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Research Article

Susanne C. Brenner* and Li-Yeng Sung

An Interior Maximum Norm Error Estimate for the Symmetric Interior Penalty Method on Planar Polygonal Domains

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Abstract: We establish an interior maximum norm error estimate for the symmetric interior penalty method on planar polygonal domains.

Keywords: Interior Maximum Norm Error Estimates, Symmetric Interior Penalty Method

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $f \in L^2(\Omega)$. A model Dirichlet boundary value problem is to find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega fv\,dx \quad \text{for all } v \in H_0^1(\Omega). \tag{1.1}$$

Here and below, we follow the standard notation for differential operators, function spaces and norms that can be found for example in [1, 8, 13].

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω, $k \geq 1$, and let V_h be the space of discontinuous piecewise polynomial functions of degree at most k associated with $\mathcal{T}_h$, i.e.,

$$V_h = \{v \in L^2(\Omega) : v_T = v|_T \in \mathbb{P}_k(T) \text{ for all } T \in \mathcal{T}_h\}.$$

As usual, the mesh parameter h is the maximum of the diameters of the triangles in $\mathcal{T}_h$.

Remark 1.1. We will treat all triangles as open triangles.

The symmetric interior penalty (SIP) method (cf. [3, 33]) for (1.1) computes $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega fv\,dx \quad \text{for all } v \in V_h, \tag{1.2}$$

where the bilinear form $a_h(\cdot,\cdot)$ is given by

$$a_h(w, v) = \sum_{T\in\mathcal{T}_h} \int_T \nabla w \cdot \nabla v\,dx + \sum_{e\in\mathcal{E}_h} \int_e (\{\!\{\partial w/\partial n\}\!\}[\![v]\!] + \{\!\{\partial v/\partial n\}\!\}[\![w]\!])\,ds + \sigma \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \int_e [\![w]\!][\![v]\!]\,ds. \tag{1.3}$$

***Corresponding author: Susanne C. Brenner,** Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA, e-mail: brenner@math.lsu.edu
Li-Yeng Sung, Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA, e-mail: sung@math.lsu.edu

Here $\mathcal{E}_h$ is the set of the edges of $\mathcal{T}_h$, $|e|$ is the length of the edge e and σ is a positive penalty parameter. On each $e \in \mathcal{E}_h^i$ (the set of the interior edges of $\mathcal{T}_h$) shared by two triangles $T_e^{\pm}$, we define the average of the normal derivative of v across e by

$$\{\!\{\partial v/\partial n\}\!\} = \frac{\partial v_e^-}{\partial n_e} + \frac{\partial v_e^+}{\partial n_e}, \tag{1.4}$$

where $v_e^{\pm} = v|_{T_e^{\pm}}$ and n_e is a unit vector normal to e pointing from T_e^- to T_e^+. The jump $[\![v]\!]$ of v across e is defined by

$$[\![v]\!] = v_e^+ - v_e^-. \tag{1.5}$$

On an edge $e \in \mathcal{E}_h^b$ (the set of the boundary edges of $\mathcal{T}_h$) that is an edge of $T_e \in \mathcal{T}_h$, we define

$$[\![\partial v/\partial n]\!] = \frac{\partial v_e}{\partial n_e} \quad \text{and} \quad [\![v]\!] = -v_e, \tag{1.6}$$

where $v_e = v|_{T_e}$ and n_e is the unit vector normal to e pointing towards the outside of Ω.

We assume that the penalty parameter σ is sufficiently large so that the discrete problem is uniquely solvable (cf. [24]). We also assume that $\mathcal{T}_h$ is properly graded around the reentrant corners of Ω (cf. [2, 5, 17]) to ensure the optimal convergence of finite element methods.

Let K be a compact subset of the open subset D of Ω such that $D \Subset \Omega$ (i.e., $\bar{D}$ is a compact subset of Ω). Our goal is to give a self-contained derivation of the following estimate:

$$\begin{aligned}\|u - u_h\|_{L^\infty(K)} \le C\big(&\|u - \Pi_h u\|_{L^\infty(D)} + h(1 + |\ln h|)\|u - \Pi_h u\|_{W_h^{1,\infty}(D)} \\ &+ \|u - u_h\|_{L^2(D)} + h\|u - \Pi_h u\|_{W_h^{1,2}(\Omega)}\big)\end{aligned} \tag{1.7}$$

asymptotically as $h \downarrow 0$, where Π_h is the nodal interpolation operator for the $\mathbb{P}_k$ Lagrange finite element space $H_0^1(\Omega) \cap V_h$ and the positive constant C is independent of h.

The mesh-dependent (semi-)norms in (1.7) are defined as follows. Let G be a subset of Ω. We take

$$\mathcal{T}_h(G) = \{T \in \mathcal{T}_h : T \cap G \neq \emptyset\} \tag{1.8}$$

and define the (semi-)norms $\|\cdot\|_{W_h^{1,2}(G)}$ and $\|\cdot\|_{W_h^{1,\infty}(G)}$ by

$$\|v\|_{W_h^{1,2}(G)}^2 = \sum_{T \in \mathcal{T}_h(G)} \Big[\|\nabla v\|_{L^2(T)}^2 + \sum_{e \subset \partial T} \big(|e| \|\{\!\{\partial v/\partial n\}\!\}\|_{L^2(e)}^2 + |e|^{-1}\|[\![v]\!]\|_{L^2(e)}^2\big)\Big], \tag{1.9}$$

$$\|v\|_{W_h^{1,\infty}(G)} = \max_{T \in \mathcal{T}_h(G)} \big[\|\nabla v\|_{L^\infty(T)} + \max_{e \subset \partial T}\big(\|\{\!\{\partial v/\partial n\}\!\}\|_{L^\infty(e)} + |e|^{-1}\|[\![v]\!]\|_{L^\infty(e)}\big)\big]. \tag{1.10}$$

Interior maximum norm (or pointwise) error estimates for classical finite element methods (cf. [27, 32] and the references therein) were extended to the two-dimensional SIP method (with $k = 1$) in [22] under a global H^2 regularity assumption that is valid only for convex domains. Pointwise error estimates for the SIP method in arbitrary dimensions were established in [12] in terms of the global weighted norms from [25] that are in some sense localized. The results in [12] were extended in [20] to other two-dimensional discontinuous Galerkin methods. The theory in both of the papers [12, 20] requires the domain Ω to be smooth. Other related work can be found in [10, 11, 23].

However, the true interior pointwise error estimate in [26] (that improved the results in [27]) has not yet been extended to discontinuous Galerkin methods. We believe this is due to the fact that the derivations in [26, 27] require Galerkin approximations for an auxiliary Neumann problem on a local disc around the point under consideration. But it is not clear how Galerkin approximations for the Neumann problem can be obtained by using discontinuous finite element functions on a mesh that does not fit the disc exactly.

We obtain estimate (1.7) by avoiding the local Neumann problem, at the expense of involving a nonlocal term (the fourth term on the right-hand side). Nevertheless, under our assumption on $\mathcal{T}_h$, the estimate

$$\|u - u_h\|_{L_\infty(K)} \le Ch^2(1 + |\ln h|) \tag{1.11}$$

follows immediately from (1.7) (cf. (2.3), (2.10), (2.11) and (2.13)) provided that $u \in W_{\mathrm{loc}}^{2,\infty}(\Omega)$, which is valid for example if f is locally Hölder continuous (cf. [18]).

Remark 1.2. Estimate (1.11) is optimal for $k = 1$ (cf. [21]). For $k \geq 2$, we expect the term $1 + |\ln h|$ can be removed after additional (nontrivial) efforts (cf. [27, 29]).

Remark 1.3. Green's function for the boundary value problem (1.1) plays a role in the analysis in [12, 20], which is behind the smooth domain assumption in these papers. In our approach, we use instead the fundamental solution in the free space, and therefore we do not need to assume that the domain is smooth.

The rest of the paper is organized as follows. We recall some preliminary results concerning the SIP method in Section 2 and obtain a discrete Caccioppoli estimate in Section 3 that is crucial for the local energy norm error estimate in Section 4. We then use the result in Section 4 to derive an interior $W^{1,1}$ error estimate in Section 5, which provides the final tool for establishing the interior maximum norm error estimate in Section 6. We end with some concluding remarks in Section 7.

Throughout the paper, we use C (with or without subscripts) to denote a generic positive constant that is independent of h. (The dependence of C on other parameters will be mentioned in context.) We also use the notation $A \lesssim B$ to represent the statement that $A \leq (\text{constant})B$, where the hidden positive constant is independent of h. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

We will frequently use the following elementary scaling estimates, where h_T denotes the diameter of the triangle T.

- *Discrete estimates:*

$$\|\nabla v\|_{L^2(T)} \leq C h_T^{-1} \|v\|_{L^2(T)} \quad \text{for all } v \in \mathbb{P}_k, \tag{1.12}$$

$$\|v\|_{L^2(\partial T)} \leq C h_T^{-1/2} \|v\|_{L^2(T)} \quad \text{for all } v \in \mathbb{P}_k. \tag{1.13}$$

- *Trace inequality:*

$$h_T^{-1} \|v\|_{L^2(\partial T)}^2 \leq C\big(h_T^{-2} \|v\|_{L^2(T)}^2 + \|\nabla v\|_{L^2(T)}^2\big) \quad \text{for all } v \in H^1(T). \tag{1.14}$$

Finally, we record the following useful inequality.

- *Young's inequality:*

$$ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2 \quad \text{for all } \epsilon > 0. \tag{1.15}$$

2 Preliminaries

2.1 Energy Space

The energy space for the Dirichlet boundary value problem (1.1) is

$$\mathring{E}(\Delta; L^2(\Omega)) = \{v \in H_0^1(\Omega) : \Delta v \in L^2(\Omega)\},$$

where Δv is understood in the sense of distributions.

It is well known (cf. [14, 19]) that $\mathring{E}(\Delta; L^2(\Omega)) \subset H^{1+\alpha}(\Omega)$ for some $\alpha \in (1/2, 1]$. Therefore, functions in $\mathring{E}(\Delta; L^2(\Omega))$ are continuous by the Sobolev inequality (cf. [1]) and Lagrange interpolations are well-defined on $\mathring{E}(\Delta; L^2(\Omega))$. If Ω is convex, then $\alpha = 1$ and $\mathring{E}(\Delta; L^2(\Omega))$ coincides with the space $H^2(\Omega) \cap H_0^1(\Omega)$.

Note that

$$\mathring{E}(\Delta; L^2(\Omega)) \subset H^2_{\text{loc}}(\Omega) \tag{2.1}$$

by interior elliptic regularity (cf. [16]).

We can include both $\mathring{E}(\Delta; L^2(\Omega))$ and V_h in the space

$$H^{1+\alpha}(\Omega; \mathcal{T}_h) = \{v \in L^2(\Omega) : v_T = v|_T \in H^{1+\alpha}(T) \text{ for all } T \in \mathcal{T}_h\},$$

and (1.3)–(1.6) are well-defined for functions in $H^{1+\alpha}(\Omega; \mathcal{T}_h)$.

2.2 Interpolation Errors

Let Π_T be the nodal interpolation operator for the $\mathbb{P}_k$ Lagrange finite element on the triangle T. We have the following standard error estimates (cf. [8, 13]):

$$\|\zeta - \Pi_T\zeta\|_{L^2(T)} + h_T|\zeta - \Pi_T\zeta|_{H^1(T)} + h_T^2|\zeta - \Pi_h\zeta|_{H^2(T)} \le Ch_T^2|\zeta|_{H^2(T)} \quad \text{for all } \zeta \in H^2(T), \tag{2.2}$$

$$\|\zeta - \Pi_T\zeta\|_{L^\infty(T)} + h_T|\zeta - \Pi_T\zeta|_{W^{1,\infty}(T)} \le Ch_T^2|\zeta|_{W^{2,\infty}(T)} \quad \text{for all } \zeta \in W^{2,\infty}(T). \tag{2.3}$$

Moreover, estimates (1.14) and (2.2) imply

$$\|\zeta - \Pi_T\zeta\|_{L^2(\partial T)} \le Ch_T^{3/2}|\zeta|_{H^2(T)} \quad \text{for all } \zeta \in H^2(T). \tag{2.4}$$

2.3 Results for the SIP Method

It follows from (1.3), (1.9) and the Cauchy–Schwarz inequality that

$$a_h(w, v) \le C\|w\|_{W_h^{1,2}(\Omega)}\|v\|_{W_h^{1,2}(\Omega)} \quad \text{for all } v, w \in H^{1+\alpha}(\Omega; \mathcal{T}_h), \tag{2.5}$$

and for a sufficiently large σ, we also have (cf. [24])

$$a_h(v, v) \ge C\|v\|^2_{W_h^{1,2}(\Omega)} \quad \text{for all } v \in V_h. \tag{2.6}$$

In view of (2.5) and (2.6), we can define the Riesz projection operator $P_h\colon H^{1+\alpha}(\Omega; \mathcal{T}_h) \to V_h$ by

$$a_h(P_h\zeta, v) = a_h(\zeta, v) \quad \text{for all } v \in V_h. \tag{2.7}$$

It follows from (2.5)–(2.7) that

$$\|P_h\zeta\|_{W_h^{1,2}(\Omega)} \le C\|\zeta\|_{W_h^{1,2}(\Omega)} \quad \text{for all } \zeta \in H^{1+\alpha}(\Omega; \mathcal{T}_h). \tag{2.8}$$

The SIP method is consistent (cf. [24]) in the sense that

$$a_h(\zeta, v) = \int_\Omega (-\Delta\zeta)v\,dx \quad \text{for all } \zeta \in \mathring{E}(\Delta; L^2(\Omega)),\ v \in V_h, \tag{2.9}$$

which together with (1.1) and (1.2) implies

$$u_h = P_h u. \tag{2.10}$$

Remark 2.1. Relation (2.9), which comes from integration by parts, is also valid for

$$\zeta \in H^2(\Omega) \quad \text{and} \quad v \in H^{1+\alpha}(\Omega; \mathcal{T}_h)$$

as long as $\operatorname{supp} v \Subset \Omega$.

Under the assumption that $\mathcal{T}_h$ is properly graded around the reentrant corners, we have (cf. [7])

$$\|\zeta - \Pi_h\zeta\|_{W_h^{1,2}(\Omega)} \lesssim h\|\Delta\zeta\|_{L^2(\Omega)} \quad \text{for all } \zeta \in \mathring{E}(\Delta; L^2(\Omega)). \tag{2.11}$$

Combining (2.5), (2.6), (2.9) and (2.11), we see that (cf. [24])

$$\|\zeta - P_h\zeta\|_{W_h^{1,2}(\Omega)} \le Ch\|\Delta\zeta\|_{L^2(\Omega)} \quad \text{for all } \zeta \in \mathring{E}(\Delta; L^2(\Omega)), \tag{2.12}$$

and then a standard duality argument and (2.12) yield the estimate

$$\|\zeta - P_h\zeta\|_{L^2(\Omega)} \le Ch\|\zeta - P_h\zeta\|_{W_h^{1,2}(\Omega)} \le Ch^2\|\Delta\zeta\|_{L^2(\Omega)} \quad \text{for all } \zeta \in \mathring{E}(\Delta; L^2(\Omega)). \tag{2.13}$$

2.4 Mesh-Dependent Norms

Besides the mesh-dependent (semi-)norms defined in (1.9) and (1.10), we will also use the norm $\|\cdot\|_{W_h^{1,1}(G)}$ defined by

$$\|v\|_{W_h^{1,1}(G)} = \sum_{T\in\mathcal{T}_h(G)} \Big[\|\nabla v\|_{L^1(T)} + \sum_{e\subset\partial T} \big(|e|\|\{\!\{\partial v/\partial n\}\!\}\|_{L^1(e)} + \|[\![v]\!]\|_{L^1(e)}\big)\Big]. \tag{2.14}$$

We can bound $\|v\|_{W_h^{1,1}(G)}$ by $\|v\|_{W_h^{1,2}(G)}$, as indicated by the following lemma.

Lemma 2.2. *We have*

$$\|v\|_{W_h^{1,1}(G)} \le C\Big(\sum_{T\in\mathcal{T}_h(G)} |T|\Big)^{\frac{1}{2}} \|v\|_{W_h^{1,2}(G)} \quad \textit{for all } v \in \mathring{E}(\Delta; L^2(\Omega)). \tag{2.15}$$

Proof. It follows from (2.14) and the Cauchy–Schwarz inequality for integrals that

$$\begin{aligned}\|v\|_{W_h^{1,1}(G)} &\le \sum_{T\in\mathcal{T}_h(G)} \Big[|T|^{\frac{1}{2}}\|\nabla v\|_{L^2(T)} + \sum_{e\subset\partial T}\big(|e|^{\frac{3}{2}}\|\{\!\{\partial v/\partial n\}\!\}\|_{L^2(e)} + |e|^{\frac{1}{2}}\|[\![v]\!]\|_{L^2(e)}\big)\Big]\\ &\lesssim \sum_{T\in\mathcal{T}_h(G)} \Big[|T|^{\frac{1}{2}}\|\nabla v\|_{L^2(T)} + \sum_{e\subset\partial T}|T|^{\frac{1}{2}}\big(|e|^{\frac{1}{2}}\|\{\!\{\partial v/\partial n\}\!\}\|_{L^2(e)} + |e|^{-\frac{1}{2}}\|[\![v]\!]\|^2_{L^2(e)}\big)\Big],\end{aligned}$$

which together with (1.9) and the discrete Cauchy–Schwarz inequality yields (2.15). □

It is also convenient to introduce the semi-norm

$$|z|_{W_h^{2,2}(G)} = \Big(\sum_{T\in\mathcal{T}_h(G)} |z|^2_{W^{2,2}(T)}\Big)^{\frac{1}{2}}. \tag{2.16}$$

2.5 Mesh-Subdomains

A subdomain D of Ω is a mesh-subdomain of $\mathcal{T}_h$ if (cf. (1.8))

$$\bar{D} = \bigcup_{T\in\mathcal{T}_h(D)} \bar{T}.$$

Let D be a mesh-subdomain. We define the bilinear form $a_{h,D}(\cdot,\cdot)$ by

$$\begin{aligned}a_{h,D}(w,v) = \sum_{T\subset D}\Big(\int_T \nabla w\cdot\nabla v\,dx + \sum_{e\subset\partial T}\mu_e\Big[\int_e(\{\!\{\partial w/\partial n\}\!\}[\![v]\!] + \{\!\{\partial v/\partial n\}\!\}[\![w]\!])\,ds\\ + \frac{\sigma}{|e|}\int_e [\![w]\!][\![v]\!]\,ds\Big]\Big),\end{aligned} \tag{2.17}$$

where

$$\mu_e = \begin{cases}\frac{1}{2} & \text{if } e\in\mathcal{E}_h^i,\\ 1 & \text{if } e\in\mathcal{E}_h^b.\end{cases}$$

Then we have, by (1.3) and (2.17),

$$a_h(w,v) = a_{h,D}(w,v) + a_{h,\Omega\setminus D}(w,v) \quad \text{for all } v,w\in H^{1+\alpha}(\Omega;\mathcal{T}_h). \tag{2.18}$$

Remark 2.3. The two bilinear forms $a_h(\cdot,\cdot)$ and $a_{h,\Omega}(\cdot,\cdot)$ are identical on $H^{1+\alpha}(\Omega;\mathcal{T}_h)$.

It follows from (1.9), (1.10), (2.14) and Hölder's inequality that

$$|a_{h,D}(w,v)| \le C\|w\|_{W_h^{1,2}(D)}\|v\|_{W_h^{1,2}(D)} \quad \text{for all } v,w\in H^{1+\alpha}(\Omega;\mathcal{T}_h), \tag{2.19}$$

$$|a_{h,D}(w,v)| \le C\|w\|_{W_h^{1,\infty}(D)}\|v\|_{W_h^{1,1}(D)} \quad \text{for all } v,w\in H^{1+\alpha}(\Omega;\mathcal{T}_h). \tag{2.20}$$

2.6 A Commutation Formula

The following lemma provides a useful relation for moving a smooth function inside the bilinear form $a_h(\cdot,\cdot)$. A more general form of this relation can be found in [12, displays (4.6) and (4.7)].

Lemma 2.4. *Let $\omega \in C^\infty(\mathbb{R}^2)$. We have*

$$\begin{aligned} a_h(\omega v, z) = a_h(v, \omega z) &+ \sum_{T\in\mathcal{T}_h} \int_T v[\nabla\omega\cdot\nabla z + \nabla\cdot(z\nabla\omega)]\,dx \\ &+ 2\sum_{e\in\mathcal{E}_h}\int_e (\partial\omega/\partial n)\{\!\{v\}\!\}[\![z]\!]\,ds \quad \text{for all } v, z \in H^{1+\alpha}(\Omega;\mathcal{T}_h), \end{aligned} \tag{2.21}$$

where $\{\!\{v\}\!\}$ is the average of the values of v from the two triangles that share the edge e as a common edge if $e \in \mathcal{E}_h^i$, and $\{\!\{v\}\!\} = v$ if $e \in \mathcal{E}_h^b$.

Proof. Firstly, it follows from (1.3)–(1.6), the smoothness of ω and the product rule that

$$\begin{aligned} a_h(\omega v, z) = &\sum_{T\in\mathcal{T}_h}\int_\Omega v(\nabla\omega\cdot\nabla z)\,dx - \sum_{T\in\mathcal{T}_h}\int_T \nabla v\cdot(z\nabla\omega)\,dx + \sum_{T\in\mathcal{T}_h}\int_T \nabla v\cdot\nabla(\omega z)\,dx \\ &+ \sum_{e\in\mathcal{E}_h}\int_e (\{\!\{\partial v/\partial n\}\!\}[\![\omega z]\!] + (\partial\omega/\partial n)\{\!\{v\}\!\}[\![z]\!] + \omega\{\!\{\partial z/\partial n\}\!\}[\![v]\!])\,ds \\ &+ \sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e [\![v]\!][\![\omega z]\!]\,ds. \end{aligned} \tag{2.22}$$

Secondly, it follows from integration by parts that

$$-\sum_{T\in\mathcal{T}_h}\int_T \nabla v\cdot(z\nabla\omega)\,dx = \sum_{T\in\mathcal{T}_h}\int_T v\nabla\cdot(z\nabla\omega)\,dx + \sum_{e\in\mathcal{E}_h}\int_e [\![v]\!]\{\!\{z\}\!\}(\partial\omega/\partial n)\,ds + \sum_{e\in\mathcal{E}_h}\int_e \{\!\{v\}\!\}[\![z]\!](\partial\omega/\partial n)\,ds. \tag{2.23}$$

Relation (2.21) is obtained by substituting (2.23) into (2.22). □

3 A Discrete Caccioppoli Estimate

First we recall a superapproximation result. Let T be a triangle, d a positive parameter and ρ a smooth function on $\mathbb{R}^2$ that satisfies

$$|\rho|_{W^{\ell,\infty}(\mathbb{R}^2)} \le C_\dagger d^{-\ell} \quad \text{for } \ell = 0, 1, 2, \ldots. \tag{3.1}$$

We have (cf. [6, 15, 20])

$$|\rho^2\chi - \Pi_T(\rho^2\chi)|_{H^1(T)} + h_T|\rho^2\chi - \Pi_T(\rho^2\chi)|_{H^2(T)} \le Ch_T d^{-2}(\|\chi\|_{L^2(T)} + d|\rho\chi|_{H^1(T)}) \tag{3.2}$$

for all $\chi \in \mathbb{P}_k$, where Π_T is the nodal interpolation operator for the $\mathbb{P}_k$ Lagrange finite element on T, and the positive constant C depends on $C_\dagger$, k and the shape of T.

Let Ω_0 be an open subset of Ω, and let $\chi_h \in V_h$ satisfy

$$a_h(\chi_h, v) = 0 \quad \text{for all } v \in V_h, \quad v = 0 \text{ on } \Omega\setminus\Omega_d, \tag{3.3}$$

where

$$\Omega_d = \{x \in \Omega : \operatorname{dist}(x, \Omega_0) < d\}. \tag{3.4}$$

Let $\rho \in C^\infty(\mathbb{R}^2)$ satisfy (3.1) and

$$\rho = \begin{cases} 1 & \text{on } \Omega_0, \\ 0 & \text{on } \Omega\setminus\Omega_\delta, \end{cases} \tag{3.5}$$

where

$$c_1 \le (\delta/d) \le c_2 \quad \text{for some } c_1, c_2 \in (0,1). \tag{3.6}$$

The following result is a discrete analog of the Caccioppoli estimate for second order elliptic partial differential equations (cf. [9]).

Lemma 3.1. *We have*

$$\|\rho\chi_h\|_{W_h^{1,2}(\Omega)} \le Cd^{-1}\|\chi_h\|_{L^2(\Omega_d)} \tag{3.7}$$

provided that (h/d) *is sufficiently small.*

Proof. Let $\rho_I \in H_0^1(\Omega)$ be the nodal interpolant of ρ in the $\mathbb{P}_1$ conforming finite element space associated with $\mathcal{T}_h$. It follows from (1.9), (3.1) and standard interpolation error estimates (cf. [8, 13]) that

$$\begin{aligned}
\|(\rho-\rho_I)\chi_h\|_{W_h^{1,2}(\Omega)}^2 &\lesssim \sum_{T\in\mathcal{T}_h(\Omega_\delta)} \big(\|\chi_h\nabla(\rho-\rho_I)\|_{L^2(T)}^2 + \|(\rho-\rho_I)\nabla\chi_h\|_{L^2(T)}^2\big) \\
&\quad + \sum_{T\in\mathcal{T}_h(\Omega_\delta)}\sum_{e\subset\partial T} |e|\big(\|\{\!\{\chi_h(\partial(\rho-\rho_I)/\partial n)\}\!\}\|_{L^2(e)}^2 + \|(\rho-\rho_I)\{\!\{\partial\chi_h/\partial n\}\!\}\|_{L^2(e)}^2\big) \\
&\quad + \sum_{T\in\mathcal{T}_h(\Omega_\delta)}\sum_{e\subset\partial T} |e|^{-1}\|(\rho-\rho_I)[\![\chi_h]\!]\|_{L^2(e)}^2 \\
&\lesssim \sum_{T\in\mathcal{T}_h(\Omega_\delta)} \big[(h_T/d^2)^2\|\chi_h\|_{L^2(T)}^2 + (h_T/d)^2\|\nabla\chi_h\|_{L^2(T)}^2\big] \\
&\quad + \sum_{T\in\mathcal{T}_h(\Omega_\delta)}\sum_{e\subset\partial T} |e|\big[(|e|/d^2)^2\|\{\!\{|\chi_h|\}\!\}\|_{L^2(e)}^2 + (|e|/d)^2\|\{\!\{\partial\chi_h/\partial n\}\!\}\|_{L^2(e)}^2\big] \\
&\quad + \sum_{T\in\mathcal{T}_h(\Omega_\delta)}\sum_{e\subset\partial T} |e|^{-1}(|e|/d)^2\|[\![\chi_h]\!]\|_{L^2(e)}^2,
\end{aligned}$$

which together with (1.12), (1.13), (3.6) and $(h/d) \ll 1$ implies

$$\|(\rho-\rho_I)\chi_h\|_{W_h^{1,2}(\Omega)} \lesssim d^{-1}\|\chi_h\|_{L^2(\Omega_d)}. \tag{3.8}$$

Using (2.6) and (3.8), we find

$$\|\rho\chi_h\|_{W_h^{1,2}(\Omega)}^2 \lesssim \|\rho_I\chi_h\|_{W_h^{1,2}(\Omega)}^2 + \|(\rho-\rho_I)\chi_h\|_{W_h^{1,2}(\Omega)}^2 \lesssim a_h(\rho_I\chi_h,\rho_I\chi_h) + d^{-2}\|\chi_h\|_{L^2(\Omega_d)}^2, \tag{3.9}$$

and, in view of (2.5),

$$\begin{aligned}
a_h(\rho_I\chi_h,\rho_I\chi_h) &= a_h(\rho\chi_h,\rho\chi_h) - 2a_h((\rho-\rho_I)\chi_h,\rho\chi_h) + a_h((\rho-\rho_I)\chi_h,(\rho-\rho_I)\chi_h) \\
&\lesssim a_h(\rho\chi_h,\rho\chi_h) + \|(\rho-\rho_I)\chi_h\|_{W_h^{1,2}(\Omega)}\|\rho\chi_h\|_{W_h^{1,2}(\Omega)} + \|(\rho-\rho_I)\chi_h\|_{W_h^{1,2}(\Omega)}^2 \\
&\lesssim a_h(\rho\chi_h,\rho\chi_h) + d^{-1}\|\chi_h\|_{L^2(\Omega_d)}\|\rho\chi_h\|_{W_h^{1,2}(\Omega)} + d^{-2}\|\chi_h\|_{L^2(\Omega_d)}^2.
\end{aligned} \tag{3.10}$$

It then follows from (1.15), (3.9) and (3.10) that

$$\|\rho\chi_h\|_{W_h^{1,2}(\Omega)}^2 \lesssim a_h(\rho\chi_h,\rho\chi_h) + d^{-2}\|\chi_h\|_{L^2(\Omega_\delta)}^2, \tag{3.11}$$

and it only remains to estimate the first term on the right-hand side of (3.11).

According to Lemma 2.4, we can write

$$a_h(\rho\chi_h,\rho\chi_h) = a_h(\chi_h,\rho^2\chi_h) + R, \tag{3.12}$$

where

$$R = \sum_{T\in\mathcal{T}_h}\int_T \chi_h[\nabla\rho\cdot\nabla(\rho\chi_h) + \nabla\cdot(\rho\chi_h\nabla\rho)]\,dx + 2\sum_{e\in\mathcal{E}_h}\int_e (\partial\rho/\partial n)\{\!\{\chi_h\}\!\}[\![\rho\chi_h]\!]\,ds,$$

and we have, by (1.9), (1.12), (1.13), (3.1), (3.4)–(3.6) and the Cauchy–Schwarz inequality,

$$\begin{aligned}
|R| &\lesssim \sum_{T\in\mathcal{T}_h(\Omega_\delta)} \|\chi_h\|_{L^2(T)}[d^{-1}|\rho\chi_h|_{H^1(T)} + d^{-2}\|\chi_h\|_{L^2(T)}] \\
&\quad + d^{-1}\sum_{T\in\mathcal{T}_h(\Omega_\delta)}\sum_{e\subset\partial T} (|e|^{\frac12}\|\{\!\{\chi_h\}\!\}\|_{L^2(e)})(|e|^{-1/2}\|[\![\rho\chi_h]\!]\|_{L^2(e)}) \\
&\lesssim d^{-1}\|\chi_h\|_{L^2(\Omega_d)}\|\rho\chi_h\|_{W_h^{1,2}(\Omega)} + d^{-2}\|\chi_h\|_{L^2(\Omega_d)}^2.
\end{aligned} \tag{3.13}$$

Now we use (3.3), (3.5) and $(h/d) \ll 1$ to write

$$a_h(\chi_h, \rho^2\chi_h) = a_h(\chi_h, \rho^2\chi_h - \Pi_h(\rho^2\chi_h)), \tag{3.14}$$

and it follows from (1.3), (1.9), (3.5) and the Cauchy–Schwarz inequality that

$$\begin{aligned}
&a_h(\chi_h, \rho^2\chi_h - \Pi_h(\rho^2\chi_h)) \\
&\quad = \sum_{T\in\mathcal{T}_h} \int_T \nabla\chi_h \cdot \nabla(\rho^2\chi_h - \Pi_h(\rho^2\chi_h))\,dx \\
&\qquad + \sum_{e\in\mathcal{E}_h} \int_e [\{\!\{\partial\chi_h/\partial n\}\!\} [\![\rho^2\chi_h - \Pi_h(\rho^2\chi_h)]\!] + \{\!\{\partial(\rho^2\chi_h - \Pi_h(\rho^2\chi_h))/\partial n\}\!\} [\![\chi_h]\!]]\,ds \\
&\qquad + \sigma \sum_{e\in\mathcal{E}_h} |e|^{-1} \int_e [\![\chi_h]\!] [\![\rho^2\chi_h - \Pi_h(\rho^2\chi_h)]\!]\,ds \\
&\quad \lesssim \sum_{T\in\mathcal{T}_h(\Omega_\delta)} |\chi_h|_{H^1(T)} |\rho^2\chi_h - \Pi_h(\rho^2\chi_h)|_{H^1(T)} \\
&\qquad + \sum_{T\in\mathcal{T}_h(\Omega_\delta)} \sum_{e\subset\partial T} (|e|^{\frac{3}{2}} \|\{\!\{\partial\chi_h/\partial n\}\!\}\|_{L^2(e)})(|e|^{-\frac{3}{2}} \|[\![\rho^2\chi_h - \Pi_h(\rho^2\chi_h)]\!]\|_{L^2(e)}) \\
&\qquad + \sum_{T\in\mathcal{T}_h(\Omega_\delta)} \sum_{e\subset\partial T} (|e|^{-\frac{1}{2}} \|\{\!\{\partial(\rho^2\chi_h - \Pi_h(\rho^2\chi_h))/\partial n\}\!\}\|_{L^2(e)})(|e|^{\frac{1}{2}} \|[\![\chi_h]\!]\|_{L^2(e)}) \\
&\qquad + \sum_{T\in\mathcal{T}_h(\Omega_\delta)} \sum_{e\subset\partial T} (|e|^{\frac{1}{2}} \|[\![\chi_h]\!]\|_{L^2(e)})(|e|^{-\frac{3}{2}} \|[\![\rho^2\chi_h - \Pi_h(\rho^2\chi_h)]\!]\|_{L^2(e)}),
\end{aligned}$$

which together with (1.12)–(1.14), (2.4), (3.2) and (3.4)–(3.6) gives the estimate

$$\begin{aligned}
a_h(\chi_h, \rho^2\chi_h - \Pi_h(\rho^2\chi_h)) &\lesssim \sum_{T\in\mathcal{T}_h(\Omega_\delta)} \|\chi_h\|_{L^2(T)} (d^{-2}\|\chi_h\|_{L^2(T)} + d^{-1}|\rho\chi_h|_{H^1(T)}) \\
&\lesssim d^{-2}\|\chi_h\|^2_{L^2(\Omega_d)} + d^{-1}\|\chi_h\|_{L^2(\Omega_d)} \|\rho\chi_h\|_{W_h^{1,2}(\Omega)}.
\end{aligned} \tag{3.15}$$

Putting (3.11)–(3.15) together, we arrive at the estimate

$$\|\rho\chi_h\|^2_{W_h^{1,2}(\Omega)} \lesssim d^{-2}\|\chi_h\|_{L^2(\Omega_d)} + d^{-1}\|\chi_h\|_{L^2(\Omega_d)} \|\rho\chi_h\|_{W_h^{1,2}(\Omega)},$$

which implies (3.7) through (1.15). □

4 A Local Energy Norm Error Estimate

We derive a local energy norm error estimate that is needed in Section 5 and Section 6. A similar result can also be found in [12, Section 4].

We will use the notation Ω_0 and Ω_d introduced in Section 3.

Lemma 4.1. *We have*

$$\|\zeta - P_h\zeta\|_{W_h^{1,2}(\Omega_0)} \le C\Big(\inf_{v\in V_h} [\|\zeta - v\|_{W_h^{1,2}(\Omega_d)} + d^{-1}\|\zeta - v\|_{L^2(\Omega_d)}] + d^{-1}\|\zeta - P_h\zeta\|_{L^2(\Omega_d)}\Big) \tag{4.1}$$

for all $\zeta \in \mathring{E}(\Delta; L^2(\Omega))$, provided that (h/d) is sufficiently small.

Proof. Let ω be a smooth function defined on $\mathbb{R}^2$ such that

$$\omega = \begin{cases} 1 & \text{on } \Omega_{d/3}, \\ 0 & \text{on } \Omega \setminus \Omega_{2d/3}, \end{cases} \tag{4.2}$$

and

$$\|\nabla\omega\|_{L^\infty(\Omega)} \le Cd^{-1}. \tag{4.3}$$

We have

$$\|\zeta - P_h\zeta\|_{W_h^{1,2}(\Omega_0)} = \|\omega\zeta - P_h\zeta\|_{W_h^{1,2}(\Omega_0)} \tag{4.4}$$

by (1.9), (4.2) and $(h/d) \ll 1$, and we can write

$$\omega\zeta - P_h\zeta = [\omega\zeta - P_h(\omega\zeta)] + [P_h(\omega\zeta) - P_h\zeta]. \tag{4.5}$$

There is also a straightforward estimate

$$\begin{aligned}
\|\omega\zeta - P_h(\omega\zeta)\|^2_{W_h^{1,2}(\Omega_0)} &\le \|\omega\zeta - P_h(\omega\zeta)\|^2_{W_h^{1,2}(\Omega)} \lesssim \|\omega\zeta\|^2_{W_h^{1,2}(\Omega)} \\
&\lesssim \sum_{T\in\mathcal{T}_h(\Omega_{2d/3})} \Big(\|\nabla(\omega\zeta)\|^2_{L^2(T)} + \sum_{e\subset\partial T} |e| \|\{\!\{\partial(\omega\zeta)/\partial n\}\!\}\|^2_{L^2(e)}\Big) \\
&\lesssim \sum_{T\in\mathcal{T}_h(\Omega_{2d/3})} (d^{-2}\|\zeta\|^2_{L^2(T)} + \|\nabla\zeta\|^2_{L^2(T)}) \\
&\quad + \sum_{T\in\mathcal{T}_h(\Omega_{2d/3})} \sum_{e\subset\partial T} |e| (d^{-2}\|\{\!\{\zeta\}\!\}\|^2_{L^2(e)} + \|\{\!\{\partial\zeta/\partial n\}\!\}\|^2_{L^2(e)}) \\
&\lesssim \|\zeta\|^2_{W_h^{1,2}(\Omega_d)} + d^{-2}\|\zeta\|^2_{L^2(\Omega_d)}
\end{aligned} \tag{4.6}$$

that follows from (1.9), (1.13), (2.8), (4.2), (4.3) and $(h/d) \ll 1$.

Note also that

$$\|\omega\zeta - P_h(\omega\zeta)\|_{L^2(\Omega)} \lesssim h\|\omega\zeta - P_h(\omega\zeta)\|_{W_h^{1,2}(\Omega)} \lesssim h(\|\zeta\|_{W_h^{1,2}(\Omega_d)} + d^{-1}\|\zeta\|_{L^2(\Omega_d)}) \tag{4.7}$$

by (2.13) and (4.6).

It only remains to estimate the function $\chi_h = P_h(\omega\zeta) - P_h\zeta \in V_h$ that satisfies

$$a_h(\chi_h, v) = 0 \quad \text{for all } v \in V_h, \quad v = 0 \text{ on } \Omega \setminus \Omega_{d/3} \tag{4.8}$$

because of (2.7) and (4.2).

Let ρ be a smooth function on $\mathbb{R}^2$ such that

$$\rho = \begin{cases} 1 & \text{on } \Omega_0, \\ 0 & \text{on } \Omega \setminus \Omega_{d/4}, \end{cases} \tag{4.9}$$

and

$$\|\rho\|_{W^{\ell,\infty}(\mathbb{R}^2)} \le Cd^{-\ell} \quad \text{for} \quad \ell = 0, 1, 2, \dots.$$

In view of Lemma 3.1 and (4.8), we have

$$\|\rho\chi_h\|_{W_h^{1,2}(\Omega)} \lesssim d^{-1}\|\chi_h\|_{L^2(\Omega_{d/3})}. \tag{4.10}$$

Combining (4.2), (4.4)–(4.7), (4.9)–(4.10) and $(h/d) \ll 1$, we find

$$\begin{aligned}
\|\zeta - P_h\zeta\|_{W_h^{1,2}(\Omega_0)} &\le \|\omega\zeta - P_h(\omega\zeta)\|_{W_h^{1,2}(\Omega_0)} + \|\rho\chi_h\|_{W_h^{1,2}(\Omega_0)} \\
&\lesssim \|\zeta\|_{W_h^{1,2}(\Omega_d)} + d^{-1}\|\zeta\|_{L^2(\Omega_d)} + d^{-1}\|\chi_h\|_{L^2(\Omega_{d/3})} \\
&\lesssim \|\zeta\|_{W_h^{1,2}(\Omega_d)} + d^{-1}\|\zeta\|_{L^2(\Omega_d)} + d^{-1}(\|\omega\zeta - P_h(\omega\zeta)\|_{L^2(\Omega_{d/3})} + \|\omega\zeta - P_h\zeta\|_{L^2(\Omega_{d/3})}) \\
&\lesssim \|\zeta\|_{W_h^{1,2}(\Omega_d)} + d^{-1}\|\zeta\|_{L^2(\Omega_d)} + d^{-1}\|\zeta - P_h\zeta\|_{L^2(\Omega_d)},
\end{aligned}$$

which implies (4.1) if we replace ζ by $\zeta - v$ for an arbitrary $v \in V_h$. □

Corollary 4.2. *In the case where $\Omega_0 \Subset \Omega$, we have*

$$\|\zeta - P_h\zeta\|_{W_h^{1,2}(\Omega_0)} \le C(h|\zeta|_{W_h^{2,2}(\Omega_d)} + d^{-1}\|\zeta - P_h\zeta\|_{L^2(\Omega_d)})$$

for all $\zeta \in \mathring{E}(\Delta; L^2(\Omega))$, provided that (h/d) is sufficiently small.

Proof. First we note that we can adjust the value of d in (4.1) so that

$$\|\zeta - P_h\zeta\|_{W_h^{1,2}(\Omega_0)} \lesssim \|\zeta - \Pi_h\zeta\|_{W_h^{1,2}(\Omega_{d/2})} + d^{-1}\|\zeta - \Pi_h\zeta\|_{L^2(\Omega_{d/2})} + d^{-1}\|\zeta - P_h\zeta\|_{\Omega_{d/2}}.$$

Next, in view of (1.9), (1.14), (2.1), (2.2) and (2.16), we have

$$\begin{aligned}\|\zeta - \Pi_h\zeta\|^2_{W_h^{1,2}(\Omega_{d/2})} &= \sum_{T\in\mathcal{T}_h(\Omega_{d/2})}\Big[\|\nabla(\zeta - \Pi_h\zeta)\|^2_{L^2(T)} + \sum_{e\subset\partial T}|e|\|\{\!\{\partial(\zeta - \Pi_h\zeta)/\partial n\}\!\}\|^2_{L^2(e)}\Big]\\ &\lesssim \sum_{T\in\mathcal{T}_h(\Omega_{d/2})} h_T^2|\zeta|^2_{H^2(T)} + \sum_{T\in\mathcal{T}_h(\Omega_d)}\big(|\zeta - \Pi_h\zeta|^2_{H^1(T)} + h_T^2|\zeta - \Pi_h\zeta|^2_{H^2(T)}\big)\\ &\lesssim \sum_{T\in\mathcal{T}_h(\Omega_d)} h_T^2|\zeta|^2_{H^2(T)} \le h^2|\zeta|_{W_h^{2,2}(\Omega_d)},\\ d^{-2}\|\zeta - \Pi_h\zeta\|^2_{L^2(\Omega_{d/2})} &\le d^{-2}\sum_{T\in\mathcal{T}_h(\Omega_d)}\|\zeta - \Pi_h\zeta\|^2_{L^2(T)} \lesssim d^{-2}\sum_{T\in\mathcal{T}_h(\Omega_d)} h_T^4|\zeta|^2_{H^2(T)} \le h^2|\zeta|^2_{W_h^{2,2}(\Omega_d)}.\end{aligned}$$

□

5 An Interior $W^{1,1}$ Error Estimate

Let $T_* \in \mathcal{T}_h$ such that $\bar{T}_*$ is a compact subset of the open set $D \Subset \Omega$, $\phi_* \in C_c^\infty(T_*)$, and let the function ζ be the Newtonian potential with density ϕ_* defined by

$$\zeta(x) = \int_{T_*} N(x-y)\phi_*(y)\,dy, \tag{5.1}$$

where

$$N(x) = -\frac{1}{2\pi}\ln|x| \tag{5.2}$$

is the fundamental solution for $-\Delta$ in the free space $\mathbb{R}^2$.

Then ζ belongs to $C^\infty(\mathbb{R}^2)$,

$$-\Delta\zeta = \phi_*, \tag{5.3}$$

and direct calculations produce the following estimates:

$$|\zeta(x)| \le C[1 + |\ln \operatorname{dist}(x, T_*)|]\|\phi_*\|_{L^1(T_*)} \quad \text{for all } x \in \mathbb{R}^2, \tag{5.4a}$$

$$\|\nabla\zeta(x)\| \le C[\operatorname{dist}(x, T_*)]^{-1}\|\phi_*\|_{L^1(T_*)} \quad \text{for all } x \in \mathbb{R}^2, \tag{5.4b}$$

$$\|\nabla^2\zeta(x)\| \le C[\operatorname{dist}(x, T_*)]^{-2}\|\phi_*\|_{L^1(T_*)} \quad \text{for all } x \in \mathbb{R}^2. \tag{5.4c}$$

Lemma 5.1. *Let ζ be defined by* (5.1). *Given any $\omega \in C_c^\infty(\Omega)$ such that*

$$\omega = 1 \quad \text{on } D, \tag{5.5}$$

we have

$$\|\omega\zeta - P_h(\omega\zeta)\|_{W_h^{1,1}(D)} \le Ch^2(1 + |\ln h|)\|\phi_*\|_{L^2(T_*)}, \tag{5.6}$$

where the positive constant C is independent of h but increases as $\operatorname{dist}(T_*, \Omega\setminus D)$ *decreases.*

Proof. Observe that $\omega\zeta \in C_c^\infty(\Omega)$ and

$$\|\Delta(\omega\zeta)\|_{L^2(\Omega)} \lesssim \|\phi_*\|_{L^2(T_*)} \tag{5.7}$$

by (5.3)–(5.5), where the hidden constant increases as $\operatorname{dist}(T_*, \Omega\setminus D)$ decreases.

We follow the approach in [27] to employ a dyadic decomposition

$$A_j = \{x \in \Omega : 2^{-j-1}d < \operatorname{dist}(x, T_*) < 2^{-j}d\}, \quad \text{where } d = \max_{x\in\partial D}\operatorname{dist}(x, T_*).$$

Let J be the largest integer such that

$$2^{-J}d \ge mh \tag{5.8}$$

for a sufficiently large positive integer m (independent of h), $\Omega_h = D \setminus \bigcup_{j=0}^{J} A_j$ and

$$A_{-1} = \Big\{x \in \Omega : d < \operatorname{dist}(x, T_*) < d + \frac{1}{2}\operatorname{dist}(D, \mathbb{R}^2 \setminus \Omega)\Big\}.$$

Note that $|A_j| \approx 2^{-j}d$ for $0 \le j \le J$ and (5.8) implies $|\Omega_h| \approx h^2$.

We have

$$\|\omega\zeta - P_h(\omega\zeta)\|_{W_h^{1,1}(D)} \le \|\omega\zeta - P_h(\omega\zeta)\|_{W_h^{1,1}(\Omega_h)} + \sum_{j=0}^{J}\|\omega\zeta - P_h(\omega\zeta)\|_{W_h^{1,1}(A_j)}, \tag{5.9}$$

$$\begin{aligned}\|\omega\zeta - P_h(\omega\zeta)\|_{W_h^{1,1}(\Omega_h)} &\lesssim h\|\omega\zeta - P_h(\omega\zeta)\|_{W_h^{1,2}(\Omega_h)}\\ &\le h\|\omega\zeta - P_h(\omega\zeta)\|_{W_h^{1,2}(\Omega)} \lesssim h^2\|\Delta(\omega\zeta)\|_{L^2(\Omega)} \lesssim h^2\|\phi_*\|_{L^2(T_*)}\end{aligned} \tag{5.10}$$

by (2.12), Lemma 2.2 and (5.7).

For $j = 0, \dots, J$ and $d_j = 2^{-j}d$, we have

$$\begin{aligned}\|\omega\zeta - P_h(\omega\zeta)\|_{W_h^{1,1}(A_j)} &\lesssim d_j\|\omega\zeta - P_h(\omega\zeta)\|_{W^{1,2}(A_j)}\\ &\lesssim d_j\big(h|\omega\zeta|_{W_h^{2,2}(A_{j-1}\cup A_j\cup A_{j+1})} + d_j^{-1}\|\omega\zeta - P_h(\omega\zeta)\|_{L^2(A_{j-1}\cup A_j\cup A_{j+1})}\big)\end{aligned} \tag{5.11}$$

by Lemma 2.2, Corollary 4.2 (with $\Omega_0 = A_j$ and $d \approx d_j$), and

$$|\omega\zeta|_{W_h^{2,2}(A_{j-1}\cup A_j\cup A_{j+1})} \lesssim d_j^{-1}\|\phi_*\|_{L^1(T_*)} \lesssim d_j^{-1}h\|\phi_*\|_{L^2(T_*)} \tag{5.12}$$

by (5.4) and the Cauchy–Schwarz inequality.

It follows from (2.13), (5.7), (5.8), (5.11) and (5.12) that

$$\sum_{j=0}^{J}\|\omega\zeta - P_h(\omega\zeta)\|_{W_h^{1,1}(A_j)} \lesssim \sum_{j=0}^{J} h^2\|\phi_*\|_{L^2(T_*)} + \sum_{j=0}^{J}\|\omega\zeta - P_h(\omega\zeta)\|_{L^2(\Omega)} \lesssim h^2(1 + |\ln h|)\|\phi_*\|_{L^2(T_*)},$$

which together with (5.9) and (5.10) implies (5.6). □

6 An Interior Maximum Norm Error Estimate

Let $T_* \in \mathcal{T}_h$ and $x_* \in \bar{T}_*$. We follow [28, 29] to introduce a smoothed Dirac delta function $\phi_* \in C_c^\infty(T)$ such that

$$\int_{T_*} p\phi_*\,dx = p(x_*) \quad \text{for all } p \in \mathbb{P}_k. \tag{6.1}$$

More precisely, we start with the construction in the reference simplex $\hat{T}$ with vertices (0, 0), (1, 0) and (0, 1). Let λ be a nonnegative function in $C_c^\infty(\hat{T})$ such that

$$\int_{\hat{T}} \lambda\,dx = 1.$$

The formula

$$(\!(p, q)\!) = \int_{\hat{T}} pq\lambda\,dx$$

defines an inner product on $\mathbb{P}_k$, and there exists $q \in \mathbb{P}_k$ such that

$$(\!(p, q)\!) = p(\hat{x}_*) \quad \text{for all } p \in \mathbb{P}_k,$$

where $\hat{x}_*$ is the point in the closure of $\hat{T}$ corresponding to $x_* \in \bar{T}_*$ under an orientation-preserving affine transformation A_* that maps $\hat{T}$ to T_*. We have

$$\int_{\hat{T}} p\hat{\phi}_*\,dx = p(\hat{x}_*) \quad \text{for all } p \in \mathbb{P}_k,$$

where $\hat{\phi}_* = q\lambda \in C_c^\infty(\hat{T})$. We then take ϕ_* to be the function $(2|T|)^{-1}(\hat{\phi}_* \circ A_*^{-1})$. In particular, we have

$$h_{T_*}\|\phi_*\|_{L^2(T_*)} + \|\phi_*\|_{L^1(T_*)} \approx 1. \tag{6.2}$$

Theorem 6.1. *Let K be a compact subset of the open subset $D \Subset \Omega$. We have*

$$\|u-u_h\|_{L^\infty(K)} \le C(\|u-\Pi_h u\|_{L^\infty(D)} + h(1+|\ln h|)\|u-\Pi_h u\|_{W_h^{1,\infty}(D)} + \|u-u_h\|_{L^2(D)} + h\|u-\Pi_h u\|_{W_h^{1,2}(\Omega)})$$

asymptotically as $h \downarrow 0$, where the positive constant C is independent of h.

Proof. We may take D to be a mesh-subdomain of $\mathcal{T}_h$ without loss of generality in the following arguments. (Otherwise, we replace D by a mesh-subdomain that is a subset of D, which is possible because $h \downarrow 0$.)

Let $T_* \in \mathcal{T}_h(K)$, let $x_* \in \bar{T}_*$ be one of the nodes for the $\mathbb{P}_k$ Lagrange element, and let $\phi_* \in C_c^\infty(T_*)$ satisfy (6.1) and (6.2). We can assume that h is sufficiently small so that $\operatorname{dist}(T_*, \Omega \setminus D) \ge \frac{1}{2}\operatorname{dist}(K, \Omega \setminus D)$.

Let $\omega \in C_c^\infty(\Omega)$ satisfy (5.5). We have, by (6.1) and (5.5),

$$\begin{aligned} u(x_*) - u_h(x_*) &= (\Pi_h u)(x_*) - u_h(x_*) \\ &= \int_\Omega \omega(\Pi_h u - u_h)\phi_*\,dx \\ &= \int_\Omega \omega(\Pi_h u - u)\phi_*\,dx + \int_\Omega \omega(u-u_h)\phi_*\,dx, \end{aligned} \tag{6.3}$$

and, in view of (5.5) and (6.2),

$$\left|\int_\Omega \omega(\Pi_h u - u)\phi_*\,dx\right| \le \|\Pi_h u - u\|_{L^\infty(D)}\|\phi_*\|_{L^1(T_*)} \lesssim \|\Pi_h u - u\|_{L^\infty(D)}. \tag{6.4}$$

Let $N(x)$ be the fundamental solution of $-\Delta$ in (5.2), and let

$$g(x) = \int_{T_*} N(x-y)\phi_*(y)\,dy$$

be the Newtonian potential with density ϕ_*.

We have $g \in C^\infty(\mathbb{R}^2)$,

$$-\Delta g = \phi_* \tag{6.5}$$

and

$$\|g\|_{W^{2,\infty}(\Omega\setminus\tilde{D})} \lesssim \|\phi_*\|_{L^1(T_*)} \approx 1, \tag{6.6}$$

where $\tilde{D}$ is a mesh-subdomain of $\mathcal{T}_h$ such that $K \subset \tilde{D} \Subset D$ and that $\operatorname{dist}(\tilde{D}, \Omega \setminus D) \approx \frac{1}{2}\operatorname{dist}(K, \Omega\setminus D)$.

The estimate

$$\|\Delta(\omega g)\|_{L^2(\Omega)} \lesssim h^{-1} \tag{6.7}$$

is then a simple consequence of (5.5), (6.2), (6.5) and (6.6).

It follows from Remark 2.1, Lemma 2.4 and (6.5) that

$$\int_\Omega \omega(u-u_h)\phi_*\,dx = a_h(\omega(u-u_h), g) = a_h(u-u_h, \omega g) + I, \tag{6.8}$$

where

$$I = \sum_{T\in\mathcal{T}_h} \int_T (u-u_h)[\nabla\omega\cdot\nabla g + \nabla\cdot(g\nabla\omega)]\,dx$$

satisfies

$$|I| \lesssim \|u-u_h\|_{L^2(D)} \tag{6.9}$$

because of (5.5) and (6.6).

Next we use (2.7), (2.10) and (2.18) to write

$$\begin{aligned} a_h(u-u_h, \omega g) &= a_h(u-\Pi_h u, \omega g - P_h(\omega g)) \\ &= a_{h,D}(u-\Pi_h u, \omega g - P_h(\omega g)) + a_{h,\Omega\setminus D}(u-\Pi_h u, \omega g - P_h(\omega g)), \end{aligned} \tag{6.10}$$

and we find, by (2.19) and (2.20),

$$|a_{h,D}(u - \Pi_h u, \omega g - P_h(\omega g))| \lesssim \|u - \Pi_h u\|_{W_h^{1,\infty}(D)} \|\omega g - P_h(\omega g)\|_{W_h^{1,1}(D)}, \tag{6.11}$$

$$|a_{h,\Omega\setminus D}(u - \Pi_h u, \omega g - P_h(\omega g))| \lesssim \|u - \Pi_h u\|_{W_h^{1,2}(\Omega\setminus D)} \|\omega g - P_h(\omega g)\|_{W_h^{1,2}(\Omega\setminus D)}. \tag{6.12}$$

It only remains to estimate the two terms $\|\omega g - P_h(\omega g)\|_{W_h^{1,1}(D)}$ and $\|\omega g - P_h(\omega g)\|_{W_h^{1,2}(\Omega\setminus D)}$.

First we take $\Omega_0 = \Omega \setminus D$ and $d = \frac{1}{2}\operatorname{dist}(\tilde{D}, \Omega \setminus D)$ in Lemma 4.1 to obtain

$$\|\omega g - P_h(\omega g)\|_{W_h^{1,2}(\Omega\setminus D)} = \|\omega g - P_h(\omega g)\|_{W_h^{1,2}(\Omega_0)} \lesssim \|\omega g - \Pi_h(\omega g)\|_{W_h^{1,2}(\Omega_d)} + \|\omega g - P_h(\omega g)\|_{L^2(\Omega)}. \tag{6.13}$$

From (1.14), (2.2), (6.6) and the same calculation in the proof of Corollary 4.2, we have

$$\|\omega g - \Pi_h(\omega g)\|_{W_h^{1,2}(\Omega_d)} \lesssim h, \tag{6.14}$$

and, from (2.13) and (6.7),

$$\|\omega g - P_h(\omega g)\|_{L^2(\Omega)} \lesssim h. \tag{6.15}$$

Combining (6.13)–(6.15), we see that

$$\|\omega g - P_h(\omega g)\|_{W_h^{1,2}(\Omega\setminus D)} \lesssim h,$$

which together with (6.12) gives

$$|a_{h,\Omega\setminus D}(u - \Pi_h u, \omega g - P_h(\omega g))| \lesssim h\|u - \Pi_h u\|_{W_h^{1,2}(\Omega)}. \tag{6.16}$$

Finally, using Lemma 5.1 and (6.2), we obtain

$$\|\omega g - P_h(\omega g)\|_{W_h^{1,1}(D)} \lesssim h^2(1 + |\ln h|)\|\phi_*\|_{L^2(T_*)} \lesssim h(1 + |\ln h|),$$

which, in view of (6.11), implies

$$|a_{h,D}(u - \Pi_h u, \omega g - P_h(\omega g))| \lesssim h(1 + |\ln h|)\|u - \Pi_h u\|_{W_h^{1,\infty}(D)}. \tag{6.17}$$

Putting (6.3), (6.4), (6.8)–(6.10), (6.16) and (6.17) together, we arrive at the estimate

$$\begin{aligned} |u(x_*) - u_h(x_*)| &\lesssim \|u - \Pi_h u\|_{L^\infty(D)} + h(1 + |\ln h|)\|u - \Pi_h u\|_{W_h^{1,\infty}(D)} \\ &\quad + \|u - u_h\|_{L^2(D)} + h\|u - \Pi_h u\|_{W_h^{1,2}(\Omega)}. \end{aligned} \tag{6.18}$$

On the other hand, we have, by scaling,

$$\|\Pi_h u - u_h\|_{L^\infty(T)} \lesssim \sum_{i=1}^{N_k} |(\Pi_h u - u_h)(x_i)|, \tag{6.19}$$

where $x_1, \dots, x_{N_k} \in \bar{T}$ are the nodes for the $\mathbb{P}_k$ Lagrange finite element.

It then follows from (6.18) and (6.19) that

$$\begin{aligned} \|u - u_h\|_{L^\infty(T)} &\lesssim \|u - \Pi_h u\|_{L^\infty(D)} + h(1 + |\ln h|)\|u - \Pi_h u\|_{W_h^{1,\infty}(D)} \\ &\quad + \|u - u_h\|_{L^2(D)} + h\|u - \Pi_h u\|_{W_h^{1,2}(\Omega)} \quad \text{for all } T \in \mathcal{T}_h(K). \end{aligned}$$

□

7 Concluding Remarks

We have established an interior maximum norm error estimate for the SIP method for a simple model problem in two dimensions. The derivation is self-contained (up to the standard results for the SIP method and the superapproximation result in (3.2)). The results in this paper can be extended along the lines in [20] to other discontinuous Galerkin methods in [4].

Our approach can also be applied to elliptic problems with variable coefficients where $-\Delta$ is replaced by an elliptic operator $p(x, D)$ in divergence form. One only has to replace the Newtonian potential

$$\int_{T_*} N(x-y)\phi_*(y)\,dy$$

by $q(x, D)\phi_*$, where the pseudo-differential operator $q(x, D)$ of order -2 is a parametrix for $p(x, D)$ in the free space (cf. [31, Theorem 3.1.3 and Lemma 12.3.1] and [30, Section 6.4]).

Note that the approach in this paper does not work properly in three dimensions because in that case estimate (6.2) takes the form

$$h^{3/2}\|\phi_*\|_{L^2(T_*)} + \|\phi_*\|_{L^1(T_*)} \approx 1$$

so that

$$\|\omega g - P_h(\omega g)\|_{L^2(\Omega)} \lesssim h^{\frac{1}{2}}.$$

Consequently, estimate (6.16) now reads

$$|a_{h,\Omega\setminus D}(u - \Pi_h u, \omega g - \Pi_h(\omega g))| \lesssim h^{\frac{1}{2}}\|u - \Pi_h u\|_{W_h^{1,2}(\Omega)},$$

and the fourth term that appears on the right-hand side of (1.7) becomes $h^{\frac{1}{2}}\|u - \Pi_h u\|_{W_h^{1,2}(\Omega)}$, which is suboptimal.

We believe interior pointwise error estimates in three dimensions can still be established without using a local Neumann problem. However, the correct order of convergence can only be achieved if no global term appears on the right-hand side of the estimate, which means more of the techniques in [27] would have to be adopted.

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