THE JOURNAL OF SYMBOLIC LOGIC
Volume 57, Number 2, June 1992

THEORIES OF MODULES CLOSED UNDER DIRECT PRODUCTS

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Abstract. We generalize to theories of modules (complete or not) a result of U. Felgner stating that a complete theory of abelian groups is a Horn theory if and only if it is closed under products. To prove this we show that a reduced product of modules $\prod_F M_i$ ($i \in I$) is elementarily equivalent to a direct product of ultraproducts of the modules M_i ($i \in I$).

§1. Introduction. Let L be some first-order language. By an L-theory we mean any consistent set of L-sentences. Furthermore we say that a theory is closed under some operation on models if this is the case for its class of models. For example, a theory T is *closed under direct products* if for any models M_i of T ($i \in I$), $\prod_I M_i$ (the cartesian product) is a model of T. We will use the book [1] of Chang and Keisler as a general reference on model theory.

One may ask if there is any relationship between the fact of being closed under direct products and the fact of being closed under binary products, i.e. if M and N are both models then $M \times N$ is also a model. This question is answered by the following classical theorem of Vaught (see [1, Theorem 6.3.14])

THEOREM 1.1 (Vaught). *A theory T is closed under direct products if and only if it is closed under binary products.*

The following class of formulas plays an important role in the analysis of theories closed under direct products.

DEFINITION. Let L be some first-order language. The set of *Horn formulas* of L is the smallest set of formulas containing every disjunction of finitely many negations of atomic formulas with at most one atomic formula, which is closed under conjunction and both quantifiers. Furthermore a theory is said to be a *Horn theory* if it is axiomatized by Horn sentences.

It has been proved by Horn that any Horn theory is closed under direct products. Unfortunately the converse is not true. Chang and Morel showed (see [1, Example 6.2.3]) that the theory of Boolean algebras having at least one atom is closed under direct products but that it is not a Horn theory. Nevertheless Horn theories are exactly the theories closed under *reduced products*, an algebraic operation which we will now define.

DEFINITION. Let L be any first-order language, and let M_i ($i \in I$) be some L-structures. For F a filter over I we define the *reduced product* $\prod_F M_i$ to be as follows.

Received January 16, 1991; revised April 29, 1991.

— The universe of the reduced product is the cartesian product $\prod_I M_i$ modulo the equivalence relation $\sim$, where $m \sim n$ if $\{i \in I;\ m(i) = n(i)\} \in F$ (here $m(i)$ is the ith component of m).

— In the same way we say that a relation or an operation is satisfied in the reduced product if the set of indices of I where it is satisfied is in F.

The following result was proved by Keisler in [4, Result A, p. 307] using the continuum hypothesis. Galvin in [3, Theorem 6.1] showed that the continuum hypothesis was not necessary for the result to hold. The next theorem can also be seen as a consequence of [1, Lemma 6.2.5 and 6.2.5′] using the fact, which follows from the existence for any formula of an autonomous set containing it (see [1, p. 426] for the definition of autonomous set and [1, Theorem 6.3.6(i)] for a proof of this fact), that a theory closed under reduced products is axiomatized by reduced products formulas, i.e. formulas which by themselves form theories closed under reduced products.

THEOREM 1.2 (Galvin and Keisler). *Let L be any first-order language and let T be an L-theory. The following conditions are equivalent.*

(a) *T is a Horn theory.*

(b) *T is closed under reduced products.*

Hence to show that the theory of Boolean algebras with at least one atom is not a Horn theory Chang and Morel considered a reduced product of an atomic Boolean algebra over the Fréchet filter on the natural numbers, i.e. the filter of cofinite set. As it is easily shown, this reduced product has no atom; hence the theory of Boolean algebras with at least one atom is not a Horn theory.

In the following section we will show that the situation is much simpler for modules, namely that the theory of modules is closed under products if and only if it is a Horn theory.

§2. Theories of modules closed under direct products. In the remainder of this paper, let L be the language of the theory of modules over some fixed ring. Every module that we will consider in this section will be over this fixed ring. As a general reference on model theory of modules we use the book [5] of M. Prest.

DEFINITION. A *positive primitive formula* is an L-formula of the form

$$\exists \bar{y}\left(\bigwedge_i \varphi_i(\bar{x}, \bar{y})\right)$$

where the conjunction is finite and $\varphi_i(\bar{x}, \bar{y})$ is an atomic L-formula.

Let us first recall that direct products and direct sums are elementarily equivalent (see [5, Lemma 2.24(a)]). Hence a theory of modules is closed under direct products if and only if it is closed under direct sums.

Let M be a module. The structures M and $M \oplus M$ are elementarily equivalent if and only if each of the Baur-Monk invariants of M are either equal to 1 or infinite (see [5, Corollary 2.18]). U. Felgner noticed this fact, and he furthermore proved the following result.

THEOREM 2.1 [2, Theorem 2.1]. *Let T be a complete theory of abelian groups. The following conditions are equivalent.*

(a) *T is closed under direct products.*

(b) *T is a Horn theory.*

(c) *For some (any) model M of T, we have that M and $M \oplus M$ are elementarily equivalent.*

This result is somewhat surprising since, as mentioned in §1, it is not the case for all first-order languages that every theory closed under direct products is a Horn theory.

We will now show that this result generalizes to any theory of modules (complete or not). To prove this we will show that a reduced product of modules $\prod_F M_i$ $(i \in I)$ is elementarily equivalent to a product of ultraproducts of the modules M_i $(i \in I)$.

DEFINITION. Let $\prod_F M_i$ $(i \in I)$ be a reduced product. The Boolean algebra $\mathscr{P}(I)/F$, where $\mathscr{P}(I)$ is the power set of I, is called the *quotient* of this reduced product.

We will first show that a reduced product is elementarily equivalent to the direct sum of a reduced product with atomless quotient and products of ultraproducts.

DEFINITION. Let A and B be two modules. A homomorphism α of A in B is said to be a *pure embedding* if for any tuple $\bar{a} \in A$ and any positive primitive formula $\varphi(\bar{x})$, we have that $\varphi(\bar{a})$ is true in A whenever $\varphi(\alpha(\bar{a}))$ is true in B.

DEFINITION. Let A, B and C be modules, and let α: $A \to B$ and β: $B \to C$ be homomorphisms. The sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is said to be *pure exact* if α is a pure embedding, β is surjective and the kernel of β is equal to the image of α.

NOTATION. Let I be a set, and let F be a filter over I. Let E be a subset of I. The equivalence class of E modulo F will be written E/F. It is clear that for subsets E and E' of I, $E/F = E'/F$ if and only if $(E \Delta E')^c$ (the complement of the symmetric difference of E with E') is in F.

DEFINITION. Let F be a filter over a set I, and let E and E' be subsets of I. We say that E and E' are *F-disjoint* if $(E \cap E')/F = \varnothing/F$.

In the following proofs we will work with *representatives* in $\prod_I M_i$ of elements of $\prod_F M_i$. The *support* of an element m of $\prod_I M_i$ is the subset of I for which the component of m is nonzero. For a tuple of elements $\bar{m}$ it is the union of the supports of the elements. Furthermore for some m of $\prod_I M_i$ and some subset X of I, the *restriction* $m|_X$ is the element of $\prod_I M_i$ which is equal to m on X and equal to 0 outside X; for a tuple it is the tuple of the restrictions. As before, we will write $m(i)$ for the ith component of m, for some m in $\prod_I M_i$. Furthermore if $\bar{m} \in \prod_I M_i$, then $\bar{m}/F$ will be the canonical image of $\bar{m}$ in $\prod_F M_i$.

LEMMA 2.2. *Let $\prod_F M_i$ be a reduced product of modules and let $\bigwedge_i \eta_i(\bar{x}, \bar{y})$ be a finite conjunction of atomic formulas. If $\bigwedge_i \eta_i(\bar{m}/F, \bar{m}'/F)$ for some $\bar{m}/F, \bar{m}'/F \in \prod_F M_i$, then there exists $\bar{m}''$ in $\prod_I M_i$ such that $\bigwedge_i \eta_i(\bar{m}/F, \bar{m}''/F)$ and the support of $\bar{m}''$ is included in the support of $\bar{m}$.*

PROOF. Suppose $\bigwedge_i \eta_i(\bar{m}/F, \bar{m}'/F)$ is satisfied in $\prod_F M_i$. Take X to be the support of $\bar{m}$. Let $\bar{m}'' = \bar{m}'|_X$. Let me show that $\bigwedge_i \eta_i(\bar{m}/F, \bar{m}''/F)$ is satisfied in $\prod_F M_i$. Let $Y = \{i \in I;\ \bigwedge_i \eta_i(\bar{m}(i), \bar{m}'(i))\}$. By definition of the reduced product, $Y \in F$. Now $\{i \in I\colon \bigwedge_i \eta(\bar{m}(i), \bar{m}''(i))\} = X^c \cup (X \cap Y)$ since on X^c both $\bar{m}(i)$ and $\bar{m}''(i)$ are 0 ($\bigwedge_i \eta_i(\bar{0}, \bar{0})$ is always true), and $\bar{m}'(i)$ and $\bar{m}''(i)$ are equal on X. Hence since $Y \subseteq X^c \cup (X \cap Y)$ it follows that $X^c \cup (X \cap Y) \in F$, and the result is proved.

LEMMA 2.3. *Let F be a filter over a set I, $E \subseteq I$, and F' the filter generated by $\{a \cap E;\ a \in F\}$. If E/F is an atom in $\mathscr{P}(I)/F$, then F' is an ultrafilter.*

PROOF. Let X be any subset of I. Since E/F is an atom, it follows that either $(X \cap E)/F = E/F$ or $(X \cap E)/F = \varnothing/F$. Suppose we are in the first case; then $((X \cap E)\,\Delta E)^c$ is in F. Therefore since $((X \cap E)\,\Delta E)^c \cap E \subseteq X$ it follows that X is in F'. In the second case we have that $((X \cap E)\,\Delta\varnothing)^c = (X \cap E)^c$ is in F; hence $(X \cap E)^c \cap E$ is in F'. Therefore since $(X \cap E)^c \cap E \subseteq X^c$ it follows that X^c is in F', proving that F' is an ultrafilter.

LEMMA 2.4. *Let $\prod_F M_i (i \in I)$ be a reduced product of modules. Let $\{E_j;\ j \in J\}$ be a maximal set of pairwise F-disjoint subsets of I such that E_j/F is an atom in $\mathscr{P}(I)/F$ for every $j \in J$ (there is such a set by Zorn's lemma). Let F_j be the filter generated by $\{a \cap E_j;\ a \in F\}$ $(j \in J)$. Then the F_j $(j \in J)$ are ultrafilters on E, and there exists a homomorphism α such that the following sequence is pure-exact:*

$$0 \to \bigoplus_{j \in J}\left[\prod_{F_j} M_i\right] \xrightarrow{\alpha} \prod_F M_i \xrightarrow{\beta} \prod_{F'} M_i \to 0,$$

where F' is the filter generated by F and the set $\{E_j^c;\ j \in J\}$ of complements of the E_j $(j \in J)$, and the mapping β is canonical.

Furthermore, $\mathscr{P}(I)/F'$ is atomless.

REMARK. This result should be seen as a slight generalization of the well-known fact that for any index set J and modules N_j $(j \in J)$ the following sequence is pure exact:

$$0 \to \bigoplus_{j \in J} N_j \to \prod_J N_j \to \prod_{\mathrm{Fr}} N_j \to 0,$$

where all mappings are canonical. Furthermore the N_j in the direct sum is thought of as the ultraproduct over the principal ultrafilter (j) and Fr is the Fréchet filter of cofinite sets over J. In this case it is clear that $\mathscr{P}(J)/\mathrm{Fr}$ is atomless.

PROOF OF LEMMA 2.4. The F_j $(j \in J)$ are ultrafilters by Lemma 2.3.

Let us now define α. Let α_j be the map from $\prod_{F_j} M_i$ to $\prod_F M_i$ which sends an element with representative m to $m|_{E_j}$. This map is well defined. To show this, suppose that m and m' represent the same element in $\prod_{F_j} M_i$, and let X be the set of elements of I such that $m(i) = m'(i)$. Hence X is in F_j. Therefore X contains $a \cap E_j$ for some a in F. Now the set Y of $i \in I$ such that $m|_{E_j}(i) = m'|_{E_j}(i)$ contains also $a \cap E_j$. Furthermore, $E_j^c \subseteq Y$ by the definition of restriction. Hence $a \subseteq Y$ and Y is in F. This shows that α_j is well defined. It is clear that α_j is also a homomorphism. We now define α to be the sum over $j \in J$ of the various α_j.

We can now prove that α and β possess the properties stated. First it is clear, since $\prod_{F'} M_i$ is a quotient of $\prod_F M_i$, that β is surjective.

Let us now show that the kernel of β is equal to the image of α. Let m be a representative of an element of the image of α. The support of m is included in a finite union $E_{i_1} \cup \cdots \cup E_{i_k}$. Therefore this element is sent in $\prod_{F'} M_i$ to 0. Hence the image of α is included in the kernel of β.

Now let m be a representative which is sent by β to 0. Hence its support is included in a finite union $a^c \cup E_{i_1} \cup \cdots \cup E_{i_k}$, where $a \in F$. Let $m_{i_1}, \ldots, m_{i_k}$ be such that they coincide with m on $E_{i_1}\backslash(E_{i_2}\cup\cdots\cup E_{i_k})$, $E_{i_2}\backslash(E_{i_3}\cup\cdots\cup E_{i_k}),\ldots,E_{i_k}$ respectively, and are 0 elsewhere. Let us show that m/F and $(m_{i_1}+\cdots+m_{i_k})/F$ are equal. The representatives m and $m_{i_1}+\cdots+m_{i_k}$ coincide in every ith component, except maybe for $i\in a^c$. Hence they coincide in a set containing $a\in F$; hence m/F and $m_{i_1}+\cdots+m_{i_k}/F$ are equal, proving that m represents an element of the image of α. Hence the kernel of β is included in the image of α.

We will now show that α is a pure embedding. Let $\bar{m}$ be the representative of a tuple of elements of the image of α. Suppose $\bar{m}/F$ satisfies some positive primitive formula $\varphi(\bar{x})=\exists\bar{y}\bigwedge_i\eta_i(\bar{x},\bar{y})$ in $\prod_F M_i$. Then there exists $\bar{y}/F$ such that $\bigwedge_i\eta_i(\bar{m}/F,\bar{y}/F)$ is satisfied, and by Lemma 2.2 we can suppose that the support of $\bar{y}$ is included in the support of $\bar{m}$. Since the support of $\bar{m}$ is included in some finite union $E_{i_1}\cup\cdots\cup E_{i_k}$, let as before $\bar{y}_{i_1},\ldots,\bar{y}_{i_k}$ be such that they coincide with $\bar{y}$ on $E_{i_1}\backslash(E_{i_2}\cup\cdots\cup E_{i_k})$, $E_{i_2}\backslash(E_{i_3}\cup\cdots\cup E_{i_k}),\ldots,E_{i_k}$ respectively, and are 0 elsewhere. Hence $\bar{y}=\bar{y}_{i_1}+\cdots+\bar{y}_{i_k}$, and it follows that $\bar{y}$ is in the image of α. Hence α is a pure embedding.

It is now left to show that $\mathscr{P}(I)/F'$ is atomless. Suppose a subset X of I was the representative of an atom in $\mathscr{P}(I)/F'$. If $X'=X\backslash\bigcup_{j\in J}E_j$ was not equal to $\varnothing/F'$, then X'/F' would also be an atom; hence X'/F also would be an atom, contradicting the maximality of $\{E_j;\ j\in J\}$. Hence $X'/F'=\varnothing/F'$. Let $X''=X\cap\bigcup_j E_j$. Suppose that there are only finitely many $E_{i_1},\ldots,E_{i_k}$ for which $(X''\cap E_j)/F$ is different from $\varnothing/F$. Then $(X''\backslash E_{i_1}\cup\cdots\cup E_{i_k})/F$ would be equal to $\varnothing/F$ and $X''/F'=\varnothing/F'$, a contradiction to the fact that $X''/F'=X/F'$ is an atom. Therefore there is an infinite subset $J'\subseteq J$ such that $X''\cap E_j/F\neq\varnothing/F$ for all $j\in J'$. Write J' as a disjoint union $J'_1\cup J'_2$ of two infinite subsets. I now claim that $(X''\cap\bigcup_{J'_1}E_j)/F'$ and $(X''\cap\bigcup_{J'_2}E_j)/F'$ are both different from $\varnothing/F'$. Suppose this was not the case, i.e. suppose that $(X''\cap\bigcup_{J'_1}E_j)/F'=\varnothing/F'$. Then $Y''=(X''\cap\bigcup_{J'_1}E_j)\subseteq a^c\cup E_{i_1}\cup\cdots\cup E_{i_k}$ for some a in F and $i_1,\ldots,i_k$ of J. But $Y''\cap E_l\subseteq a^c\cup(E_{i_1}\cap E_l)\cup\cdots\cup(E_{i_k}\cap E_l)$ and for $l\in J$ different from $i_1,\ldots,i_k$ it follows from $E_{i_j}\cap E_l/F=\varnothing/F$ that $Y''\cap E_l\subseteq a^c\cup a^c_{i_1}\cup\cdots\cup a^c_{i_k}$ for $a_{i_1},\ldots,a_{i_k}$ in F. Hence $Y''\cap E_l/F=\varnothing/F$, which is a contradiction to the fact that there are infinitely many E_i such that $X''\cap E_i/F\neq\varnothing/F$. In the same way one can show that $X''\cap\bigcup_{j\in J'_2}E_j/F'$ is different from $\varnothing/F'$. Furthermore $X''\cap\bigcup_{j\in J'_1}E_j/F'$ and $X''\cap\bigcup_{j\in J'_2}E_j/F'$ are disjoint, which contradicts the fact that $X''/F'=X/F'$ is an atom. Hence $\mathscr{P}(I)/F'$ is atomless.

COROLLARY 2.5. *Under the same hypothesis $\prod_F M_i$ is elementarily equivalent to*

$$\bigoplus_{j\in J}\left[\prod_{F_j}M_i\right]\oplus\prod_{F'}M_i.$$

PROOF. It follows from Theorem 2.4 using the fact (see [5, Lemma 2.23]) that for any pure exact sequence $0\to A\to B\to C\to 0$ the modules $A\oplus C$ and B are elementarily equivalent.

We will now prove that a reduced product with atomless quotient $\prod_F M_i$ $(i\in I)$ is elementarily equivalent to a product of ultraproducts of the modules M_i $(i\in I)$.

LEMMA 2.6. *Let $\prod_F M_i(i\in I)$ be a reduced product of modules, and let $\varphi(\bar{x})$ be some positive primitive formula. For any $\bar{m}/F\in\prod_F M_i$ we have that $\prod_F M_i\models\varphi(\bar{m}/F)$ if and only if $\{i\in I;\ M_i\models\varphi(\bar{m}(i))\}\in F$.*

PROOF. Let $\varphi(\bar{x}) = \exists \bar{y} \bigwedge_j \eta_j(\bar{x}, \bar{y})$, where $\eta_j(\bar{x}, \bar{y})$ is atomic. Hence

$$\prod_F M_i \models \varphi(\bar{m}/F)$$

if and only if there exists a $\bar{y}/F$ in $\prod_F M_i$ such that $\prod_F M_i \models \bigwedge_j \eta_j(\bar{m}/F, \bar{y}/F)$. Since $\prod_F M_i \models \bigwedge_j \eta_j(\bar{m}/F, \bar{y}/F)$ is equivalent to $\{i \in I;\ M_i \models \bigwedge_j \eta_j(\bar{m}(i), \bar{y}(i))\} \in F$, it follows that $\prod_F M_i \models \varphi(\bar{m}/F)$ if and only if $\{i \in I;\ M_i \models \exists \bar{y} \bigwedge_j \eta_j(\bar{m}(i), \bar{y})\} \in F$.

We can now prove the following result.

THEOREM 2.7. *Let $\prod_F M_i$ be a reduced product with an atomless quotient. Then for any positive primitive formulas $\varphi(x)$ and $\psi(x)$ we have that* $\mathrm{Inv}(\prod_F M_i, \varphi, \psi)$ *is either 1 or infinite.*

PROOF. Let $\mathscr{P}(I)/F$ be the quotient of $\prod_F M_i$, and let X/F be a nonzero element. Let $F_X = \{a \cap X;\ a \in F\}$. It is clear that $\mathscr{P}(X)/F_X$ is also an atomless Boolean algebra. Furthermore, since $\mathscr{P}(X)/F_X$ is infinite, the structure $\langle \mathscr{P}(X)/F_X, \Delta, \varnothing/F_X \rangle$, where Δ is the symmetric difference, is an infinite abelian group of exponent 2.

Now let φ and ψ be two positive primitive formulas such that $\mathrm{Inv}(\prod_F M_i, \varphi, \psi) > 1$. Let m be an element of $\prod_I M_i$ representing an element m/F of $\varphi(\prod_F M_i)$ which is not in $\psi(\prod_F M_i)$. Let $X = \{i;\ M_i \models \neg \psi(m(i))\}$. Since $\neg \psi(\prod_F M_i)$ holds by hypothesis, it follows that X/F is a nonzero element. By the above argument $\langle \mathscr{P}(X)/F_X, \Delta, \varnothing/F_X \rangle$ is an infinite abelian group of exponent 2, hence an infinite-dimensional vector space over the two-element field. Let X_i/F $(i \in S)$ be a basis of this space, and let $m_i = m|_{X_i}$.

By Lemma 2.6 we know that for any m'/F of $\prod_F M_i$ and any positive primitive formula η, $\eta(m'/F)$ holds in $\prod_F M_i$ if and only if the set of components of m' satisfying η is in F. Hence for any $i \in S$ the formula $\varphi(m_i/F)$ holds. Let i and j be in S. Now since $m_i + m_j$ is equal to m on $X_i \Delta X_j$ and since $(X_i \Delta X_j)/F \neq \varnothing/F$ (because X_i/F and X_j/F are linearly independent in $\langle \mathscr{P}(X)/F_X, \Delta, \varnothing/F_X \rangle$), it follows that $\psi(m_i/F + m_j/F)$ does not hold. Hence $\mathrm{Inv}(\prod_F M_i, \varphi, \psi)$ is infinite.

PROPOSITION 2.8. *Let $\prod_F M_i$ be some reduced product of modules. For any invariant* $\mathrm{Inv}(-, \varphi, \psi)$ *such that* $\mathrm{Inv}(\prod_F M_i, \varphi, \psi) > 1$ *there exists an ultrafilter U containing F such that* $\mathrm{Inv}(\prod_U M_i, \varphi, \psi) > 1$.

PROOF. Let $\mathrm{Inv}(\prod_F M_i, \varphi, \psi) > 1$, and let m/F be an element of $\varphi(\prod_F M_i)$ which is not in $\psi(\prod_F M_i)$. Let X be the set of $i \in I$ such that $\psi(m(i))$ does not hold in M_i. Since m/F is not in $\psi(\prod_F M_i)$, the set X^c cannot be in F. Take U to be an ultrafilter extending F and containing X. Hence $\varphi(m/U)$, but $\psi(m/U)$ does not hold since $X \in U$.

COROLLARY 2.9. *If the quotient of a reduced product $\prod_F M_i$ is atomless, then $\prod_F M_i$ is elementarily equivalent to $(\prod_{i \in J}[\prod_{U_j} M_i])^\omega$ (the countable direct power), where $\{U_j;\ j \in J\}$ is the set of all ultrafilters extending F.*

PROOF. It is sufficient to show that the Baur-Monk invariants of $(\prod_{i \in J}[\prod_{U_j} M_i])^\omega$ and $\prod_F M_i$ are equal. By Theorem 2.7 each invariant of $\prod_F M_i$ is either 1 or infinite. Since $(\prod_{i \in J}[\prod_{U_j} M_i])^\omega$ is an infinite direct product, here also each invariant is either 1 or infinite. Furthermore, for each ultrafilter U extending F there is a canonical projection π_U: $\prod_F M_i \to \prod_U M_i$. I claim that the kernel of this projection is pure in $\prod_F M_i$. This follows from the fact that a tuple $\bar{m}$ of $\prod_{i \in I} M_i$ represents an element of the kernel of π_U if and only if its support is equal to $\varnothing$ modulo U. Now if $\bar{m}$ satisfies a positive primitive formula $\exists\bar{y}\bigwedge_i \eta_i(\bar{x},\bar{y})$ in $\prod_F M_i$, then by Lemma 2.2 it is possible to find a $\bar{y}/F$ such that $\bigwedge_i \eta_i(\bar{m}/F, \bar{y}/F)$ and such that the support of $\bar{y}$ is included in the support of $\bar{m}$; hence $\bar{y}/F$ is also in the kernel of π_U. Since the kernel of π_U is pure in $\prod_F M_i$, it follows that (see [5, Lemma 3.23(a)]) $\mathrm{Inv}(\prod_U M_i, \varphi, \psi) \leq \mathrm{Inv}(\prod_F M_i, \varphi, \psi)$ for every pair of positive primitive formulas φ and ψ.

Therefore to show that $\prod_F M_i$ and $(\prod_{i\in J}[\prod_{U_j} M_i])^{\omega}$ are elementarily equivalent it is sufficient to show that for any Baur-Monk invariant greater than 1 in $\prod_F M_i$ there exists an ultrafilter U extending F such that this invariant is also greater than 1 in $\prod_U M_i$. This is exactly the statement of Proposition 2.8.

THEOREM 2.10. *Any reduced product $\prod_F M_i$ $(i \in I)$ is elementarily equivalent to a direct product of ultraproducts of the modules M_i $(i \in I)$.*

PROOF. Let $\prod_F M_i$ be a reduced product of modules. By Corollary 2.5 it is elementarily equivalent to a direct sum of utlraproducts of the modules M_i $(i \in I)$ and of a reduced product with an atomless quotient. Since direct sums and direct products are elementarily equivalent (see [5, Lemma 2.24(a)]), it is sufficient to prove the result for reduced products with atomless quotient. Now Corollary 2.9 completes the proof.

THEOREM 2.11. *Let T be a theory of modules. The following conditions are equivalent.*

(a) *T is closed under direct product.*

(b) *T is a Horn theory.*

PROOF. The second condition implies the first by Theorem 1.2. By Theorem 2.10, if the first condition is satisfied, i.e. if T is closed under direct products, then it is also closed under reduced products; hence the second condition holds again by Theorem 1.2.

Acknowledgments. I would like to thank Professor Ulrich Felgner for suggesting this research thema and for many stimulating discussions (not only on this topic) as I was studying under his direction in Tübingen. This paper was written as I held a postdoctoral fellowship of the Conseil de Recherches en Sciences Naturelles et en Génie du Canada at McGill University.

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