The Quadratic-Quadratic Regulator Problem:
Approximating feedback controls for quadratic-in-state nonlinear systems

Jeff Borggaard and Lizette Zietsman

Abstract— **Feedback control problems involving autonomous quadratic systems are prevalent, yet there are only a limited number of software tools available for approximating their solution due to the complexity of the problem. This paper represents a step forward in the special case where both the state equation and the control costs are quadratic. As it represents the natural extension of the** *linear-quadratic regulator* **(LQR) problem, we describe this setting as the** *quadratic-quadratic regulator* **(QQR) problem. This is significantly more challenging and holds the LQR as special case that must be solved along the way. We describe an algorithm that exploits the structure of the QQR problem that arises when implementing Al'Brekht's method. This approach is amenable to feedback laws with low degree polynomials but have a relatively modest model dimension that could be achieved by modern model reduction methods. This problem has an elegant formulation and a solution that introduces several linear systems where the structure suggests modern tensor-based linear solvers. We demonstrate this algorithm on a suite of random test problems then apply it to a distributed parameter control problem that fits the QQR framework. Comparisons to linear feedback control laws show a modest benefit using the QQR formulation.**

I. MOTIVATION

Linear feedback control of autonomous nonlinear systems, such as those describing the behavior of fluids, can be sufficient to achieve stabilization–even for an unstable steady-state solution [6], [9]–[11]. There is a shortage of software tools for nonlinear problems in control and systems theory. The general Matlab Nonlinear Systems Toolbox (NST) by Krener [19] takes a broad step toward delivering useful tools for a number of important problems. Since we inherently encounter the *curse of dimensionality* in these problems, there is also a need to develop specialized tools for important classes of problems. This paper addresses this by specifically solving the quadratic-quadratic regulator problem: minimizing a quadratic cost subject to a state equation with a quadratic nonlinearity.

For example, linear feedback laws found by solving the linear-quadratic regulator (LQR) problem compute the linear feedback law as the solution to a single algebraic Riccati equation and have the property that the linear portion of the nonlinear system becomes stable [3]–[5], [23]. Unfortunately,

Jeff Borggaard and Lizette Zietsman are with the Interdisciplinary Center for Applied Mathematics and the Department of Mathematics at Virginia Tech, Blacksburg, VA 24061, USA.
`jborggaard@vt.edu`, `lzietsma@vt.edu`.

for nonlinear systems, this only guarantees local stability. Thus the ability of linear feedback to stabilize the steady-state solution depends on the initial condition, which must be sufficiently close to the steady-state. An alternative would be to develop nonlinear feedback control laws that could offer the ability to expand the radius of convergence (shown with a simple example in [12]). However, these require us to approximate solutions to the Hamilton-Jacobi-Bellman (HJB) equations, e.g. [13], [21]. The HJB equations are notoriously complex in the general case. Nevertheless, if one considers the quadratic-quadratic regulator (QQR) problem, with autonomous quadratic state equations and a quadratic control objective, there is sufficient structure in polynomial approximations based on Al'Brekht's method [22] for polynomial feedback laws to be computable for modest problem sizes. The QQR problem also happens to be exactly what is needed to solve discretized versions of distributed parameter control problems where the nonlinearity is quadratic (such as the Navier-Stokes equations used as our motivation above). This is particularly true when linear feedback laws are being based on LQR problems. As in the LQR case, suitable model reduction methods [1], [2], [17] are essential to forming a solution methodology for distributed parameter control problems with quadratic nonlinearities. First of all, the Riccati equation is still needed to compute the linear term [7], [25], [26] and the curse-of-dimensionality still appears with higher-order polynomial approximations of the feedback law.

In this paper, we briefly outline the HJB equations, the QQR problem, and polynomial approximations to the value function and the feedback control operators. Our formulation leads to a sequence of linear systems in Kronecker product form after an initial solution to the algebraic Riccati equation. As we shall see, a naïve construction of these matrices and other terms would quickly become prohibitive. However, the structure lends itself to newly developed recursive tensor linear algebra that avoids assembly and other taxing of computer memory. We present a numerical study with a set of randomly selected control problems to compare solutions obtained by Krener's NST [19], direct assembly and solution to the Kronecker system, and the recursive tensor-based algorithm using the tensor toolbox [18] and a recursive blocked algorithm for systems with a special Kronecker sum form [16].

II. BACKGROUND

For simplicity of exposition, we describe the nonlinear optimal control problem and its computational challenges for

systems modeled by autonomous systems of ordinary differential equations. This can be widely found in the literature and we are reintroducing it here to set up our notation. The problem is to find a control $\mathbf{u}(\cdot) \in L_2(0, \infty; \mathbb{R}^m)$ that solves

$$\min_{\mathbf{u}} J(\mathbf{x}, \mathbf{u}) = \int_0^\infty \ell(\mathbf{x}(t), \mathbf{u}(t)) \, dt, \tag{1}$$

where $\ell : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow [0, \infty)$ is a prescribed control objective, and minimization occurs subject to

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \tag{2}$$

from $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times m}$ are constant matrices and $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is Lipschitz continuous and satisfies $\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $\nabla_{\mathbf{x}}\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

We define the *value function* $v(\mathbf{x}_0) = J(\mathbf{x}^*(\cdot; \mathbf{x}_0), \mathbf{u}^*(\cdot))$ to be the value of (1) when the optimal control $\mathbf{u}^*$ and corresponding state $\mathbf{x}^*$ are found from the initial point $\mathbf{x}_0$. Assume that the optimal control is given by the feedback relation

$$\mathbf{u}(t) = \mathcal{K}(\mathbf{x}(t)). \tag{3}$$

For $\mathbf{f}$, ℓ, and v smooth enough, and v convex, the feedback relation (3) satisfies the HJB partial differential equations

$$0 = \frac{\partial v}{\partial \mathbf{x}}(\mathbf{x}) \left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathcal{K}(\mathbf{x}) + \mathbf{f}(\mathbf{x}, \mathcal{K}(\mathbf{x})) \right) + \ell(\mathbf{x}, \mathcal{K}(\mathbf{x})), \tag{4}$$

$$0 = \frac{\partial v}{\partial \mathbf{x}}(\mathbf{x}) \left(\mathbf{B} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}, \mathcal{K}(\mathbf{x})) \right) + \frac{\partial \ell}{\partial \mathbf{u}}(\mathbf{x}, \mathcal{K}(\mathbf{x})). \tag{5}$$

Ideally, we could solve the HJB equations simultaneously for v and $\mathcal{K}$. This would provide the desired feedback relation $\mathbf{u}(t) = \mathcal{K}(\mathbf{x}(t))$. The value function v can often serve as a Lyapunov function to examine the region of attraction for the controlled system. Unfortunately, the HJB equations suffer from the curse of dimensionality as they are partial differential equations in $\mathbb{R}^n$. There have been studies that use *model reduction* to replace the nonlinear dynamics in (2) with high-fidelity, yet much lower order, reduced dynamics for simple problems [21]. However, direct solution of the HJB equations will still be a computational challenge even when we can replace the Navier-Stokes equations with modest reduced-order models on the order of 30–50.

Therefore, even with optimal reduced models, solving the nonlinear optimal control problem (1)-(2) is intractable for general nonlinearities. Fortunately, as we outline below, for problems where $\mathbf{f}$ and ℓ have quadratic nonlinearities, there is enough structure to compute nonlinear feedback laws for modest sized problems (at least $n = 50$). This provides a useful tool for reduced systems of flow equations. Many linear feedback laws are computed from linearized and reduced models of this size in the literature, e.g. [11].

Our simplification comes from using Kronecker products (which have a long history in the control literature [14], [24]. Let $\mathbf{X} \in \mathbb{R}^{i_x \times j_x}$ and $\mathbf{Y} \in \mathbb{R}^{i_y \times j_y}$, with entries x_{ij} and y_{ij},

respectively. Then $\mathbf{X} \otimes \mathbf{Y} \in \mathbb{R}^{i_x i_y \times j_x j_y}$ is the block matrix

$$\mathbf{X} \otimes \mathbf{Y} \equiv \begin{bmatrix} x_{11}\mathbf{Y} & x_{12}\mathbf{Y} & \cdots & x_{1j_x}\mathbf{Y} \\ x_{21}\mathbf{Y} & x_{22}\mathbf{Y} & \cdots & x_{2j_x}\mathbf{Y} \\ \vdots & & & \vdots \\ x_{i_x 1}\mathbf{Y} & x_{i_x 2}\mathbf{Y} & \cdots & x_{i_x j_x}\mathbf{Y} \end{bmatrix}.$$

III. THE QUADRATIC-QUADRATIC REGULATOR

To simplify the expressions, we use the Kronecker product description of the quadratic-quadratic regulator (QQR) problem. Thus, we seek the control $\mathbf{u}(t) = \mathcal{K}(\mathbf{x}(t))$ that is the solution to

$$\min_{\mathbf{u}} \int_0^\infty \mathbf{q}_2' \left(\mathbf{x}(t) \otimes \mathbf{x}(t) \right) + \mathbf{r}_2' \left(\mathbf{u}(t) \otimes \mathbf{u}(t) \right) \, dt \tag{6}$$

subject to

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{N} \left(\mathbf{x}(t) \otimes \mathbf{x}(t) \right), \tag{7}$$

from any $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$. In this formulation, the matrices above are time-invariant with dimensions

$$\mathbf{A} \in \mathbb{R}^{n \times n}, \qquad \mathbf{B} \in \mathbb{R}^{n \times m}, \qquad \mathbf{N} \in \mathbb{R}^{n \times n^2},$$

$$\mathbf{q}_2 \in \mathbb{R}^{n^2 \times 1}, \qquad \text{and} \qquad \mathbf{r}_2 \in \mathbb{R}^{m^2 \times 1}$$

(and $'$ denotes the transpose). This is merely for a convenient representation for the algorithm below. Note that we require the standard control systems properties required by the linear-quadratic regulator (LQR) problem and these can be readily checked with the identities $\mathbf{q}_2 = \text{vec}(\mathbf{Q}_2)$ and $\mathbf{r}_2 = \text{vec}(\mathbf{R}_2)$ with the usual quadratic cost integrand being $\mathbf{x}'\mathbf{Q}_2\mathbf{x} + \mathbf{u}'\mathbf{R}_2\mathbf{u}$.

We now expand the *value function* as

$$v(\mathbf{x}) = \underbrace{\mathbf{v}_2' \left(\mathbf{x} \otimes \mathbf{x} \right)}_{v^{[2]}(\mathbf{x})} + \underbrace{\mathbf{v}_3' \left(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} \right)}_{v^{[3]}(\mathbf{x})} + \cdots$$

and the feedback operator as

$$\mathcal{K}(\mathbf{x}) = \underbrace{\mathbf{k}_1' \mathbf{x}}_{\mathbf{k}^{[1]}(\mathbf{x})} + \underbrace{\mathbf{k}_2' \left(\mathbf{x} \otimes \mathbf{x} \right)}_{\mathbf{k}^{[2]}(\mathbf{x})} + \underbrace{\mathbf{k}_3' \left(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} \right)}_{\mathbf{k}^{[3]}(\mathbf{x})} + \cdots.$$

Note that $\mathbf{v}_d \in \mathbb{R}^{n^d \times 1}$ and $\mathbf{k}_d \in \mathbb{R}^{n^d \times m}$. The Hamiltonian-Jacobi-Bellman equations for this problem now has the form

$$\frac{\partial v}{\partial \mathbf{x}}(\mathbf{x}) \left[\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{N}(\mathbf{x} \otimes \mathbf{x}) \right]$$
$$+ \mathbf{q}_2'(\mathbf{x} \otimes \mathbf{x}) + \mathbf{r}_2'(\mathcal{K}(\mathbf{x}) \otimes \mathcal{K}(\mathbf{x})) = \mathbf{0}, \tag{8}$$

$$\frac{\partial v}{\partial \mathbf{x}}(\mathbf{x}) \left[\mathbf{B} \right] + \mathbf{r}_2'\mathcal{K}(\mathbf{x}) = \mathbf{0}. \tag{9}$$

Substituting in the expansions for the value function v and the feedback operator $\mathcal{K}$ into (8), then collecting $O(\mathbf{x}^2)$ terms, we have

$$\mathbf{v}_2' \left((\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{k}_1\mathbf{x}) \otimes \mathbf{x} + \mathbf{x} \otimes (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{k}_1\mathbf{x}) \right)$$
$$+ \mathbf{q}_2'(\mathbf{x} \otimes \mathbf{x}) + \mathbf{r}_2'((\mathbf{k}_1\mathbf{x}) \otimes (\mathbf{k}_1\mathbf{x})) = \mathbf{0} \tag{10}$$

which, using $\mathbf{k}_1 = -\mathbf{R}_2^{-1}\mathbf{B}'\mathbf{V}_2$ is equivalent to the algebraic Riccati equation (ARE) for finding $\mathbf{V}_2$

$$\mathbf{A}'\mathbf{V}_2 + \mathbf{V}_2\mathbf{A} - \mathbf{V}_2\mathbf{B}\mathbf{R}_2^{-1}\mathbf{B}'\mathbf{V}_2 + \mathbf{Q}_2 = \mathbf{0}.$$

Note that it is natural to use the efficient algorithms for solving the ARE and set $\mathbf{v}_2 = \mathrm{vec}(\mathbf{V}_2)$.

A. Coefficients of $\mathbf{v}_{d+1}$

When we gather higher degree terms, we ignore those terms in (8) that involve components of $\mathcal{K}$ that have yet to be computed. Those terms will be updated from (9). The degree three terms in (8) can then be written using the definition $\mathbf{A}_c = \mathbf{A} + \mathbf{B}\mathbf{k}_1$ as

$$(\mathbf{A}_c \otimes \mathbf{I}_n \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{A}_c \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{I}_n \otimes \mathbf{A}_c)' \mathbf{v}_3 \\ = -(\mathbf{N}' \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{N}') \mathbf{v}_2. \quad (11)$$

Note that this is a simplification that is independent of $\mathbf{k}_2$ since collecting those terms then factoring leaves us with identities following from (9) involving the $\mathbf{v}_2$, $\mathbf{r}_2$ and $\mathbf{k}_1$ terms that define $\mathbf{k}_1$. For example,

$$((\mathbf{B}\mathbf{k}_2)' \otimes \mathbf{I}_n)\mathbf{v}_2 + (\mathbf{k}_2' \otimes \mathbf{k}_1')\mathbf{r}_2 = 0.$$

This identity appears in all subsequent collections of similar degree terms since the $\mathbf{k}_1$ term will always be matched up with $\mathbf{k}_d$ terms. This fact serves to decouple equations for $\mathbf{v}_{d+1}$ in (8) from the equations for $\mathbf{k}_d$ in (9).

To write the equations from matching higher degree terms in a more compact way, we define the *N-way Lyapunov matrix* or a special *Kronecker sum* [8] matrix,

$$\mathcal{L}_d(\mathbf{X}) \equiv \underbrace{\mathbf{X} \otimes \cdots \otimes \mathbf{I}_n}_{d \text{ terms}} + \cdots + \underbrace{\mathbf{I}_n \otimes \cdots \otimes \mathbf{X}}_{d \text{ terms}}. \quad (12)$$

Then the calculation of $\mathbf{v}_3$ follows from solving an equation of the form

$$\mathcal{L}_3(\mathbf{A}_c')\mathbf{v}_3 = -\mathcal{L}_2(\mathbf{N}')\mathbf{v}_2. \quad (13)$$

Once we have $\mathbf{v}_3$, we can readily compute $\mathbf{k}_2$ as shown in Section III-B. The other terms in the series expansion of the value function lead to equations that have a similar form. All of the left-hand-sides are generically the same $\mathcal{L}_{d+1}(\mathbf{A}_c')\mathbf{v}_{d+1}$. However, the right-hand-sides of the equations gather more terms due to the $\mathbf{r}_2$ term in (8) and the interactions of the previously computed nonlinear feedback terms with previously computed terms of the value function (that are known and moved to the right-hand-side). This process is clarified from explicitly writing the next two terms for $v(\mathbf{x})$ below. For $O(\mathbf{x}^4)$, we have

$$\mathcal{L}_4(\mathbf{A}_c')\mathbf{v}_4 = -\mathcal{L}_3((\mathbf{B}\mathbf{k}_2 + \mathbf{N})')\mathbf{v}_3 - (\mathbf{k}_2' \otimes \mathbf{k}_2')\mathbf{r}_2, \quad (14)$$

which can be solved for $\mathbf{v}_4$ once $\mathbf{k}_2$ is computed from the solution $\mathbf{v}_3$ from (13), and

$$\mathcal{L}_5(\mathbf{A}_c')\mathbf{v}_5 = -\mathcal{L}_4((\mathbf{B}\mathbf{k}_2 + \mathbf{N})')\mathbf{v}_4 - \mathcal{L}_3((\mathbf{B}\mathbf{k}_3)')\mathbf{v}_3 \\ - (\mathbf{k}_2 \otimes \mathbf{k}_3 + \mathbf{k}_3 \otimes \mathbf{k}_2)'\mathbf{r}_2. \quad (15)$$

Again, once we compute $\mathbf{k}_3$ from $\mathbf{v}_4$, we have everything we need to compute $\mathbf{v}_5$.

In general, while calculation of the coefficients $\mathbf{v}_d$ are described by large linear systems ($\mathcal{L}_d(\mathbf{A}_c') \in \mathbb{R}^{n^d \times n^d}$), there is a great deal of structure. For example, consistent with the remarks in [20], the eigenvalues of $\mathcal{L}_d(\mathbf{A}_c')$ are merely sums of combinations of the eigenvalues of $\mathbf{A}_c$. Therefore, since $\mathbf{A}_c$ is a stable matrix, $\mathcal{L}_d(\mathbf{A}_c')$ will also be. It is also immediately obvious that *without* the nonlinear term $\mathbf{N}$ in our state equation, the right-hand-side in (13) would vanish leading to $\mathbf{v}_3 = \mathbf{0}$. The remaining equations for $\mathbf{v}_{d+1}$ would have homogeneous right-hand-sides and thus $\mathbf{v}_{d+1} = \mathbf{0}$ for $d = 2$ and higher. This is consistent with the LQR theory.

B. Coefficients of $\mathbf{k}_d$

We now turn our attention to using (9) to calculate $\mathbf{k}_d$ from $\mathbf{v}_{d+1}$. This is again straight-forward using the specialized Kronecker sum operator,

$$\mathbf{k}_d = -\mathbf{R}_2^{-1}\left(\mathcal{L}_{d+1}(\mathbf{B}')\mathbf{v}_{d+1}\right)'. \quad (16)$$

C. Computing Right-Hand-Side Vectors

The assembly and solution of linear systems with the form

$$\mathcal{L}_{d+1}(\mathbf{A}_c)\mathbf{v}_{d+1} = \mathbf{c} \quad (17)$$

is only feasible for small values of d and n. We denote these computations by *full Kronecker* in Section IV. The advantage of the Kronecker product structure is that we can perform operations with Kronecker product matrices *without* actually forming the large block matrix. The main issue that we deal with in this section is calculating the terms on the right-hand-sides of e.g. (13)–(15) or (16). Solution of the system (17) is described in the next section.

To calculate $\mathbf{c}$ for (13)–(16) involves two types of terms. The first involves the multiplication of a Kronecker form with a vector $\mathbf{r}_2$. Recall, e.g. [14], that

$$(\mathbf{X} \otimes \mathbf{Y})\mathbf{r}_2 = \mathrm{vec}(\mathbf{Y}'\mathbf{R}\mathbf{X}), \quad (18)$$

where $\mathbf{R}$ has the appropriate dimensions and $\mathbf{r}_2 = \mathbf{vec}(\mathbf{R})$. Therefore, the terms involving $\mathbf{r}_2$ only require matrix multiplications and no assembly of the Kronecker product is required.

The second type of term are products of the Kronecker sum with a $\mathbf{v}_{d+1}$: $\mathcal{L}_{d+1}(\mathbf{X})\mathbf{v}_{d+1}$. Using the definition of (12), we have to calculate $d + 1$ different multiplications of the Kronecker products with $\mathbf{v}_{d+1}$. Using the fact that the Kronecker product is associative, we write

$$\mathbf{I}_{n^\ell} = \underbrace{\mathbf{I}_n \otimes \cdots \otimes \mathbf{I}_n}_{\ell \text{ terms}}.$$

The multiplications can be reduced to three different cases

$$(\mathbf{X} \otimes \mathbf{I}_{n^d})\mathbf{v}_{d+1}, \quad (\mathbf{I}_{n^{d-\ell}} \otimes \mathbf{X} \otimes \mathbf{I}_{n^\ell})\mathbf{v}_{d+1}, \quad \text{and} \quad (\mathbf{I}_{n^d} \otimes \mathbf{X})\mathbf{v}_{d+1}.$$

Here the relation (18) and the associative law for Kronecker products are useful. The first and last terms above can be handled by the appropriate reshaping of $\mathbf{v}_{d+1}$ and multiplying with $\mathbf{X}$ (the multiplication by $\mathbf{I}_{n^d}$ is trivial). The

associative law allows us to handle all of the intermediate terms recursively as

$$(\mathbf{I}_{n^{d-\ell}} \otimes \mathbf{X} \otimes \mathbf{I}_{n^\ell})\mathbf{v}_{d+1} = ((\mathbf{I}_{n^{d-\ell}} \otimes \mathbf{X}) \otimes \mathbf{I}_{n^\ell})\mathbf{v}_{d+1}$$
$$= (\mathbf{I}_{n^{d-\ell}} \otimes (\mathbf{X} \otimes \mathbf{I}_{n^\ell}))\mathbf{v}_{d+1}.$$

The grouping can be done to maximize the size of the free identity matrix.

D. Linear System Solutions

The Kronecker structure leads to larger systems (17), but are now ameneble to modern high performance algorithms [16], [18], [24]. Many of these algorithms, e.g. [16] utilize a real Schur factorization of the matrix $\mathbf{A}_c$. For this study, we used the *recursive* algorithms in [16] for Laplace-like equations. Their software was trivially modified to take advantage of the fact that the same term $\mathbf{A}_c$ appears in every block and gave the system exactly the form (12).

IV. NUMERICAL RESULTS

We present two sets of results. The first is a challenging verification test using randomly generated matrices. This is challenging since the conditioning of these systems are poor for many of our randomly generated samples. Our study is reproducible since we reset the random seed before each test. The second set a discretized control problem involving the one-dimensional Burgers equation. The systems are much better conditioned in this case and there is a notion of convergence as the problem sizes in our tests grow.

A. Random System Study

In this section, we perform computational tests for both performance and accuracy. For accuracy, we compare against the feedback matrices computed using the Nonlinear Systems Toolbox (NST) [19]. We note that the NST is designed for a wide range of control problem formulations and general nonlinearities. Thus it is not used as a performance measure but rather to provide a sense of general speed and accuracy. Since much of the use of the symbolic toolbox in NST is in the preprocessing step, we remove this calculation from our comparative timings (though provide those times separately for completeness).

For reproducibility, we provide the source to generate our numerical findings in Matlab below.

```
rng(0,'v5uniform'); % set random seed
A = rand(n,n);
B = rand(n,m);
N = rand(n,n*n);
Q = eye(n);
R = eye(m);
```

All computations were performed on a 2017 Macbook Pro with a 3.1GHz Inter Core i7 processor and 16GB of RAM using MATLAB version R2019b.

Our first set of tests computed the degree 2 feedback term, $\mathbf{k}_2$, and the required $\mathbf{v}_3$ component of the value function. Problem sizes from 6 to 20 were randomly generated and the Matlab runtimes are reported in Table I. Generally speaking,

only the first significant digits or two of the CPU times are meaningful since we only averaged the time over a small set of experimental runs. The trend is clear, however, that our QQR algorithm is exceptionally fast primarily due to the careful work assembling the right-hand-sides and the recursive blocked algorithms of Chen and Kressner [16]. The computational trend for the QQR is even more pronounced when we extend these tests to include both the degree 3 feedback term, $\mathbf{k}_3$, and the corresponding $\mathbf{v}_4$ component of the value function (seen in Table II). The calculation of the full Kronecker problem for $n = 16$ encountered a memory limit error, so computations beyond that could not be carried out. However, the recursive solver and specially tailored matrix-vector products allowed us to continue calculations well beyond the limits seen with the full Kronecker form. In fact, we were able to solve an order $n = 40$ random system with a degree $d = 3$ feedback law in 12.72 seconds and an order $n = 30$ random system with a degree $d = 4$ feedback law in 142.26 seconds. This demonstrates that modest size problems in the QQR framework can be readily incorporated into control design workflows.

TABLE I
RANDOM: CPU TIME FOR DEGREE 2 TERMS

n	recursive	full Kronecker	NST	(symbolic calc.)
6	0.03584	0.00152	0.0123	(0.5665)
8	0.03958	0.00552	0.0258	(0.9183)
10	0.03531	0.02020	0.0515	(1.4596)
12	0.04466	0.07945	0.1006	(2.2126)
14	0.05852	0.22527	0.2528	(3.6933)
16	0.06112	0.67803	0.5932	(5.6935)
18	0.07812	1.55314	1.2353	(8.8498)
20	0.09467	3.50995	2.5385	(13.3123)

TABLE II
RANDOM: CUMULATIVE CPU TIME FOR DEGREE 3 TERMS

n	recursive	full Kronecker	NST	(symbolic calc.)
6	0.00700	0.04937	0.0333	(1.6310)
8	0.04363	0.87317	0.1219	(4.0723)
10	0.06942	6.6439	0.4928	(9.1590)
12	0.09098	53.2562	1.9644	(20.2047)
14	0.21810	588.826	7.1282	(41.0284)
16	0.53792	not computed	23.9001	(85.0918)
18	0.63851	not computed	68.1008	(174.518)
20	0.82421	not computed	170.217	(352.068)

There is no truth model for testing the accuracy of the HJB series solution, so we relied on the well-tested NST software to compare against. Therefore, relative ℓ^2 errors are those reported with respect to NST solutions. This was a challenge since NST computes its solutions in a compact Taylor series format (returning coefficients of the unique monomial terms) whereas the Kronecker formulation introduces a lot of repeated monomials. Thus there are multiple correct representations for the coefficients (even though only one unique representation will be calculated). To compare to the coefficients in compact Taylor series form required us to accumulate all of the coefficients for equivalent monomials to

generate the comparisons. The relative errors in ℓ^2 are those of the summed coefficients (i.e. summing the coefficients for $x_1 x_2$ with those from $x_2 x_1$ etc.). Tables III and IV show very good agreement for small problems. Note that both the full Kronecker system and the NST systems generated condition number warnings for $n = 16$ and $n = 20$ for the $d = 2$ solutions and the *NST* generated an additional warning when $d = 3$ for the $n = 18$ case. These warnings did not occur in the recursive blocked algorithms since the full system was never assembled. However, the systems that are solved in the recursive and full Kronecker columns are the same, so many of the large relative errors can be explained by ill-conditioning. Therefore, our next numerical example in Section IV-B will have more desirable control-theoretic properties.

TABLE III
RANDOM: RELATIVE ERRORS IN $k^{[2]}$ AND $v^{[3]}$

	recursive		full Kronecker	
n	error $k^{[2]}$	error $v^{[3]}$	error $k^{[2]}$	error $v^{[3]}$
6	4.09e-12	3.09e-12	3.88e-12	2.61e-12
8	1.70e-11	1.67e-11	2.05e-11	2.66e-11
10	4.92e-08	4.95e-08	1.25e-06	1.27e-06
12	1.19e-10	1.04e-10	3.93e-10	3.72e-10
14	9.26e-07	9.95e-07	7.42e-06	7.88e-06
16	1.06e-04	1.13e-04	3.80e-03	3.97e-03
18	1.44e-04	1.50e-04	5.48e-03	5.75e-03
20	2.65e-02	3.23e-02	1.68e+00	1.70e+00

TABLE IV
RANDOM: RELATIVE ERRORS IN $k^{[3]}$ AND $v^{[4]}$

	recursive		full Kronecker	
n	error $k^{[3]}$	error $v^{[4]}$	error $k^{[3]}$	error $v^{[4]}$
6	1.68e-10	1.11e-10	3.37e-11	2.71e-11
8	9.18e-10	6.85e-10	1.17e-09	1.00e-09
10	7.52e-04	9.11e-04	2.32e-03	2.81e-03
12	7.21e-10	1.04e-09	1.15e-07	1.28e-07
14	7.95e-04	5.80e-04	1.86e-02	1.38e-02
16	2.11e-01	2.61e-01	not computed	not computed
18	7.90e+00	1.15e+00	not computed	not computed

B. Burgers Equation

As a more structured test problem, we consider the QQR problem with a discretization of the Burgers equation. This test problem has a long history in the study of control for distributed parameter systems, e.g. [27], including the development of effective computational methods, e.g. [15]. Thus, we consider

We consider the specific problem found in [12] but with two control inputs ($m = 2$) that consist of uniformly distributed sources over disjoint patches. Thus, we have a bounded input operator. The formal description of the problem is

$$\min_{\mathbf{u}} J(z, u) = \int_0^\infty \left(\int_0^1 z^2(\xi, t) \, d\xi + \mathbf{u}'(t)\mathbf{u}(t) \right) \, dt$$

subject to

$$\dot{z}(x, t) = \epsilon z_{xx}(x, t) - \frac{1}{2} \left(z^2(x, t) \right)_x$$
$$+ \sum_{k=1}^m \chi_{[(k-1)/m, \ k/m]}(x)u_k(t)$$
$$z(\cdot, 0) = z_0(\cdot) \in H^1_{\text{per}}(0, 1),$$

where $\chi_{[a,b]}(x)$ is the characteristic function over $[a, b]$. We discretized the state equations with n linear finite elements, set $m = 2$, and chose $\epsilon = 0.001$ to make the nonlinearity significant.

The discretized system fits within the QQR framework (6)-(7). The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{N}$ come from the finite element approximation. The matrix $\mathbf{Q}_2$ is the finite element mass matrix and the matrix $\mathbf{R}_2 = \mathbf{I}_m$.

With the same rationale, the NST solution is used as the truth model and used to compute relative errors. It is evident that a well-conditioned feedback control problem produces much more consistent errors. As expected, the $n = 16$ case also breaks down here (more memory is required since $m = 2$), but the accuracy of the recursive block algorithm is easily observed in Tables V and VI. There is very little growth in the discrepancy between the two solutions with increasing values of n. This suggests a convergence to an infinite-dimensional representation that we will investigate in future studies.

TABLE V
BURGERS: RELATIVE ERRORS IN $k^{[2]}$ AND $v^{[3]}$

	recursive		full Kronecker	
n	error $k^{[2]}$	error $v^{[3]}$	error $k^{[2]}$	error $v^{[3]}$
10	6.34e-12	3.99e-12	6.34e-12	3.99e-12
12	1.02e-11	4.94e-12	1.02e-11	4.94e-11
14	2.53e-11	1.85e-11	2.53e-11	1.85e-11
16	2.33e-11	1.24e-11	not computed	not computed
18	3.63e-11	1.76e-11	not computed	not computed
20	7.43e-11	4.44e-11	not computed	not computed

TABLE VI
BURGERS: RELATIVE ERRORS IN $k^{[3]}$ AND $v^{[4]}$

	recursive		full Kronecker	
n	error $k^{[3]}$	error $v^{[4]}$	error $k^{[3]}$	error $v^{[4]}$
10	1.21e-12	3.02e-12	1.21e-12	3.02e-12
12	1.74e-11	4.79e-12	1.74e-11	4.80e-11
14	3.66e-11	1.67e-11	3.66e-11	1.67e-11
16	3.07e-11	1.00e-11	not computed	not computed
18	4.76e-11	1.52e-11	not computed	not computed
20	1.22e-10	4.60e-11	not computed	not computed

Instead of presenting both degree 2 and the accumulation up to degree three, we only report the accumulated times up to degree 3 in Table VII. Again, the computational speed to solve the QQR problem for the degree 2 and degree 3 feedback terms is impressive.

To test the possibility of using this software for larger problems, we increased the degree and order to see what sizes could be computed in two to three minutes for this

TABLE VII
BURGERS: CUMULATIVE CPU TIME FOR DEGREE 3 TERMS

n	recursive	full Kronecker	NST	(symbolic calc.)
10	0.07355	6.2939	0.5367	(12.0943)
12	0.06801	49.9701	2.2008	(26.7163)
14	0.19543	474.602	7.4379	(50.843)
16	0.39385	not computed	25.9467	(108.846)
18	0.42039	not computed	69.2946	(203.07)
20	0.50770	not computed	179.731	(388.069)

problem. We found that we could solve a degree $d = 4$ feedback law for order $n = 32$ and $m = 2$ in 182.9 seconds. Furthermore, a degree $d = 3$ feedback law for order $n = 64$ and $m = 2$ was computed in 130.2 seconds. Thus, some higher degree control laws are feasible with modest discretizations in one-dimensional problems. It is reasonable to expect that this would be an attractive option to try when developing feedback control laws from modest reduced models of fluid systems.

V. CONCLUSIONS AND FUTURE WORK

We presented a special formulation of the Al'Brekht polynomial approximation for the quadratic-quadratic regulator problem. Writing the expansions in terms of Kronecker products leads to a series of progressively larger linear systems for the next terms in the expansion. While easy to write down and implement, efficiency is only achieved by exploiting new numerical linear algebra tools that avoid the assembly of the large, dense systems [16], [18]. Repeatable numerical experiments with random linear systems of different order confirm the efficiency of our approach. We performed a comparison with a general, well-developed software tool, the Nonlinear Systems Toolbox [19], to verify our implementation. Our solution method was competitive with NST in terms of CPU time even if we neglect the overhead in using Matlab's symbolic toolbox (we described an effective means to compute the derivatives of the system that are required by NST using automatic differentiation in a previous paper [12]).

Our future work will evolve down three paths. The first will be to better understand the numerical trade-offs between our use of [16] for our application and a linear system that is built for a compact Taylor series representation of the problem. This compact Taylor series removes redundant variables (e.g. coefficients of $x_1 x_2$ and $x_2 x_1$ can be combined). Since NST uses the compact Taylor series approach, we have already built the restriction and prolongation operators between polynomials up to order 5 for our verification. Part of this will include a better understanding of the discrepancies between the solutions for some of our randomly generated test cases.

Our second path is to consider natural generalizations to include in this software framework. This would include the investigation of more general control costs, addition of descriptor systems [28], and the related observer problem.

Finally, we will apply this to more significant applications than the one-dimensional Burgers equation. In particular, study how this work could be used in conjunction with reduced models of complex flows that result in quadratic-in-state systems.

This software is available for download at `https://github.com/jborggaard/QQR.git`.

REFERENCES

[1] M. I. Ahmad, P. Benner, and L. Feng. A new interpolatory model reduction approach for quadratic bilinear descriptor systems. *Proceedings in Applied Mathematics and Mechanics*, 15:589–590, 2015.
[2] N. Aubry, P. J. Holmes, J. L. Lumley, and E. Stone. The dynamics of coherent structures in the wall region of a turbulent boundary layer. *Journal of Fluid Mechanics*, 192:115–173, 1988.
[3] V. Barbu, I. Lasiecka, and R. Triggiani. *Tangential Boundary Stabilization of Navier-Stokes Equations*, volume 181 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2006.
[4] V. Barbu, I. Lasiecka, and R. Triggiani. Local exponential stabilization strategies of the Navier-Stokes equations, $d = 2, 3$, via feedback stabilization of its linearization. *International Series of Numerical Mathematics*, 155:13–46, 2007.
[5] V. Barbu and S. S. Sritharan. Flow invariance preserving feedback controllers for the Navier-Stokes equation. *Journal of Mathematical Analysis and Applications*, 255:281–307, 2001.
[6] P. Benner and J. Heiland. Robust stabilization of laminar flows in varying flow regimes. *IFAC-PapersOnLine*, 49(8):031–036, 2016.
[7] P. Benner, J. Saak, M. Stoll, and H. K. Weichelt. Efficient solution of large-scale saddle point systems arising in Riccati-based boundary feedback stabilization of incompressible Stokes flow. *SIAM Journal on Scientific Computing*, 35(5):S150—S170, 2013.
[8] M. Benzi and V. Simoncini. Approximation of functions of large matrices with Kronecker structure. *Numerische Mathematik*, 135(1):1–26, 2017.
[9] M. Bergmann, L. Cordier, and J.-P. Brancher. Optimal rotary control of the cylinder wake using proper orthogonal decomposition reduced-order model. *Physics of Fluids*, 17(097101), 2005.
[10] J. Borggaard and S. Gugercin. Model reduction for DAEs with an application to flow control. In R. King, editor, *Active Flow and Combustion Control 2014*, volume 127 of *Notes on Numerical Fluid Mechanics and Multidisciplinary Design*, pages 381–396. Springer, 2014.
[11] J. Borggaard, M. Stoyanov, and L. Zietsman. Linear feedback control of a von Kármán street by cylinder rotation. In *Proc. 2010 American Control Conference*, pages 5674–5681, 2010. FrB06.3.
[12] J. Borggaard and L. Zietsman. Computation of nonlinear feedback for flow control problems. In *2018 Annual American Control Conference, ACC 2018, Milwaukee, WI, USA, June 27-29, 2018*, pages 1726–1731, 2018.
[13] T. Breiten, K. Kunisch, and L. Pfeiffer. Feedback stabilization of the two-dimensional Navier-Stokes equations by value function approximation. *Applied Mathematics & Optimization*, 2019. https://doi.org/10.1007/s00245-019-09586-x.
[14] J. Brewer. Kronecker products and matrix calculus in system theory. *IEEE Transactions on Circuits and Systems*, CAS-25(9), 1978.
[15] J. A. Burns and S. Kang. A control problem for Burgers equation with bounded input/output. Technical report, ICASE, 1990.
[16] M. Chen and D. Kressner. Recursive blocked algorithms for linear systems with Kronecker product structure. *Numerical Algorithms*, 15(9), 2019. https://doi.org/10.1007/s11075-019-00797-5.
[17] P. J. Holmes, J. L. Lumley, and G. Berkooz. *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*. Cambridge University Press, 1996.
[18] T. G. Kolda and B. W. Bader. Tensor decompositions and applications. *SIAM Review*, 51(3):455–500, 2009.
[19] A. J. Krener. Nonlinear Systems Toolbox, 2015. Matlab-based toolbox available by request from `ajkrener@ucdavis.edu`.
[20] A. J. Krener, C. O. Aguilar, and T. W. Hunt. Series solutions of HJB equations. In *Festscrift in Honor of Uwe Helmke*, pages 247–260, 2014.
[21] K. Kunisch, S. Volkwein, and L. Xie. HJB-POD-based feedback design for the optimal control of evolution problems. *SIAM Journal on Applied Dynamical Systems*, 3(4):703–722, 2004.

[22] C. L. Navasca and A. J. Krener. Solution of Hamilton Jacobi Bellman equations. In *IEEE Conference on Decision and Control*, pages 570–574, 2000.

[23] J.-P. Raymond. Feedback boundary stabilization of the two-dimensional Navier-Stokes equations. *SIAM Journal on Control and Optimization*, 45(3):790–828, 2006.

[24] Valeria Simoncini. Computational methods for linear matrix equations. *SIAM Review*, 58:377–441, 2016.

[25] J. Singler and B. Kramer. A POD projection method for large-scale algebraic Riccati equations. *Numerical Algebra, Control and Optimization*, 6(4):413–435, 2016.

[26] J. R. Singler. Approximate low rank solutions of Lyapunov equations via proper orthogonal decomposition. In *Proceedings of the 2008 American Control Conference*, 2008.

[27] L. Thevenet, J.-M. Buchot, and J.-P. Raymond. Nonlinear feedback stabilization of a two-dimensional Burgers equation. *ESAIM: Control, Optimisation and Calculus of Variations*, 16(4):929–955, 2009.

[28] H. Xu and K. Mizukami. Hamilton-Jacobi equation for descriptor systems. *Systems & Control Letters*, 21:321–327, 1993.