite $\mathcal{R}$-height.

Theorem 3.6. *Let S be a semigroup with finite $\mathcal{R}$-height, and let A be a right ideal of S. Then*

$$H_{\mathcal{R}}(A) \leq 2n - 1,$$

where n is the maximum length of a chain of $\mathcal{R}$-classes of S contained in A.

Proof. Suppose first that $n = 1$. Then A is a union of minimal right ideals of S. Minimal right ideals are right simple subsemigroups by [3, Theorem 2.4]. It follows that A is a union of minimal right ideals of itself, and hence $H_{\mathcal{R}}(A) = 1$ by Lemma 2.2.

Now assume that $n > 1$. Suppose for a contradiction that $H_{\mathcal{R}}(A) \geq 2n$. Then, by Lemma 2.3, there exists a chain

$$a_1 <_A a_2 <_A \cdots <_A a_{2n}$$

where $a_1 \in K(S)$. Then we have a chain

$$a_1 \leq_S a_2 \leq_S \cdots \leq_S a_{2n}.$$

By assumption, the chain

$$R^S_{a_1} \leq_S R^S_{a_2} \leq_S \cdots \leq_S R^S_{a_{2n}}$$

has size at most n. Since $a_1 \in K$, we have $a_1 \, \mathcal{R}_S \, a_1^2$, so that $a_1 \in a_1^2 S^1 \in a_1 A$, using the fact that A is a right ideal of S. We must then have $a_1 <_S a_2$, for otherwise we would have $a_2 \in a_1 S \subseteq a_1 A S \subseteq a_1 A$. It follows that the chain

$$R^S_{a_2} \leq_S R^S_{a_3} \leq_S \cdots \leq_S R^S_{a_{2n}}$$

has size at most $n-1$. Since $\frac{2n-1}{n-1} = 2 + \frac{1}{n-1} > 2$, by the generalised pigeonhole principle we deduce that there exists $i \in \{1, \dots, 2n-2\}$ such that $a_i \, \mathcal{R}_S \, a_{i+2}$. We then have

$$a_{i+2} \in a_i S \subseteq a_{i+1} A S \subseteq a_{i+1} A,$$

using the fact that A is a right ideal of S. But this contradicts that $a_{i+1} <_A a_{i+2}$. □

Corollary 3.7. *Let S be a semigroup with finite $\mathcal{R}$-height, and let A be a right ideal of S. Then $H_{\mathcal{R}}(A) \leq 2H_{\mathcal{R}}(S) - 1$.*

We now turn our attention to left ideals.

Theorem 3.8. *Let S be a semigroup with finite $\mathcal{R}$-height, and let A be a left ideal of S. Then*

$$H_{\mathcal{R}}(A) \leq 2n,$$

where n is the maximal length of a chain of $\mathcal{R}$-classes of S that intersect A.

Proof. Suppose for a contradiction that there exists a chain

$$a_1 <_A a_2 <_A \cdots <_A a_{2n+1}.$$

Then we have a chain

$$a_1 \leq_S a_2 \leq_S \cdots \leq_S a_{2n+1}.$$

By assumption, the chain

$$R^S_{a_1} \leq_S R^S_{a_2} \leq_S \cdots \leq_S R^S_{a_{2n+1}}$$

has size at most n. Since $\frac{2n+1}{n} = 2 + \frac{1}{n} > 2$, by the generalised pigeonhole principle we obtain $i \in \{1, \dots, 2n-1\}$ such that $a_i \,\mathcal{R}_S\, a_{i+2}$. Thus $a_{i+2} \in a_i S$. We then have

$$a_{i+1} \in a_{i+2}A \subseteq a_i SA \subseteq a_i A,$$

using the fact that A is a left ideal of S. But this contradicts that $a_i <_A a_{i+1}$. □

Corollary 3.9. *Let S be a semigroup with finite $\mathcal{R}$-height, and let A be a left ideal of S. Then $H_{\mathcal{R}}(A) \leq 2H_{\mathcal{R}}(S)$.*

Again, we obtain a slightly shorter bound in the case that $K(S)$ is completely simple.

Theorem 3.10. *Let S be a semigroup with finite $\mathcal{R}$-height whose kernel is completely simple, and let A be a left ideal of S. Then*

$$H_{\mathcal{R}}(A) \leq 2n - 1,$$

where n is the maximal length of a chain of $\mathcal{R}$-classes of S that intersect A.

Proof. Suppose for a contradiction that $H_{\mathcal{R}}(A) \geq 2n$. Then, by Lemma 2.3, there exists a chain

$$a_1 <_A a_2 <_A \cdots <_A a_{2n},$$

where $a_1 \in K(S)$. Then we have a chain

$$a_1 \leq_S a_2 \leq_S \cdots \leq_S a_{2n}.$$

By assumption, the chain

$$R^S_{a_1} \leq_S R^S_{a_2} \leq_S \cdots \leq_S R^S_{a_{2n}}$$

has size at most n. We cannot have $a_1 \,\mathcal{R}_S\, a_2$, since that would imply that $a_1 \,\mathcal{R}_A\, a_2$ by Corollary 2.5, so $a_1 <_S a_2$. Therefore, the chain

$$R^S_{a_2} \leq_S R^S_{a_3} \leq_S \cdots \leq_S R^S_{a_{2n}}$$

has size at most $n-1$. Since $\frac{2n-1}{n-1} > 2$, by the generalised pigeonhole principle we obtain $i \in \{2, \dots, 2n-2\}$ such that $a_i \,\mathcal{R}_S\, a_{i+2}$. But then the same argument as that of Theorem 3.8 yields a contradiction. □

Corollary 3.11. *Let S be a semigroup with finite $\mathcal{R}$-height whose kernel is completely simple, and let A be a left ideal of S. Then $H_{\mathcal{R}}(A) \leq 2H_{\mathcal{R}}(S) - 1$.*

If a left ideal A of S is contained in the set $\mathrm{Reg}(S)$ of regular elements, then for any $a \in S$ we have $a \in aSa \in aA$, using the fact that a is regular and A is a left ideal, so every element of A has a local right identity in A. Thus, by Proposition 3.5, we have:

Proposition 3.12. *Let S be a semigroup with finite $\mathcal{R}$-height, and let A be a left ideal of S such that $A \subseteq \operatorname{Reg}(S)$. Then*

$$H_{\mathcal{R}}(A) = n,$$

where n is the maximal length of a chain of $\mathcal{R}$-classes of S that intersect A.

Proposition 3.12 does not hold if we replace 'left ideal' by 'right ideal', as the following example demonstrates.

Example 3.13. Let S be the semigroup with universe $\{(1,1),(1,2),(2,1),(2,2),0\}$ and multiplication given by

$$(i,j)(k,l) = \begin{cases} (i,l) & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

and $0(i,j) = (i,j)0 = 00 = 0$. Then S is a completely 0-simple inverse semigroup, where the inverse of each (i,j) is (j,i). Consider the right ideal

$$A = (1,1)S^1 = \{(1,1),(1,2),0\}.$$

Certainly $A \subseteq \operatorname{Reg}(S) = S$. It is straightforward to verify that the posets of $\mathcal{R}$-classes of S and A are as presented in Figure 1 below, and hence $H_{\mathcal{R}}(S) = 2$ and $H_{\mathcal{R}}(A) = 3$.

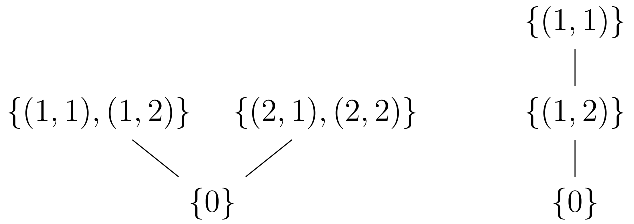

FIGURE 1. The poset of $\mathcal{R}_S$-classes (left) and the poset of $\mathcal{R}_A$-classes (right).

We now provide a bound on the $\mathcal{R}$-height of an ideal.

Theorem 3.14. *Let S be a semigroup with finite $\mathcal{R}$-height, and let A be an ideal of S. Then*

$$H_{\mathcal{R}}(A) \leq n,$$

where n is the maximum length of a chain of $\mathcal{R}$-classes of S contained in A.

Proof. Suppose for a contradiction that there exists a chain

$$a_1 <_A a_2 <_A \cdots <_A a_{n+1}.$$

Then we have a chain

$$a_1 \leq_S a_2 \leq_S \cdots \leq_S a_{n+1}.$$

By assumption, the chain

$$R^S_{a_1} \leq_S R^S_{a_2} \leq_S \cdots \leq_S R^S_{a_{n+1}}$$

has size at most n. By the pigeonhole principle there exists $i \in \{1,\dots,n\}$ such that $a_i \,\mathcal{R}_S\, a_{i+1}$. Then we have

$$a_{i+1} \in a_i S \subseteq a_{i+1} AS \subseteq a_i SAS \subseteq a_i A,$$

using the fact A is an ideal of S. But this contradicts that $a_i <_A a_{i+1}$. □

Corollary 3.15. *Let S be a semigroup with finite $\mathcal{R}$-height, and let A be an ideal of S. Then $H_{\mathcal{R}}(A) \leq H_{\mathcal{R}}(S)$.*

We conclude this section by demonstrating that the $\mathcal{R}$-height of an ideal can be substantially lower than the bound provided in Theorem 3.14.

Example 3.16. For any $n \in \mathbb{N}$, let T be a semigroup such that $H_{\mathcal{R}}(T) = n$. Let $N = \{x_a : a \in T\} \cup \{0\}$ be a set disjoint from T. We turn N into a null semigroup by defining $xy = 0$ for all $x, y \in N$. Let $S = T \cup N$, and define a multiplication on S, extending those on T and N, as follows:

$$ax_b = x_a b = x_{ab} \text{ and } a0 = 0a = 0$$

for all $a, b \in T$. Then N is an ideal of S. It is straightforward to show that for any $a, b \in T$, we have $a <_T b$ if and only if $x_a <_S x_b$; and clearly $0 <_S x_a$. It follows that the maximum length of a chain of $\mathcal{R}$-classes of S contained in N is $n+1$. On the other hand, it is easy to see that $H_{\mathcal{R}}(N) = 2$.

4. Attaining the Bounds

Given Theorems 3.1-3.14, an immediate question arises: Can the bounds established in these results be attained? In fact, one can ask a stronger question: Can these bounds be attained for every natural number n and in such a way that $H_{\mathcal{R}}(S) = n$? More precisely, we have the following problems.

(1) For each $n \in \mathbb{N}$, does there exist a semigroup S with a bi-ideal B such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(B) = 3n - 1$?
(2) For each $n \in \mathbb{N}$, does there exist a semigroup S with a completely simple kernel and a bi-ideal B such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(B) = 3n - 2$?
(3) For each $n \in \mathbb{N}$, does there exist a semigroup S with a right ideal A such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n - 1$?
(4) For each $n \in \mathbb{N}$, does there exist a semigroup S with a left ideal A such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n$?
(5) For each $n \in \mathbb{N}$, does there exist a semigroup S with a completely simple kernel and a left ideal A such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n - 1$?
(6) For each $n \in \mathbb{N}$, does there exist a semigroup S with an ideal A such that $H_{\mathcal{R}}(S) = H_{\mathcal{R}}(A) = n$?

Unfortunately, we have not been able to answer question (1).

Open Problem 4.1. For each $n \in \mathbb{N}$, does there exist a semigroup S with a bi-ideal B such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(B) = 3n - 1$?

We shall answer questions (2)-(6) in the positive. Question (6) is easily dealt with: take S to be any semigroup with $H_{\mathcal{R}}(S) = n$ and set $A = S$.

We now consider question (2). The case $n = 1$ is trivial: we can just take S to be the trivial semigroup and $B = S$; then $H_{\mathcal{R}}(S) = H_{\mathcal{R}}(B) = 1 = 3(1) - 2$. For $n \geq 2$, the following result provides the desired semigroups.

Theorem 4.2. *Let $n \geq 2$. Let S be the finite semigroup defined by the presentation*

$$\langle x, y, z, t \,|\, xyzt = x,\ yzty = y,\ ztyz = z,\ tyzt = t, w = 0 \\ (w \in \{x^n,\ y^2,\ z^2,\ t^2,\ xz,\ xt,\ yx,\ yt,\ zx,\ zy,\ tz,\ tx^{n-1}\})\rangle.$$

Let $X = \{x, y, z, tx\} \subseteq S$ and let $B = X \cup XS^1X$ (B is the smallest bi-ideal of S containing X). Then $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(B) = 3n - 2$.

Proof. We begin by finding a normal form for S. It is straightforward to show that the associated rewriting system of the presentation for S is complete. That it is noetherian follows from the fact that all the rewriting rules are length-reducing. For confluence, it suffices to check that all the critical pairs resolve. For instance, $(0tyz, xz)$ is a critical pair, since $xztyz \to 0tyz$ and $xztyz \to xz$, and clearly both sides of this pair reduce to 0. This rewriting system, therefore, yields the following normal form for S, consisting of all words over $\{x, y, z, t\}$ that do not contain as a subword the left-hand side of any of the rewriting rules, along with 0:

$$\{x^i,\, x^iy,\, x^iyz : 1 \leq i \leq n-1\} \cup \{y,\, yz,\, yzt,\, z,\, zt,\, zty,\, t,\, ty,\, tyz\} \cup U \cup \{0\},$$

where

$$U = \{yztx^i,\, yztx^iy,\, yztx^iyz,\, ztx^i,\, ztx^iy,\, ztx^iyz,\, tx^i,\, tx^iy,\, tx^iyz : 1 \leq i \leq n-2\}.$$

We note that $U = \emptyset$ if $n = 2$. It is easy to calculate that $|S| = 12(n-1)+1$. Using the above normal form and the relations of the presentation, it is easy to show that $t, zt, ty, yzt, tyz \notin B$. All other elements in the normal form have the form u or uwv where $u, v \in X$ and $w \in \{x, y, z, t\}^*$, so they belong to B. Thus $B = S \backslash \{t, zt, ty, yzt, tyz\}$.

From the relations of the presentation, it follows that for each generator $u \in \{x, y, z, t\}$, the principal right ideal uS^1 consists of precisely the words in the normal form whose first letter is u, along with 0. Thus, for any two generators $u, v \in \{x, y, z, t\}, u \neq v$, there are no elements of the form uw and vw' in S such that $uw \, \mathcal{R}_S \, vw'$.

Now let $R_i = \{x^i, x^iy, x^iyz\}$ $(1 \leq i \leq n-1)$, $S_1 = \{y, yz, yzt\}$, $T_1 = \{z, zt, zty\}$ and $U_1 = \{t, ty, tyz\}$. It is easy to see from the presentation that each of the sets is an $\mathcal{R}_S$-class. For $i \in \{2, \ldots, n-1\}$, let

$$S_i = yztR_{i-1} = \{yztx^{i-1}, yztx^{i-1}y, yztx^{i-1}yz\},$$

$T_i = ztR_{i-1}$ and $U_i = tR_{i-1}$. Since $\mathcal{R}_S$ is a left congruence on S, and R_{i-1} is an $\mathcal{R}_S$-class, each S_i, T_i and U_i is also an $\mathcal{R}_S$-class. Of course, $\{0\}$ is both an $\mathcal{R}_S$-class and an $\mathcal{R}_B$-class.

Consider $i \in \{1, \ldots, n-2\}$. Clearly $R_i \geq R_{i+1}$. Also, we have $yztx^{i-1}yz(tx) = yztx^i \in S_{i+1}$, so $S_i \geq S_{i+1}$. Similarly, $T_i \geq T_{i+1}$ and $U_i \geq U_{i+1}$. It is easy to see from the presentation that for any $v \in S$ and $s \in S^1$ with $vs \neq 0$, we have $|v|_x \leq |vs|_x$, where $|w|_x$ denotes the number of appearances of x in w. Thus $x^i \notin x^{i+1}S^1$, so $R_i > R_{i+1}$. Similarly, we have $S_i > S_{i+1}$, $T_i > T_{i+1}$ and $U_i > U_{i+1}$. We conclude that the poset of $\mathcal{R}_S$-classes is as displayed in Figure 2 below, so that $H_{\mathcal{R}}(S) = n$.

Turning our attention to B, we have

$$x^i \geq_B x^iy \geq_B x^iyz \geq_B x^iyz(tx) = x^{i+1}.$$

We certainly have $x^iyz >_B x^{i+1}$ since $x^iyz >_S x^{i+1}$. Also, it is easy to calculate that

$$x^iyB = \{x^iy,\, x^iyz,\, x^j,\, x^jy,\, x^jyz,\, 0 : i+1 \leq j \leq n-1\}$$

and

$$x^iyzB = \{x^j,\, x^jy,\, x^jyz,\, 0 : i+1 \leq j \leq n-1\},$$

so $(x^i, x^iy), (x^iy, x^iyz) \notin \mathcal{R}_B$. Thus we have a chain

$$x >_B xy >_B xyz >_B x^2 >_B x^2y >_B x^2yz >_B \cdots >_B x^{n-1} >_B x^{n-1}y >_B x^{n-1}yz >_B 0,$$

so $H_{\mathcal{R}}(B) \geq 3n-2$. By Theorem 3.3, we have $H_{\mathcal{R}}(B) \leq 3n-2$. Thus $H_{\mathcal{R}}(B) = 3n-2$. □

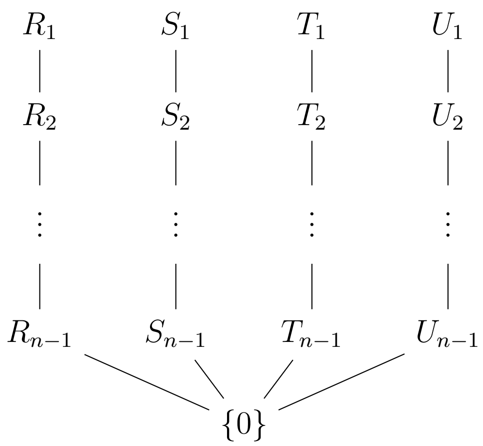

FIGURE 2. The poset of $\mathcal{R}$-classes of the semigroup S given in the statement of Theorem 4.2.

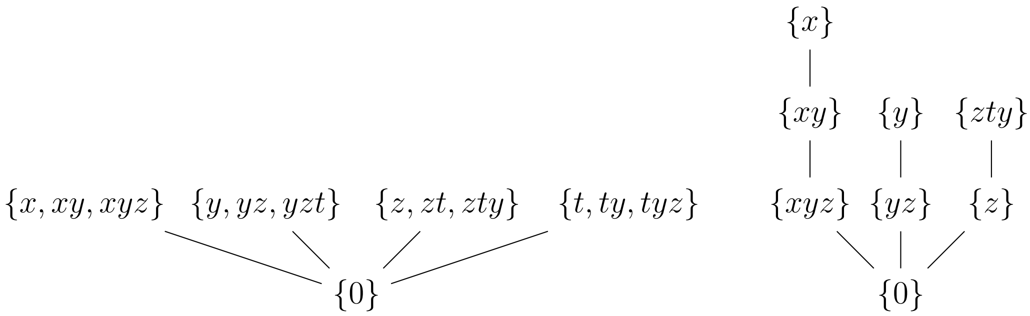

FIGURE 3. Let S and B be as given in Theorem 4.2 in the case $n = 2$. The poset of $\mathcal{R}_S$-classes is displayed on the left, and the poset of $\mathcal{R}_B$-classes is displayed on the right.

We now move on to problem (3). To solve this, we utilise the following construction.

Definition 4.3. Let S be a semigroup and let I be a non-empty set. The *Brandt extension of S by I*, denoted by $\mathcal{B}(S, I)$, is the semigroup with universe $(I \times S \times I) \cup \{0\}$ and multiplication given by

$$(i, s, j)(k, t, l) = \begin{cases} (i, st, l) & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

and $0x = x0 = 0$ for all $x \in (I \times S \times I) \cup \{0\}$.

Remark 4.4. Brandt extensions of groups, known as *Brandt semigroups*, are precisely the completely 0-simple inverse semigroups [10, Theorem 5.1.8]. The semigroup S from Example 3.13 is (isomorphic to) the 5-element Brandt semigroup over the trivial group. We note also that the subsemigroup $\langle y, z, t\rangle$ of the semigroup S from Theorem 4.2 is the 10-element Brandt semigroup over the trivial group.

Theorem 4.5. *Let S be a semigroup with finite $\mathcal{R}$-height, and let $A = aS^1$ be a principal right ideal of S. Let I be any set with $|I| \geq 2$, and let $T = \mathcal{B}(S, I)$. Fix $1 \in I$, and consider the principal right ideal $B = (1, a, 1)T^1$ of T. Then $H_{\mathcal{R}}(T) = H_{\mathcal{R}}(S) + 1$ and $H_{\mathcal{R}}(B) = H_{\mathcal{R}}(A) + 2$.*

Proof. It can be easily proved that for any $s, t \in S$ and $i, j, k, l \in I$, we have

$$(i, s, j) <_T (k, t, l) \Leftrightarrow i = k \text{ and } s <_S t. \tag{1}$$

Let $H_{\mathcal{R}}(S) = n$. Then there exists a chain

$$s_1 <_S s_2 <_S \cdots <_S s_n.$$

Letting $t_i = (1, s_i, 1)$, by (1) we have a chain

$$0 <_T t_1 <_T t_2 <_T \cdots <_T t_n,$$

so $H_{\mathcal{R}}(T) \geq n + 1$. Now suppose for a contradiction that $H_{\mathcal{R}}(T) > n + 1$. Then there exists a chain

$$0 <_T x_1 <_T x_2 <_T \cdots <_T x_{n+1}.$$

By (1) there exists $i \in I$ such that each x_k has the form (i, y_k, j_k) for some $y_k \in S$ and $j_k \in I$. But then, by (1), we have a chain

$$y_1 <_S y_2 <_S \cdots <_S y_{n+1},$$

contradicting the fact that $H_{\mathcal{R}}(S) = n$. Thus $H_{\mathcal{R}}(T) = n + 1$.

Now let $H_{\mathcal{R}}(A) = m$. Then there exists a chain

$$a_1 <_A a_2 <_A \cdots <_A a_m.$$

We have that $B = \{(1, a, 1)\} \cup \{(1, as, i) : s \in S, i \in I\} \cup \{0\}$. Let $b_i = (1, a_i, 1)$. Then $b_i \in B$. For each $i \in \{1, \ldots, m\}$, there exists $c_i \in A$ such that $a_i = a_{i+1}c_i$. Therefore, we have that $b_i = b_{i+1}(1, c_i, 1) \in b_{i+1}B$, so $b_i \leq_B b_{i+1}$. Clearly, if $b_i \, \mathcal{R}_B \, b_{i+1}$ then $a_i \, \mathcal{R}_A \, a_{i+1}$, contradicting that $a_i <_A a_{i+1}$, so $b_i <_B b_{i+1}$. Now choose $s \in S$ and $j \in I \backslash \{1\}$, and let $b_0 = (1, a_1(as), j)$. Then $b_0 = b_1(1, as, j) \in b_1 B$. Since $j \neq 1$, we have that $b_0 B = \{0\}$, so $b_0 <_B b_1$. Clearly $0 <_B b_0$. In conclusion, we have a sequence

$$0 <_B b_0 <_B b_1 <_B b_2 <_B \cdots <_B b_m,$$

so $H_{\mathcal{R}}(B) \geq m + 2$. Now suppose for a contradiction that $H_{\mathcal{R}}(B) > m + 2$. Then there exists a chain

$$0 <_B d_1 <_B d_2 <_B \cdots <_B d_{m+2}.$$

Let $d_i = (1, c_i, j_i)$. Then we have a chain

$$c_1 \leq_A c_2 \leq_A \cdots \leq_A c_{m+2}.$$

Since $H_{\mathcal{R}}(A) = m$, it follows that there exist $k, l \in \{1, \ldots, m + 1\}$ with $k < l$ such that $c_k \, \mathcal{R}_A \, c_{k+1}$ and $c_l \, \mathcal{R}_A \, c_{l+1}$. In particular, we have $c_{l+1} \in c_l A^1$. Since $d_l <_B d_{l+1}$, we must have that $c_l \in c_{l+1}A$, and hence

$$c_{l+1} \in c_l A^1 \subseteq c_{l+1} A A^1 \subseteq c_l A^1 A A^1 = c_l A.$$

So, there exists $u \in A$ such that $c_{l+1} = c_l u$. We cannot have $j_l = 1$, for then $d_{l+1} = d_l(1, u, j_{l+1}) \in d_l B$, contradicting that $d_l <_B d_{l+1}$. But then $d_l B = \{0\}$, contradicting that $d_k \in d_l B$. We conclude that $H_{\mathcal{R}}(B) = m + 2$. $\square$

Corollary 4.6. *For any $n \in \mathbb{N}$, there exists a finite semigroup S with a principal right ideal A such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n - 1$.*

Proof. We prove the result by induction. For $n = 1$, take S to be the trivial semigroup and $A = S$. Now let $n \geq 1$, and assume that there exists a finite semigroup S with a principal right ideal $A = aS^1$ such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n - 1$. Let T and B be as given in the statement of Theorem 4.5. Then, by Theorem 4.5, we have that $H_{\mathcal{R}}(T) = n + 1$ and $H_{\mathcal{R}}(B) = (2n - 1) + 2 = 2(n + 1) - 1$. This completes the proof. $\square$

We now turn our attention to question (4), beginning with the case $n = 1$.

Proposition 4.7. *Let S be a right simple semigroup with no idempotent (e.g. a Baer-Levi semigroup). (Then $H_{\mathcal{R}}(S) = 1$.) Let $a \in S$ be arbitrary, and consider the principal left ideal $A = S^1a$. Then the $\mathcal{R}_A$-classes are $\{a\}$ and $A\backslash\{a\} = Sa$, and hence $H_{\mathcal{R}}(A) = 2$.*

Proof. Let $s, t \in S$. Since S is right simple, there exists $x \in S^1$ such that $s = (ta)x$. Thus $sa = (ta)(xa) \in (ta)A$. Similarly, we have $ta \in (sa)A$, so $sa\,\mathcal{R}_A\,ta$. Now suppose for a contradiction that $a\,\mathcal{R}_A\,ua$ for some $u \in S$. Then, together with the fact just proved that $ua\,\mathcal{R}_A\,a^2$, we have $a\,\mathcal{R}_A\,a^2$ by transitivity. Therefore, there exists $y \in S^1$ such that $a = a^2(ya)$. But then a^2y is an idempotent, so we have a contradiction. Thus the $\mathcal{R}_A$-classes are $\{a\}$ and Sa. Clearly $a^2 <_A a$, so we conclude that $H_{\mathcal{R}}(A) = 2$. □

Theorem 4.8. *Let S be a semigroup with finite $\mathcal{R}$-height, and let A be a left ideal of S. Let T be any right simple semigroup with no idempotent, and let U be the semigroup defined by the presentation*

$$\langle S \cup T \,|\, ab = a \cdot b,\ cd = c \cdot d,\ ac = c\ \ (a, b \in S, c, d \in T)\rangle.$$

Fix $c \in T$, and let B denote the left ideal $T^1(A \cup \{c\})$ of U. Then $H_{\mathcal{R}}(U) = H_{\mathcal{R}}(S) + 1$ and $H_{\mathcal{R}}(B) = H_{\mathcal{R}}(A) + 2$.

Proof. The semigroup U has a normal form $S \cup T \cup TS$. Let $K = T \cup TS$. It is easy to see that K is an ideal of U. All elements of T are $\mathcal{R}_U$-related since T is right simple, and for any $a \in S$ and $t \in T$ we have that $ta = t \cdot a$ and $(ta)T = tT = T$. Thus K is an $\mathcal{R}$-class of U. It follows that K is the minimal ideal of U. Now, since $U\backslash S = K$ is an ideal, it follows that the restriction of $\leq_U$ to S is $\leq_S$. Thus the poset of $\mathcal{R}_U$-classes can be viewed as the poset of $\mathcal{R}_S$-classes along with the minimum element K. This is depicted in Figure 4 below. It follows that $H_{\mathcal{R}}(U) = H_{\mathcal{R}}(S) + 1$.

We now consider the left ideal B of U. We have $B \cap S = A$. Since $B\backslash A$ is an ideal, the restriction of $\leq_B$ to A is $\leq_A$. We claim that the sets $\{c\}$ and $TA \cup Tc$ are $\mathcal{R}$-classes of B. First, let $t, t' \in T$ and $a, a' \in A$. Since T is right simple, there exist $x, y, z \in T$ such that

$$t = t'x,\ t' = ty,\ t = (t'c)z.$$

(We can assume that $x, y, z \in T$ even if $t = t'$ or $t = t'c$, since every element of T has a local right identity.) Using the defining relations $a'x = x$ and $ay = y$, we deduce that

$$ta = (t'a')(xa),\ t'a' = (ta)(ya)\ \text{ and }\ ta = (t'c)(za),\ t'c = (ta)(yc);$$

so $ta\,\mathcal{R}_B\,t'a'$ and $ta\,\mathcal{R}_B\,t'c$. Since t, t', a, a' were chosen arbitrarily, it follows by transitivity that all elements in $TA \cup Tc$ are $\mathcal{R}_B$-related. Now suppose for a contradiction that $c\,\mathcal{R}_B\,c^2$. Then there exists $b \in B$ such that $c = c^2b$. We cannot have $b \in T^1A$, for this would imply that $c \in T \cap TS$, contradicting the fact that $S \cup T \cup TS$ is a normal form for U. Thus $b = wc$ for some $w \in T^1$, and hence $c = c^2wc$. But then c^2w is an idempotent of T, so we have a contradiction. This proves the claim.

For any $a \in A$ we have $ac = c$, so $c <_B a$. Also, we have $c^2 <_B c$. Thus the poset of $\mathcal{R}_B$-classes can be viewed as the poset of $\mathcal{R}_A$-classes along with the elements $\{c\}$ and $TA \cup TC$, where $\{c\}$ is below all the $\mathcal{R}_A$-classes and $TA \cup TC$ is the minimum element; see Figure 4 for an illustration. It follows that $H_{\mathcal{R}}(B) = H_{\mathcal{R}}(A) + 2$. □

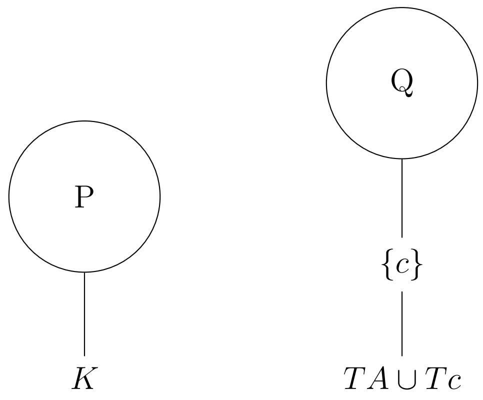

FIGURE 4. Let S, A, U and B be as given in Theorem 4.8. The poset of $\mathcal{R}_U$-classes is displayed on the left, where P denotes the poset of $\mathcal{R}_S$-classes. The poset of the $\mathcal{R}_B$-classes is displayed on the right, where Q denotes the poset of $\mathcal{R}_A$-classes.

Corollary 4.9. *For any $n \in \mathbb{N}$, there exists a semigroup S with a left ideal A such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n$.*

Proof. We prove the result by induction. Proposition 4.7 deals with the base case. Now let $n \geq 1$, and assume that there exists a semigroup S with a left ideal A such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n$. Let U and B be as given in the statement of Theorem 4.8. Then, by Theorem 4.8, we have $H_{\mathcal{R}}(U) = n + 1$ and $H_{\mathcal{R}}(B) = 2n + 2 = 2(n + 1)$. This completes the proof. □

Finally, we solve problem (5) with the following result, the case $n = 1$ being trivial.

Theorem 4.10. *Let $n \geq 2$. Let S be the finite semigroup defined by the presentation*

$$\langle x, y, z \,|\, xyz = x,\ yzy = y,\ zyz = z,\ w = 0\ \ (w \in \{x^n,\ y^2,\ z^2,\ xz,\ yx,\ zx^{n-1}\})\rangle,$$

and let $A = S^1\{x, y\}$. Then $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n - 1$.

Proof. The proof of this result is similar to that of Theorem 4.2, so we will not go into as much detail.

The associated rewriting system of the presentation for S is complete, yielding the following normal form for S:

$$\{x^i,\ x^iy : 1 \leq i \leq n-1\} \cup \{yzx^j,\ yzx^jy,\ zx^j,\ zx^jy : 0 \leq j \leq n-2\} \cup \{0\},$$

It is straightforward to calculate that $|S| = 6(n-1)+1$. It can also be easily shown that $A = S \backslash \{z, yz\}$.

Let $R_i = \{x^i, x^iy\}$ for $i \in \{1, \ldots, n-1\}$. Let $S_1 = \{y, yz\}$ and $T_1 = \{z, zy\}$, and for $i \in \{2, \ldots, n-1\}$ let $S_i = yzR_{i-1}$ and $T_i = zR_{i-1}$. Then each R_i, S_i and T_i is an $\mathcal{R}_S$-class. The poset of $\mathcal{R}_S$-classes is as displayed in Figure 5 below, so that $H_{\mathcal{R}}(S) = n$. Turning our attention to A, we have

$$x^i \geq_A x^iy \geq_A x^iy(zx) = x^{i+1}.$$

We certainly have $x^iy >_A x^{i+1}$ since $x^iy >_S x^{i+1}$. Also, we have

$$x^iyA = \{x^iy,\ x^j,\ x^jy,\ 0 : i+1 \leq j \leq n-1\},$$

so $x^i >_A x^iy$. Thus we have a chain

$$x >_A xy >_A x^2 >_A x^2y >_A \cdots >_A x^{n-1} >_A x^{n-1}y >_A 0,$$

so $H_{\mathcal{R}}(A) \geq 2n-1$. By Theorem 3.10, we have $H_{\mathcal{R}}(A) \leq 2n-1$. We conclude that $H_{\mathcal{R}}(A) = 2n-1$. □

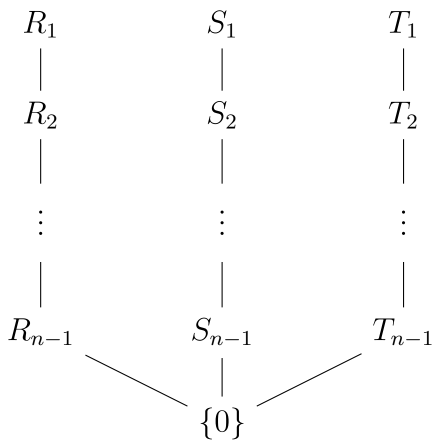

Figure 5. The poset of $\mathcal{R}$-classes of the semigroup S given in the statement of Theorem 4.10.

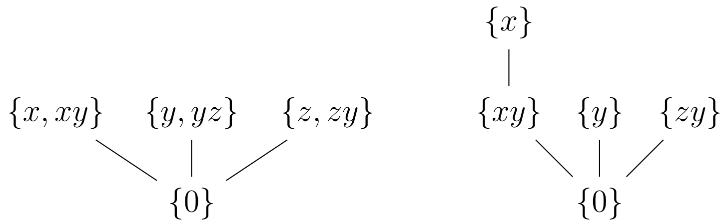

Figure 6. Let S and A be as given in Theorem 4.10 in the case $n = 2$. The poset of $\mathcal{R}_S$-classes is displayed on the left, and the poset of $\mathcal{R}_A$-classes is displayed on the right.

5. Open Problems and Future Research

As with the $\mathcal{R}$-height, one can of course define the $\mathcal{L}$-height, $\mathcal{H}$-height and $\mathcal{J}$-height of a semigroup S, which we denote by $H_{\mathcal{L}}(S)$, $H_{\mathcal{H}}(S)$ and $H_{\mathcal{J}}(S)$, respectively. It would potentially be interesting to consider the relationship between these heights. We note that for stable semigroups S, since any two $\mathcal{R}$-classes within the same $\mathcal{J}(=\mathcal{D})$-class are incomparable, we have $H_{\mathcal{R}}(S) \leq H_{\mathcal{J}}(S)$. It is easy to find stable semigroups for which $H_{\mathcal{R}}(S) = H_{\mathcal{J}}(S)$. Indeed, for any finite full transformation semigroup $S = \mathcal{T}_n$, we have $H_{\mathcal{R}}(S) = H_{\mathcal{L}}(S) = H_{\mathcal{H}}(S) = H_{\mathcal{J}}(S) = n$. On the other hand, if S is the semigroup from Theorem 4.10 (which is stable since it is finite), then $H_{\mathcal{R}}(S) = n$ by that theorem, but it turns out that $H_{\mathcal{J}}(S) = 2n-1$; indeed, the $\mathcal{J}$-classes of S form a chain

$$J_1 > K_1 > \cdots > J_{n-1} > K_{n-1} > \{0\},$$

where $J_i = \{yzx^{i-1}, yzx^{i-1}y, zx^{i-1}, zx^{i-1}y\}$ and $K_i = \{x^i, x^iy\}$ for $1 \leq i \leq n-1$. However, it is not the case that $H_{\mathcal{R}}(S) \leq H_{\mathcal{J}}(S)$ for every semigroup S. For example, for the bicyclic monoid B we have $H_{\mathcal{J}}(B) = 1$ and $H_{\mathcal{R}}(B) = \infty$. We raise the following question.

Open Problem 5.1. Is there a general upper bound for $H_{\mathcal{J}}(S)$ in terms of $H_{\mathcal{R}}(S)$?

It is perhaps also worth considering the relationship between the $\mathcal{J}$-height of a semigroup and that of its bi-ideals, one-sided ideals, etc. In particular, we ask:

Open Problem 5.2. Is the property of having finite $\mathcal{J}$-height inherited by bi-ideals?

Another possible direction for future research would be to study the $\mathcal{R}$-height more systematically. In particular, one could consider the $\mathcal{R}$-height of semigroups lying in certain special classes, such as regular semigroups. We note that for an inverse semigroup S, we have $H_{\mathcal{R}}(S)(= H_{\mathcal{L}}(S) = H_{\mathcal{H}}(S))$ is equal to the height of the semilattice $E(S)$ of idempotents of S. Moreoever, one could investigate the behaviour of the $\mathcal{R}$-height under various semigroup-theoretic constructions, such as quotients, ideal extensions, direct products, free products, etc.

Acknowledgements

This work was supported by the Engineering and Physical Sciences Research Council [EP/V002953/1]. The author would like to thank the referee for a careful reading of the paper, and for observations and questions that led to Section 5.

References

[1] J.-C. Birget. Iteration of expansions - unambiguous semigroups. *J. Pure Appl. Algebra*, 34:1-55, 1984.
[2] R. Book and F. Otto. *String Rewriting Systems*. Springer-Verlag, New York, 1993.
[3] A. Clifford. Semigroups containing minimal ideals. *Amer. J. Math.*, 70:521-526, 1948.
[4] J. East and P. Higgins. Green's relations and stability for subsemigroups. *Semigroup Forum*, 101:77-86, 2020.
[5] J. East. Structure of principal one-sided ideals. *Inter. J. Algebra Comput.*, 31:1093-1145, 2021.
[6] J. East and N. Ruškuc. Congruence lattices of ideals in categories and (partial) semigroups. *Memoirs of the Amer. Math. Soc.*
[7] S. Eilenberg. *Automata, Languages and Machines, Volume B*. Academic Press, 1976.
[8] L. Fleischer and M. Kufleitner. Green's Relations in Finite Transformation Semigroups. Pp. 112-27 in *Computer Science – Theory and Applications*, CSR 2017, Lecture Notes in Computer Science, Springer, Cham, 2017.
[9] R. Good and D. Hughes. Associated groups for a semigroup. *Bull. Amer. Math. Soc.*, 58:624-625, 1952.
[10] J. Howie. *An Introduction to Semigroup Theory*. Oxford University Press, 1995.
[11] G. Huet. Confluent reductions: abstract properties and applications to term rewriting systems. *J. Assoc. Comput. Mach.*, 27:797-821, 1980.
[12] S. Lajos. On the bi-ideals in semigroups. *Proc. Japan Acad.*, 45:710-712, 1969.
[13] S. Lajos. On the bi-ideals in semigroups II. *Proc. Japan Acad.*, 47:837-839, 1971.

Department of Mathematics, University of York, UK, YO10 5DD
Email address: craig.miller@york.ac.uk