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Research Article

Interval Valued *m*-polar Fuzzy *BCK*/*BCI*-Algebras

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ABSTRACT

The notion of interval-valued *m*-polar fuzzy sets (abbreviated *IVmPF*) is much wider than the notion of *m*-polar fuzzy sets. In this paper, we apply the theory of *IVmPF* on *BCK/BCI*-algebras. We introduce the concepts of *IVmPF* subalgebras, *IVmPF* ideals and *IVmPF* commutative ideals and some essential properties are discussed. We characterize *IVmPF* subalgebras in terms of fuzzy subalgebras and subalgebras of *BCK/BCI*-algebras. We show that in *BCK*-algebra, *IVmPF* ideals are *IVmPF* subalgebras and that the converse is not valid. We provide a condition under which an *IVmPF* subalgebra becomes an *IVmPF* ideal. Further, we characterize *IVmPF* ideals in terms of fuzzy ideals and ideals of *BCK/BCI*-algebras. Moreover, we prove that in any *BCK*-algebra, an *IVmPF* commutative ideal is an *IVmPF* fuzzy ideal but not the converse. Also, we provide conditions under which an *IVmPF* ideal becomes an *IVmPF* commutative ideal.

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1. INTRODUCTION

In 1966, Imai and Iséki introduced the concept of *BCK*/*BCI*-algebras, which is a generalization of propositional calculus and the set-theoretic difference. The literature on the theory of *BCK*/*BCI*-algebras has been developed since then, and more focus has been placed on the ideal theory of *BCK*/*BCI*-algebras in particular. In *BCK*/*BCI*-algebras and other related algebraic structures, different kinds of concepts were investigated in various ways (see, e.g., [1–8]).

The fuzzy set theory proposed by Zadeh [9] has been extended to a lot of areas. In addition, a variety of extensions and generalizations of fuzzy sets have been introduced such as the following well known sets: bipolar fuzzy sets, hesitant fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets and fuzzy multisets, etc. The interval-valued fuzzy set introduced by Zadeh takes the values of the membership functions as intervals instead of numbers. The study of interval-valued fuzzy algebraic structures started in [10] by introducing the concept of interval-valued fuzzy subgroups. Jun [11] extended the concept of interval-valued fuzzy sets to *BCK*/*BCI*-algebras and introduced the notions of interval-valued subalgebras and ideals. After that, the notion of interval-valued fuzzy sets in *BCK*/*BCI*-algebras with different aspects has been studied by several authors, for example, see [12–14].

Zhang introduced the notion of bipolar fuzzy sets which permits the membership degree of an element over two intervals $[-1, 0]$ and $[0, 1]$, that is, every element assigns negative and positive degree of memberships. By applying the notion of bipolar fuzzy sets to *BCK*/*BCI*-algebras, Lee [15] introduced the notions of bipolar fuzzy subalgebra and bipolar fuzzy ideal of *BCK*/*BCI*-algebras. Using (α, β)-bipolar fuzzy generalized bi-ideals, Ibrar *et al.* [16] characterized regular ordered semigroups whereas Bashir *et al.* [17] characterized the regular ordered ternary semigroups. For more related concepts on bipolar fuzzy sets, we refer to [18–22].

As in many problems, information often comes from several variables and there are often multi-attribute data that cannot be handled using current theories, a lot of approaches have been done to solve this problem. For example, Chen *et al.* [23] presented the *m*-polar fuzzy set, an expansion of the bipolar fuzzy set and as a new approach Akram *et al.* [24] introduced a technique in decision making based on *m*-polar fuzzy sets.

The *m*-polar fuzzy algebraic structures study began with the concept of *m*-polar fuzzy Lie subalgebras [25]. After that, the theory of *m*-polar fuzzy Lie ideals was studied in Lie algebras [26]. The concept of the *m*-polar fuzzy groups was given in [27]. Moreover, *m*-polar fuzzy matroids have been studied in [28]. Further, *m*-polar fuzzy sets have been studied in different areas (see [29–33]). Recently, Al-Masarwah and Ahmad introduced the notion of *m*-polar fuzzy (commutative) ideals [34] and *m*-polar (α, β)-fuzzy ideals [35] in *BCK*/*BCI*-algebras. As a continues work they introduced a new form of generalized *m*-polar fuzzy ideals in [36] and studied normalization of *m*-polar fuzzy subalgebras in [37]. A new advanced extensions are formed by merging two fuzzy information in one set as neutrosophic bipolar fuzzy sets, bipolar valued hesitant fuzzy sets and interval- valued *m*-polar fuzzy sets

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(IVmPF). For some recent work on these extensions, we refer the reader to [38–43].

The power of the theory of IVmPF as an advanced extension with all the work done on different algebraic structure motivated the authors to apply the theory of IVmPF on BCK/BCI-algebras. The novelty in this study lies in using the proposed model on BCK/BCI-algebras. The authors introduced and investigated the notions of interval-valued m-polar fuzzy subalgebras, interval-valued m-polar fuzzy ideals and interval-valued m-polar fuzzy commutative ideals in Sections 3, 4, 5, respectively. A summary of proposed and future work were given in Section 6.

2. PRELIMINARIES

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if $\forall \nu, \kappa, \hbar \in X$, it satisfies

(K_1) $((\nu * \hbar) * (\nu * \kappa)) * (\kappa * \hbar) = 0$,

(K_2) $(\nu * (\nu * \hbar)) * \hbar = 0$,

(K_3) $\nu * \nu = 0$,

(K_4) $\nu * \hbar = 0$ and $\hbar * \nu = 0 \Rightarrow \nu = \hbar$,

If a BCI-algebra X satisfies

(K_5) $0 * \nu = 0 \, \forall \nu \in X$

then X is a BCK-algebra.

The following conditions hold in any BCK/BCI-algebra X and for all $\nu, \kappa, \hbar \in X$:

(P_1) $\nu * 0 = \nu$,

(P_2) $(\nu * \hbar) * \kappa = (\nu * \kappa) * \hbar$,

(P_3) $\nu \leq \hbar \Rightarrow \nu * \kappa \leq \hbar * \kappa$ and $\kappa * \hbar \leq \kappa * \nu$,

(P_4) $0 * (\nu * \hbar) = (0 * \nu) * (0 * \hbar)$,

(P_5) $0 * (0 * (\nu * \hbar)) = 0 * (\hbar * \nu)$,

(P_6) $(\nu * \kappa) * (\hbar * \kappa) \leq (\nu * \hbar)$,

(P_7) $\nu * (\nu * (\nu * \hbar)) = \nu * \hbar$,

(P_8) $0 * (0 * ((\nu * \kappa) * (\hbar * \kappa))) = (0 * \hbar) * (0 * \nu)$,

(P_9) $0 * (0 * (\nu * \hbar) = (0 * \hbar) * (0 * \nu)$,

where $\nu \leq \kappa \Leftrightarrow \nu * \kappa = 0 \, \forall \nu, \kappa \in X$. Clearly, $(X, \leq)$ is a partially ordered set.

A nonempty subset B of X is called a *subalgebra* of X if $\nu * \kappa \in B$ $\forall \nu, \kappa \in B$.

A nonempty subset L of X is called an *ideal* of X if

(L_1) $0 \in L$,

(L_2) $\forall \nu, \kappa \in X, \nu * \kappa \in L \, and \, \kappa \in L \Rightarrow \nu \in L$.

Let X be a *BCK/BCI*-algebra. A fuzzy set of X is a mapping $\xi : X \to [0,1]$. A fuzzy set ξ is called a fuzzy subalgebra if $(\forall \nu, \kappa \in X) \xi(\nu * \kappa) \geq \xi(\nu) \wedge \xi(\kappa)$ and it is called a fuzzy ideal if $\xi(0) \geq \xi(\nu)$ and $\xi(\nu) \geq \xi(\nu * \kappa) \wedge \xi(\kappa)$ for all $\nu, \kappa \in X$. Further, ξ is called a fuzzy commutative ideal if $\xi(0) \geq \xi(\nu)$ and $\xi(\nu * (\kappa * (\kappa * \nu))) \geq \xi((\nu * \kappa) * \hbar) \wedge \xi(\hbar)$.

By the interval number $\tilde{n}$, we mean an interval denoted as $[n^-, n^+]$, where $0 \leq n^- \leq n^+ \leq 1$. We write $S[0,1]$ to denote the set of all interval numbers. The interval $[n,n]$ is indicated by the number $n \in [0,1]$ for whatever follows. For the interval numbers $\tilde{n}_i = [n_i^-, n_i^+]$, $\tilde{m}_i = [m_i^-, m_i^+] \in S[0,1], i \in I$, we describe

(a) $\tilde{n}_i \wedge \tilde{m}_i = [n_i^- \wedge m_i^-, n_i^+ \wedge m_i^+]$;

(b) $\tilde{n}_i \vee \tilde{m}_i = [n_i^- \vee m_i^-, n_i^+ \vee m_i^+]$;

(c) $\tilde{n}_1 \leq \tilde{n}_2 \Leftrightarrow n_1^- \leq n_2^-$ and $n_1^+ \leq n_2^+$;

(d) $\tilde{n}_1 = \tilde{n}_2 \Leftrightarrow n_1^- = n_2^-$ and $n_1^+ = n_2^+$.

Let X be a *BCK/BCI*-algebra. A mapping $\tilde{\mathscr{G}} : X \to S[0,1]$ is an interval-valued fuzzy set (briefly, IVF set) of X, where for all $\nu \in X$, $\tilde{\mathscr{G}}(\nu) = [\tilde{\mathscr{G}}^-(\nu), \tilde{\mathscr{G}}^+(\nu)]$, $\tilde{\mathscr{G}}^-$ and $\tilde{\mathscr{G}}^+$ are fuzzy sets of X with $\tilde{\mathscr{G}}^-(\nu) \leq \tilde{\mathscr{G}}^+(\nu)$.

An *IVF* set is called an *IVF* subalgebra if $(\forall \nu, \kappa \in X) \tilde{\mathscr{G}}(\nu * \kappa) \geq \tilde{\mathscr{G}}(\nu) \wedge \tilde{\mathscr{G}}(\kappa)$ and it is called an *IVF* ideal if $\tilde{\mathscr{G}}(0) \geq \tilde{\mathscr{G}}(\nu)$ and $\tilde{\mathscr{G}}(\nu) \geq \tilde{\mathscr{G}}(\nu * \kappa) \wedge \tilde{\mathscr{G}}(\kappa) \, \forall \nu, \kappa \in X$. Moreover, $\tilde{\mathscr{G}}$ is called an *IVF* commutative ideal if $\tilde{\mathscr{G}}(0) \geq \tilde{\mathscr{G}}(\nu)$ and $\tilde{\mathscr{G}}(\nu * (\kappa * (\kappa * \nu))) \geq \tilde{\mathscr{G}}((\nu * \kappa) * \hbar) \wedge \tilde{\mathscr{G}}(\hbar) \, \forall \nu, \kappa, \hbar \in X$.

3. INTERVAL-VALUED *m*-POLAR FUZZY SUBALGEBRAS

The notion of an *IVmPF* subalgebra in BCK/BCI-algebras is introduced and characterized in terms of subalgebra and fuzzy subalgebra of BCK/BCI-algebras.

Definition 3.1. Let X be a nonempty set. An *IVmPF* set of X is a mapping $\tilde{\mathscr{G}} : X \to S[0,1]^m$ defined as

$$\tilde{\mathscr{G}}(\nu) = \left(\tilde{\pi}_1 \circ \tilde{\mathscr{G}}(\nu), \tilde{\pi}_2 \circ \tilde{\mathscr{G}}(\nu), \ldots, \widetilde{\pi_m} \circ \tilde{\mathscr{G}}(\nu)\right)$$

where for $i \in \{1,2,\ldots,m\}$, $\tilde{\pi}_i \circ \tilde{\mathscr{G}} : X \to S[0,1]$ is the i^{th}-projection mapping.

That is,

$$\tilde{\mathscr{G}}(\nu) = \left([\tilde{\mathscr{G}}_1^-(\nu), \tilde{\mathscr{G}}_1^+(\nu)], [\tilde{\mathscr{G}}_2^-(\nu), \tilde{\mathscr{G}}_2^+(\nu)], \ldots, [\tilde{\mathscr{G}}_m^-(\nu), \tilde{\mathscr{G}}_m^+(\nu)]\right)$$

for all $\nu \in X$, $\tilde{\mathscr{G}}_i^-$ and $\tilde{\mathscr{G}}_i^+$ are fuzzy sets of X with $\tilde{\mathscr{G}}_i^-(\nu) \leq \tilde{\mathscr{G}}_i^+(\nu)$ for all $\nu \in X$ and $i \in \{1,2,\ldots,m\}$.

We define an order relation on $S[0,1]^m$ as pointwise, that is,

$$\nu \leq \kappa \Leftrightarrow \tilde{\pi}_i(\nu) \leq \tilde{\pi}_i(\kappa)$$

where $\tilde{\pi}_i : S[0,1]^m \to S[0,1]$ is the i^{th}-projection mapping and $i \in \{1,2,\ldots,m\}$. For an element $\widetilde{[\alpha,\beta]} \in S[0,1]^m$, we mean that $([\alpha,\beta],[\alpha,\beta],\ldots,[\alpha,\beta])$, while the element $\widetilde{[\alpha,\beta]} = ([\alpha_1,\beta_1],[\alpha_2,\beta_2],\ldots,[\alpha_m,\beta_m])$ represents an arbitrary element of $S[0,1]^m$. Clearly, the elements $\widetilde{[0,0]}$ and $\widetilde{[1,1]}$ are the smallest and largest elements in $S[0,1]^m$.

Definition 3.2. An *IVmPF* set $\tilde{\mathscr{G}}$ of X is called an IVmPF subalgebra if

$$(\forall \nu, \kappa \in X)\ \tilde{\mathscr{G}}(\nu * \kappa) \geq \tilde{\mathscr{G}}(\nu) \wedge \tilde{\mathscr{G}}(\kappa),$$

that is,

$$(\forall\, \nu, \kappa \in X, i \in \{1, 2, \ldots, m\})\ \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \geq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa).$$

Example 1.

Consider a *BCK*-algebra in which $X = \{0, \wp, \kappa, \ell\}$ and $*$ is given by the following table:

$*$	0	$\wp$	κ	ℓ
0	0	0	0	0
$\wp$	$\wp$	0	0	$\wp$
κ	κ	$\wp$	0	κ
ℓ	ℓ	ℓ	ℓ	0

Let $\widehat{[\omega, \varphi]} = \left([\omega_1, \varphi_1], [\omega_2, \varphi_2], \ldots, [\omega_m, \varphi_m]\right), \widehat{[\alpha, \beta]} = \left([\alpha_1, \beta_1], [\alpha_2, \beta_2], \ldots, [\alpha_m, \beta_m]\right) \in S[0,1]^m$ such that $\widehat{[\omega, \varphi]} \geq \widehat{[\alpha, \beta]}$. Now define an *IVmPF* set $\tilde{\mathscr{G}}$ on X as

$$\tilde{\mathscr{G}}(\nu) = \begin{cases} \left([\omega_1, \varphi_1], [\omega_2, \varphi_2], \ldots, [\omega_m, \varphi_m]\right) & \text{if } \nu = 0, \\ \left([\alpha_1, \beta_1], [\alpha_2, \beta_2], \ldots, [\alpha_m, \beta_m]\right) & \text{if } \nu = \wp, \\ ([0,0], [0,0], \ldots, [0,0]) & \text{if } \nu \in \{\kappa, \ell\}. \end{cases}$$

It is easy to verify that $\tilde{\mathscr{G}}$ is an *IVmPF* subalgebra.

Lemma 3.3. *If $\tilde{\mathscr{G}}$ is an IVmPF subalgebra of X, then*

$$\tilde{\mathscr{G}}(0) \geq \tilde{\mathscr{G}}(\nu)\, \forall \nu \in X.$$

Proof. Let $\nu \in X$. Then, we have

$$\begin{aligned} \tilde{\pi}_i \circ \tilde{\mathscr{G}}(0) &= \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \nu) \\ &\geq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \\ &= \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu), \end{aligned}$$

as required.

Theorem 3.4. *An IVmPF set $\tilde{\mathscr{G}} = \left([\tilde{\mathscr{G}}_1^-, \tilde{\mathscr{G}}_1^+], [\tilde{\mathscr{G}}_2^-, \tilde{\mathscr{G}}_2^+], \ldots, [\tilde{\mathscr{G}}_m^-, \tilde{\mathscr{G}}_m^+]\right)$ is an IVmPF subalgebra of $X \Leftrightarrow \tilde{\mathscr{G}}_i^-$ and $\tilde{\mathscr{G}}_i^+$ are fuzzy subalgebras of X for all i's.*

Proof. ($\Rightarrow$) Assume that $\tilde{\mathscr{G}} = \left([\tilde{\mathscr{G}}_1^-, \tilde{\mathscr{G}}_1^+], [\tilde{\mathscr{G}}_2^-, \tilde{\mathscr{G}}_2^+], \ldots, [\tilde{\mathscr{G}}_m^-, \tilde{\mathscr{G}}_m^+]\right)$ is an *IVmPF* subalgebra of X. Then for any $\nu, \kappa \in X$,

$$\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \geq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa)\, \forall i \in \{1, 2, \ldots, m\},$$

implies

$$\begin{aligned} \left[\tilde{\mathscr{G}}_i^-(\nu * \kappa), \tilde{\mathscr{G}}_i^+(\nu * \kappa)\right] &\geq \left[\tilde{\mathscr{G}}_i^-(\nu), \tilde{\mathscr{G}}_i^+(\nu)\right] \wedge \left[\tilde{\mathscr{G}}_i^-(\kappa), \tilde{\mathscr{G}}_i^+(\kappa)\right] \\ &= \left[\tilde{\mathscr{G}}_i^-(\nu) \wedge \tilde{\mathscr{G}}_i^-(\kappa), \tilde{\mathscr{G}}_i^+(\nu) \wedge \tilde{\mathscr{G}}_i^+(\kappa)\right]. \end{aligned}$$

Therefore, $\tilde{\mathscr{G}}_i^-(\nu * \kappa) \geq \tilde{\mathscr{G}}_i^-(\nu) \wedge \tilde{\mathscr{G}}_i^-(\kappa)$ and $\tilde{\mathscr{G}}_i^+(\nu * \kappa) \geq \tilde{\mathscr{G}}_i^+(\nu) \wedge \tilde{\mathscr{G}}_i^+(\kappa)$. Hence, $\tilde{\mathscr{G}}_i^-$ and $\tilde{\mathscr{G}}_i^+$ are fuzzy subalgebras of X for all $i \in \{1, 2, \ldots, m\}$.

($\Leftarrow$) For the converse, suppose that $\tilde{\mathscr{G}}_i^-$ and $\tilde{\mathscr{G}}_i^+$ are fuzzy subalgebras of X for all i's. So for any $\nu, \kappa \in X$, we have

$$\begin{aligned} \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) &= \left[\tilde{\mathscr{G}}_i^-(\nu * \kappa), \tilde{\mathscr{G}}_i^+(\nu * \kappa)\right] \\ &\geq \left[\tilde{\mathscr{G}}_i^-(\nu) \wedge \tilde{\mathscr{G}}_i^-(\kappa), \tilde{\mathscr{G}}_i^+(\nu) \wedge \tilde{\mathscr{G}}_i^+(\kappa)\right] \\ &= \left[\tilde{\mathscr{G}}_i^-(\nu), \tilde{\mathscr{G}}_i^+(\nu)\right] \wedge \left[\tilde{\mathscr{G}}_i^-(\kappa), \tilde{\mathscr{G}}_i^+(\kappa)\right] \\ &= \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa). \end{aligned}$$

Hence, $\tilde{\mathscr{G}}$ is an *IVmPF* subalgebra of X.

Definition 3.5. Let $\tilde{\mathscr{G}}$ be any *IVmPF* set. For $\widehat{[\alpha, \beta]} = \left([\alpha_1, \beta_1], [\alpha_2, \beta_2], \ldots, [\alpha_m, \beta_m]\right) \in S[0,1]^m$ define a level subset $U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$ as follows:

$$U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right) = \left\{x \in X \mid \tilde{\pi}_i \circ \tilde{\mathscr{G}}(x) \geq [\alpha_i, \beta_i] \textit{ for all } i \in \{1, 2, \ldots, m\}\right\}.$$

Theorem 3.6. *An IVmPF set $\tilde{\mathscr{G}}$ is an IVmPF subalgebra of $X \Leftrightarrow$ each nonempty level subset $U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$ is a subalgebra of X $\forall \widehat{[\alpha, \beta]} = \left([\alpha_1, \beta_1], [\alpha_2, \beta_2], \ldots, [\alpha_m, \beta_m]\right) \in S[0,1]^m$.*

Proof. ($\Rightarrow$) Take any $\nu, \kappa \in U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$. Therefore, $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \geq [\alpha_i, \beta_i]$ and $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa) \geq [\alpha_i, \beta_i]$ for all $i \in \{1, 2, \ldots, m\}$. Having $\tilde{\mathscr{G}}$ an *IVmPF* subalgebra of X, implies

$$\begin{aligned} \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) &\geq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa) \\ &\geq [\alpha_i, \beta_i] \wedge [\alpha_i, \beta_i] \\ &= [\alpha_i, \beta_i]. \end{aligned}$$

Therefore, $\nu * \kappa \in U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$.

($\Leftarrow$) Assume that $U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$ is a subalgebra of X $\forall \widehat{[\alpha, \beta]} \in S[0,1]^m$. On contrary, let $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) < \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa)$ for some $\nu, \kappa \in X$. So $\exists \widehat{[\gamma, \lambda]} = \left([\gamma_1, \lambda_1], [\gamma_2, \lambda_2], \ldots, [\gamma_m, \lambda_m]\right) \in S[0,1]^m$ such that $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) < [\gamma_i, \lambda_i] \leq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa)$ for each $i \in \{1, 2, \ldots, m\}$ implies $\nu, \kappa \in U\left(\tilde{\mathscr{G}}; \widehat{[\gamma, \lambda]}\right)$ but $\nu * \kappa \notin U\left(\tilde{\mathscr{G}}; \widehat{[\gamma, \lambda]}\right)$, which is a contradiction. Therefore, $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \geq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa)$ for all $i \in \{1, 2, \ldots, m\}$ and $\nu, \kappa \in X$.

Example 2.

Consider a *BCK*-algebra in which $X = \{0, \wp, \mathscr{J}, \kappa, \ell\}$ and $*$ is defined by the following table:

$*$	0	$\wp$	$\mathscr{J}$	κ	ℓ
0	0	0	0	0	0
$\wp$	$\wp$	0	$\wp$	0	0
$\mathscr{J}$	$\mathscr{J}$	$\mathscr{J}$	0	$\mathscr{J}$	0
κ	κ	κ	κ	0	0
ℓ	ℓ	ℓ	κ	$\mathscr{J}$	0

Now define an *IVmPF* set $\tilde{\mathscr{G}}$ on X as

$$\tilde{\mathscr{G}}(\nu) = \begin{cases} \widetilde{[0.8, 0.8]} = ([0.8, 0.8], [0.8, 0.8], ..., [0.8, 0.8]) & \text{if } \nu = 0, \\ \widetilde{[0.4, 0.4]} = ([0.4, 0.4], [0.4, 0.4], ..., [0.4, 0.4]) & \text{if } \nu = \wp, \\ \widetilde{[0.5, 0.5]} = ([0.5, 0.5], [0.5, 0.5], ..., [0.5, 0.5]) & \text{if } \nu = \mathscr{J}, \\ \widetilde{[0.7, 0.7]} = ([0.7, 0.7], [0.7, 0.7], ..., [0.7, 0.7]) & \text{if } \nu = \kappa, \\ \widetilde{[0.3, 0.3]} = ([0.3, 0.3], [0.3, 0.3], ..., [0.3, 0.3]) & \text{if } \nu = \ell. \end{cases}$$

Therefore,

$$U\left(\tilde{\mathscr{G}}; \widetilde{[\alpha, \beta]}\right) = \begin{cases} X, & \text{if } \widetilde{[0, 0]} < \widetilde{[\alpha, \beta]} \leq \widetilde{[0.3, 0.3]}; \\ \{0, \wp, \kappa\}, & \text{if } \widetilde{[0.3, 0.3]} < \widetilde{[\alpha, \beta]} \leq \widetilde{[0.4, 0.4]}; \\ \{0, \kappa\}, & \text{if } \widetilde{[0.4, 0.4]} < \widetilde{[\alpha, \beta]} \leq \widetilde{[0.5, 0.5]}; \\ \{0, \kappa\}, & \text{if } \widetilde{[0.5, 0.5]} < \widetilde{[\alpha, \beta]} \leq \widetilde{[0.7, 0.7]}; \\ \{0\}, & \text{if } \widetilde{[0.7, 0.7]} < \widetilde{[\alpha, \beta]} \leq \widetilde{[0.8, 0.8]}; \\ \varnothing, & \text{if } \widetilde{[0.8, 0.8]} < \widetilde{[\alpha, \beta]} \leq \widetilde{[1, 1]}. \end{cases}$$

Since for all $\widetilde{[\alpha, \beta]} \in S[0, 1]^m$, $U\left(\tilde{\mathscr{G}}; \widetilde{[\alpha, \beta]}\right)$ is a subalgebra of X. Therefore by Theorem 3.6, $\tilde{\mathscr{G}}$ is an *IVmPF* subalgebra.

4. INTERVAL-VALUED *m*-POLAR FUZZY IDEALS

The notion of an *IVmPF* ideal in *BCK*/*BCI*-algebras is introduced and associated properties of *IVmPF* ideals and *IVmPF* subalgebras are considered.

Definition 4.1. An *IVmPF* set $\tilde{\mathscr{G}}$ is called an *IVmPF* ideal if the following conditions satisfy for all $\nu, \kappa \in X$:

(1) $\tilde{\mathscr{G}}(0) \geq \tilde{\mathscr{G}}(\nu)$,

(2) $\tilde{\mathscr{G}}(\nu) \geq \tilde{\mathscr{G}}(\nu * \kappa) \wedge \tilde{\mathscr{G}}(\kappa)$,

that is,

(1) $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(0) \geq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu)$,

(2) $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \geq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa)$,

$\forall i \in \{1, 2, \ldots, m\}$.

Example 3.

Consider a *BCI*-algebra in which $X = \{0, 1, \wp, \kappa, \ell\}$ and $*$ is defined by the following table:

$*$	0	1	$\wp$	κ	ℓ
0	0	0	$\wp$	κ	ℓ
1	1	0	$\wp$	κ	ℓ
$\wp$	$\wp$	$\wp$	0	ℓ	κ
κ	κ	κ	ℓ	0	$\wp$
ℓ	ℓ	ℓ	κ	$\wp$	0

Now define an *IV5PF* set $\tilde{\mathscr{G}}$ on X as

$$\tilde{\mathscr{G}}(\nu) =$$

$$\begin{cases} ([0.6, 0.7], [0.5, 0.8], [0.3, 0.4], [0.7, 0.8], [0.6, 0.7]) & \text{if } \nu = 0, \\ ([0.5, 0.6], [0.3, 0.5], [0.2, 0.3], [0.5, 0.6], [0.4, 0.6]) & \text{if } \nu = 1, \\ ([0.2, 0.4], [0.1, 0.2], [0.1, 0.2], [0.2, 0.3], [0.2, 0.3]) & \text{if } \nu \in \{\wp, \ell\}, \\ ([0.3, 0.4], [0.2, 0.3], [0.1, 0.2], [0.3, 0.5], [0.4, 0.5]) & \text{if } \nu = \kappa, \end{cases}$$

It is routine to verify that $\tilde{\mathscr{G}}$is an *IV5PF* ideal.

Lemma 4.2. *Let $\tilde{\mathscr{G}}$ be an IVmPF ideal of X and $\nu, \kappa \in X$ such that $\nu \leq \kappa$. Then,*

$$\tilde{\mathscr{G}}(\nu) \geq \tilde{\mathscr{G}}(\kappa).$$

Proof. Let $\nu, \kappa \in X$ such that $\nu \leq \kappa$. Then,

$$\begin{aligned} \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) &\geq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa) \\ &= \tilde{\pi}_i \circ \tilde{\mathscr{G}}(0) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa) \\ &= \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa). \end{aligned}$$

Hence, $\tilde{\mathscr{G}}(\nu) \geq \tilde{\mathscr{G}}(\kappa)$.

Lemma 4.3. *Let $\tilde{\mathscr{G}}$ be an IVmPF ideal of X and $\nu, \kappa, \hbar \in X$ such that $\nu * \kappa \leq \hbar$. Then,*

$$\tilde{\mathscr{G}}(\nu) \geq \tilde{\mathscr{G}}(\kappa) \wedge \tilde{\mathscr{G}}(\hbar).$$

Proof. Let $\nu, \kappa, \hbar \in X$ such that $\nu * \kappa \leq \hbar$. Then by Lemma 4.2, we have

$$\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \geq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\hbar).$$

As $\tilde{\mathscr{G}}$ is an *IVmPF* ideal of X, so

$$\begin{aligned} \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) &\geq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa) \\ &\geq \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\hbar) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa). \end{aligned}$$

It follows that $\tilde{\mathscr{G}}(\nu) \geq \tilde{\mathscr{G}}(\kappa) \wedge \tilde{\mathscr{G}}(\hbar)$.

Theorem 4.4. *In a BCK4-algebra X, every IVmPF ideal of X is an IVmPF subalgebra.*

Proof. Suppose that $\tilde{\mathscr{G}}$ is any *IVmPF* ideal and let $\nu, \kappa \in X$. As $\nu * \kappa \le \nu$ in X, so by Lemma 4.2, $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \le \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa)$. Since $\tilde{\mathscr{G}}$ is an *IVmPF* ideal of X, we have

$$\begin{aligned} \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) &\ge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \\ &\ge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa) \\ &= \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa). \end{aligned}$$

Hence, $\tilde{\mathscr{G}}$ is an *IVmPF* subalgebra.

Remark 1. The converse of Theorem 4.4 is not true in general.

Example 4.

Consider a *BCK*-algebra in which $X = \{0, \wp, \kappa, \ell\}$ and $*$ is described by the following table:

$*$	0	$\wp$	κ	ℓ
0	0	0	0	0
$\wp$	$\wp$	0	$\wp$	0
κ	κ	κ	0	0
ℓ	ℓ	ℓ	ℓ	0

Now define an *IV3PF* set $\tilde{\mathscr{G}}$ on X as:

$$\tilde{\mathscr{G}}(\nu) = \begin{cases} ([0.7, 0.8], [0.3, 0.5], [0.2, 0.3]) & \text{if } \nu = 0, \\ ([0.5, 0.6], [0.1, 0.3], [0.1, 0.2]) & \text{if } \nu = \wp, \\ ([0.3, 0.4], [0.1, 0.1], [0.1, 0.1]) & \text{if } \nu = \kappa, \\ ([0.6, 0.7], [0.2, 0.4], [0.1, 0.3]) & \text{if } \nu = \ell. \end{cases}$$

By routine calculation one can verify that $\tilde{\mathscr{G}}$ is an *IV3PF* subalgebra but not an *IV3PF* ideal because $[0.5, 0.6] = \tilde{\pi}_1 \circ \tilde{\mathscr{G}}(\wp) \not\ge \tilde{\pi}_1 \circ \tilde{\mathscr{G}}(\wp * \ell) \wedge \tilde{\pi}_1 \circ \tilde{\mathscr{G}}(\ell) = [0.6, 0.7]$.

The following result provides a condition for an *IVmPF* subalgebra to be an *IVmPF* ideal.

Theorem 4.5. *Let $\tilde{\mathscr{G}}$ be an IVmPF subalgebra of X. Then $\tilde{\mathscr{G}}$ is an IVmPF ideal $\Leftrightarrow$ for all $\nu, \kappa, \hbar \in X$ such that $\nu * \kappa \le \hbar$ implies $\tilde{\mathscr{G}}(\nu) \ge \tilde{\mathscr{G}}(\kappa) \wedge \tilde{\mathscr{G}}(\hbar)$.*

Proof. ($\Rightarrow$) Follows from Lemma 4.3.

($\Leftarrow$) Let $\tilde{\mathscr{G}}$ be an *IVmPF* subalgebra such that for all $\nu, \kappa, \hbar \in X$ with $\nu * \kappa \le \hbar$ implies $\tilde{\mathscr{G}}(\nu) \ge \tilde{\mathscr{G}}(\kappa) \wedge \tilde{\mathscr{G}}(\hbar)$. As $\nu * (\nu * \kappa) \le \kappa$, so by hypothesis

$$\tilde{\mathscr{G}}(\nu) \ge \tilde{\mathscr{G}}(\nu * \kappa) \wedge \tilde{\mathscr{G}}(\kappa).$$

Hence, $\tilde{\mathscr{G}}$ is an *IVmPF* ideal of X.

In the following result, we give a relation between an *IVmPF* ideal and fuzzy ideals of X.

Theorem 4.6. *An IVmPF set $\tilde{\mathscr{G}} = \left(\left[\tilde{\mathscr{G}}_1^-, \tilde{\mathscr{G}}_1^+\right], \left[\tilde{\mathscr{G}}_2^-, \tilde{\mathscr{G}}_2^+\right], \dots, \left[\tilde{\mathscr{G}}_m^-, \tilde{\mathscr{G}}_m^+\right]\right)$ in X is an IVmPF ideal of $X \Leftrightarrow \tilde{\mathscr{G}}_i^-$ and $\tilde{\mathscr{G}}_i^+$ are fuzzy ideals of X for all i's.*

Proof. ($\Rightarrow$) Assume that $\tilde{\mathscr{G}}(\nu) = \left(\left[\tilde{\mathscr{G}}_1^-(\nu), \tilde{\mathscr{G}}_1^+(\nu)\right], \left[\tilde{\mathscr{G}}_2^-(\nu), \tilde{\mathscr{G}}_2^+(\nu)\right], \dots, \left[\tilde{\mathscr{G}}_m^-(\nu), \tilde{\mathscr{G}}_m^+(\nu)\right]\right)$ in X is an *IVmPF* ideal of X. For any $\nu \in X$, we have

$$\tilde{\pi}_i \circ \tilde{\mathscr{G}}(0) \ge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \, \forall i \in \{1, 2, \dots, m\},$$

implies that

$$\left[\tilde{\mathscr{G}}_i^-(0), \tilde{\mathscr{G}}_i^+(0)\right] \ge \left[\tilde{\mathscr{G}}_i^-(\nu), \tilde{\mathscr{G}}_i^+(\nu)\right].$$

It follows that $\tilde{\mathscr{G}}_i^-(0) \ge \tilde{\mathscr{G}}_i^-(\nu)$ and $\tilde{\mathscr{G}}_i^+(0) \ge \tilde{\mathscr{G}}_i^+(\nu)$. Take any $\nu, \kappa \in X$. By hypothesis, we have

$$\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \ge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa) \, \forall i \in \{1, 2, \dots, m\},$$

implies that

$$\begin{aligned} \left[\tilde{\mathscr{G}}_i^-(\nu), \tilde{\mathscr{G}}_i^+(\nu)\right] &\ge \left[\tilde{\mathscr{G}}_i^-(\nu * \kappa), \tilde{\mathscr{G}}_i^+(\nu * \kappa)\right] \wedge \left[\tilde{\mathscr{G}}_i^-(\kappa), \tilde{\mathscr{G}}_i^+(\kappa)\right] \\ &= \left[\tilde{\mathscr{G}}_i^-(\nu * \kappa) \wedge \tilde{\mathscr{G}}_i^-(\kappa), \tilde{\mathscr{G}}_i^+(\nu * \kappa) \wedge \tilde{\mathscr{G}}_i^+(\kappa)\right]. \end{aligned}$$

Therefore, $\tilde{\mathscr{G}}_i^-(\nu) \ge \tilde{\mathscr{G}}_i^-(\nu * \kappa) \wedge \tilde{\mathscr{G}}_i^-(\kappa)$ and $\tilde{\mathscr{G}}_i^+(\nu) \ge \tilde{\mathscr{G}}_i^+(\nu * \kappa) \wedge \tilde{\mathscr{G}}_i^+(\kappa)$. Hence, $\tilde{\mathscr{G}}_i^-$ and $\tilde{\mathscr{G}}_i^+$ are fuzzy ideals of X.

($\Leftarrow$) For the converse, suppose that $\tilde{\mathscr{G}}_i^-$ and $\tilde{\mathscr{G}}_i^+$are fuzzy ideals of X. Then for all $\nu, \kappa \in X$,

$$\begin{aligned} \tilde{\pi}_i \circ \tilde{\mathscr{G}}(0) &= \left[\tilde{\mathscr{G}}_i^-(0), \tilde{\mathscr{G}}_i^+(0)\right] \\ &\ge \left[\tilde{\mathscr{G}}_i^-(\nu), \tilde{\mathscr{G}}_i^+(\nu)\right] \\ &= \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \end{aligned}$$

and

$$\begin{aligned} \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) &= \left[\tilde{\mathscr{G}}_i^-(\nu), \tilde{\mathscr{G}}_i^+(\nu)\right] \\ &\ge \left[\tilde{\mathscr{G}}_i^-(\nu * \kappa) \wedge \tilde{\mathscr{G}}_i^-(\kappa), \tilde{\mathscr{G}}_i^+(\nu * \kappa) \wedge \tilde{\mathscr{G}}_i^+(\kappa)\right] \\ &= \left[\tilde{\mathscr{G}}_i^-(\nu * \kappa), \tilde{\mathscr{G}}_i^+(\nu * \kappa)\right] \wedge \left[\tilde{\mathscr{G}}_i^-(\kappa), \tilde{\mathscr{G}}_i^+(\kappa)\right] \\ &= \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa). \end{aligned}$$

Hence, $\tilde{\mathscr{G}}$ is an *IVmPF* ideal.

The following result provides a correspondence between an *IVmPF* ideal of X and an ideal of X.

Theorem 4.7. *An IVmPF set $\tilde{\mathscr{G}}$ is an IVmPF ideal of $X \Leftrightarrow$ each nonempty level subset $U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$ is an ideal of X $\forall \widehat{[\alpha, \beta]} = \left(\left[\alpha_1, \beta_1\right], \left[\alpha_2, \beta_2\right], ..., \left[\alpha_m, \beta_m\right]\right) \in S[0, 1]^m$.*

Proof. ($\Rightarrow$) Let us suppose that $\tilde{\mathscr{G}}$ is an *IVmPF* ideal of X and $\nu \in U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$. Then $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \ge \left[\alpha_i, \beta_i\right]$. By hypothesis, $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(0) \ge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) \ge \left[\alpha_i, \beta_i\right]$. Thus, $0 \in U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$. Next, take any $\nu * \kappa \in U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$ and $\kappa \in U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$. Therefore, $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \ge \left[\alpha_i, \beta_i\right]$ and $\tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa) \ge \left[\alpha_i, \beta_i\right]$. As $\tilde{\mathscr{G}}$ is an *IVmPF* ideal, so

$$\begin{aligned} \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu) &\ge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\nu * \kappa) \wedge \tilde{\pi}_i \circ \tilde{\mathscr{G}}(\kappa) \\ &\ge \left[\alpha_i, \beta_i\right] \wedge \left[\alpha_i, \beta_i\right] \\ &= \left[\alpha_i, \beta_i\right]. \end{aligned}$$

It follows that $\nu \in U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$. Hence, $U\left(\tilde{\mathscr{G}}; \widehat{[\alpha, \beta]}\right)$ is an ideal.

($\Leftarrow$) Now let $U\left(\tilde{\mathscr{G}};\widehat{[\alpha,\beta]}\right)$ be an ideal of X $\forall\widehat{[\alpha,\beta]} \in S[0,1]^m$. If $\tilde{\pi}_i\circ\tilde{\mathscr{G}}(0) < \tilde{\pi}_i\circ\tilde{\mathscr{G}}(v)$ for some $v \in X$. So $\exists\widehat{[\delta,\gamma]} = \left([\delta_1,\gamma_1],[\delta_2,\gamma_2],...,[\delta_m,\gamma_m]\right) \in S[0,1]^m$ such that $\tilde{\pi}_i\circ\tilde{\mathscr{G}}(0) < [\delta_i,\gamma_i] \le \tilde{\pi}_i\circ\tilde{\mathscr{G}}(v)$ for all $i\in\{1,2,...,m\}$ implies $v\in U\left(\tilde{\mathscr{G}};\widehat{[\delta,\gamma]}\right)$ but $0 \notin U\left(\tilde{\mathscr{G}};\widehat{[\delta,\gamma]}\right)$, which is a contradiction. Therefore, $\tilde{\pi}_i\circ\tilde{\mathscr{G}}(0) \ge \tilde{\pi}_i\circ\tilde{\mathscr{G}}(v)$ for all $v \in X$ and $i \in \{1,2,...,m\}$. Again, if $\tilde{\pi}_i\circ\tilde{\mathscr{G}}(v) < \tilde{\pi}_i\circ\tilde{\mathscr{G}}(v*\kappa)\wedge\tilde{\pi}_i\circ\tilde{\mathscr{G}}(\kappa)$ for some $v,\kappa \in X$. So $\exists\widehat{[\delta,\gamma]} = \left([\delta_1,\gamma_1],[\delta_2,\gamma_2],...,[\delta_m,\gamma_m]\right) \in S[0,1]^m$ such that $\tilde{\pi}_i\circ\tilde{\mathscr{G}}(v) < [\delta_i,\gamma_i] \le \tilde{\pi}_i\circ\tilde{\mathscr{G}}(v*\kappa)\wedge\tilde{\pi}_i\circ\tilde{\mathscr{G}}(\kappa)$ for all $i \in \{1,2,...,m\}$ implies $v*\kappa\in U\left(\tilde{\mathscr{G}};\widehat{[\delta,\gamma]}\right)$ and $\kappa\in U\left(\tilde{\mathscr{G}};\widehat{[\delta,\gamma]}\right)$ but $v \notin U\left(\tilde{\mathscr{G}};\widehat{[\delta,\gamma]}\right)$, which is again a contradiction. Therefore, $\tilde{\pi}_i\circ\tilde{\mathscr{G}}(v) \ge \tilde{\pi}_i\circ\tilde{\mathscr{G}}(v*\kappa)\wedge\tilde{\pi}_i\circ\tilde{\mathscr{G}}(\kappa)$ for all $v,\kappa \in X$ and $i\in\{1,2,...,m\}$.

5. INTERVAL-VALUED *m*-POLAR COMMUTATIVE IDEALS

The notion of an *IVmPF* commutative ideal of *BCK*/*BCI*-algebras is defined. Relations among the *IVmPF* subalgebras, *IVmPF* ideals and *IVmPF* commutative ideals are discussed.

Definition 5.1. *An IVmPF set $\tilde{\mathscr{G}}$ is called an IVmPF commutative ideal if the following conditions satisfy for all $v,\kappa,\hbar\in X$:*

(1) $\tilde{\mathscr{G}}(0)\ge\tilde{\mathscr{G}}(v)$,

(2) $\tilde{\mathscr{G}}(v*(\kappa*(\kappa*v)))\ge\tilde{\mathscr{G}}((v*\kappa)*\hbar)\wedge\tilde{\mathscr{G}}(\hbar)$,

that is,

(1) $\tilde{\pi}_i\circ\tilde{\mathscr{G}}(0)\ge\tilde{\pi}_i\circ\tilde{\mathscr{G}}(v)$,

(2) $\tilde{\pi}_i\circ\tilde{\mathscr{G}}(v*(\kappa*(\kappa*v)))\ge\tilde{\pi}_i\circ\tilde{\mathscr{G}}((v*\kappa)*\hbar)\wedge\tilde{\pi}_i\circ\tilde{\mathscr{G}}(\hbar)$,

$\forall i\in\{1,2,\ldots,m\}$.

Example 5.

Consider the *BCK*-algebra *X* of Example 1. Let $\widehat{[\omega,\varphi]} = \left([\omega_1,\varphi_1],[\omega_2,\varphi_2],\ldots,[\omega_m,\varphi_m]\right), \widehat{[\alpha,\beta]} = \left([\alpha_1,\beta_1],[\alpha_2,\beta_2],\ldots,[\alpha_m,\beta_m]\right), \widehat{[\delta,\gamma]} = \left([\delta_1,\gamma_1],[\delta_2,\gamma_2],\ldots,[\delta_m,\gamma_m]\right) \in S[0,1]^m$ such that $\widehat{[\omega,\varphi]}\ge\widehat{[\alpha,\beta]}\ge\widehat{[\delta,\gamma]}$. Now define an *IVmPF* set $\tilde{\mathscr{G}}$ *on X as:*

$$\tilde{\mathscr{G}}(v)=\begin{cases}\widehat{[\omega,\varphi]}=\left([\omega_1,\varphi_1],[\omega_2,\varphi_2],\ldots,[\omega_m,\varphi_m]\right) & \text{if } v=0,\\ \widehat{[\alpha,\beta]}=\left([\alpha_1,\beta_1],[\alpha_2,\beta_2],\ldots,[\alpha_m,\beta_m]\right) & \text{if } v\in\{\wp,\kappa\},\\ \widehat{[\delta,\gamma]}=\left([\delta_1,\gamma_1],[\delta_2,\gamma_2],\ldots,[\delta_m,\gamma_m]\right) & \text{if } v=\ell.\end{cases}$$

It is easy to verify that $\tilde{\mathscr{G}}$ is an *IVmPF* commutative ideal.

Theorem 5.2. *In any BCK-algebra X, every IVmPF commutative ideal of X is an IVmPF ideal.*

Proof. For any *IVmPF* commutative ideal $\tilde{\mathscr{G}}$ of *X* and $v,\kappa,\hbar\in X$, we have

$$\begin{aligned}\tilde{\pi}_i\circ\tilde{\mathscr{G}}(v) &= \tilde{\pi}_i\circ\tilde{\mathscr{G}}(v*(0*(0*v)))\\ &\ge\tilde{\pi}_i\circ\tilde{\mathscr{G}}((v*0)*\kappa)\wedge\tilde{\pi}_i\circ\tilde{\mathscr{G}}(\kappa)\\ &=\tilde{\pi}_i\circ\tilde{\mathscr{G}}(v*\kappa)\wedge\tilde{\pi}_i\circ\tilde{\mathscr{G}}(\kappa).\end{aligned}$$

Hence, $\tilde{\mathscr{G}}$ is an *IVmPF* ideal.

Corollary 5.3. Every *IVmPF* commutative ideal of *X* is an *IVmPF* subalgebra of *X*.

Remark 2. In general, the converse of Theorem 5.2 is not true as shown next.

Example 6.

Consider a *BCK*-algebra in which $X = \{0,\wp,,\kappa,\ell\}$ and $*$ is described with the following table:

$*$	0	$\wp$	$\mathscr{J}$	κ	ℓ
0	0	0	0	0	0
$\wp$	$\wp$	0	$\wp$	0	0
$\mathscr{J}$	$\mathscr{J}$	$\mathscr{J}$	0	0	0
κ	κ	κ	κ	0	0
ℓ	ℓ	ℓ	κ	$\mathscr{J}$	0

Let $\widehat{[\theta,\lambda]} = \left([\theta_1,\lambda_1],[\theta_2,\lambda_2],\ldots,[\theta_m,\lambda_m]\right), \widehat{[\psi,\phi]} = \left([\psi_1,\phi_1],[\psi_2,\phi_2],\ldots,[\psi_m,\phi_m]\right), \widehat{[\rho,\sigma]} = \left([\rho_1,\sigma_1],[\rho_2,\sigma_2],\ldots,[\rho_m,,\sigma_m,]\right) \in S[0,1]^m$ such that $\widehat{[\theta,\lambda]}\ge\widehat{[\psi,\phi]}\ge\widehat{[\rho,\sigma]}$. Now define an *IVmPF* set $\tilde{\mathscr{G}}$ on *X* as:

$$\tilde{\mathscr{G}}(v)=\begin{cases}\widehat{[\theta,\lambda]}=\left([\theta_1,\lambda_1],[\theta_2,\lambda_2],\ldots,[\theta_m,\lambda_m]\right) & \text{if } v=0,\\ \widehat{[\psi,\phi]}=\left([\psi_1,\phi_1],[\psi_2,\phi_2],\ldots,[\psi_m,\phi_m]\right) & \text{if } v=\wp,\\ \widehat{[\rho,\sigma]}=\left([\rho_1,\sigma_1],[\rho_2,\sigma_2],\ldots,[\rho_m,\sigma_m]\right) & \text{if } v\in\{j,\kappa,\ell\}.\end{cases}$$

It can be shown that $\tilde{\mathscr{G}}$ is an *IVmPF* ideal but not an *IVmPF* commutative ideal because $[\rho_1,\sigma_1] = \tilde{\pi}_1\circ\tilde{\mathscr{G}}(\mathscr{J}) = \tilde{\pi}_1\circ\tilde{\mathscr{G}}(\mathscr{J}*(\kappa*(\kappa*\mathscr{J})))\ngeq\tilde{\pi}_1\circ\tilde{\mathscr{G}}((\mathscr{J}*\kappa)*0)\wedge\tilde{\pi}_1\circ\tilde{\mathscr{G}}(0) = \tilde{\pi}_1\circ\tilde{\mathscr{G}}(0) = [\theta_1,\lambda_1]$.

Following two results provide conditions for an *IVmPF* ideal to be an *IVmPF* commutative ideal.

Theorem 5.4. *Let $\tilde{\mathscr{G}}$ be an IVmPF ideal of X. Then $\tilde{\mathscr{G}}$ is an IVmPF commutative ideal $\Leftrightarrow$ for all $v,\kappa\in X$,*

$$\tilde{\mathscr{G}}(v*(\kappa*(\kappa*v)))\ge\tilde{\mathscr{G}}(v*\kappa).$$

Proof. ($\Rightarrow$) Let $\tilde{\mathscr{G}}$ be an *IVmPF* commutative ideal. Then for all $v,\kappa,\hbar\in X$, we have

$$\tilde{\mathscr{G}}(v*(\kappa*(\kappa*v)))\ge\tilde{\mathscr{G}}((v*\kappa)*\hbar)\wedge\tilde{\mathscr{G}}(\hbar).$$

Taking $\hbar=0$, we get

$$\begin{aligned}\tilde{\mathscr{G}}(v*(\kappa*(\kappa*v)))&\ge\tilde{\mathscr{G}}((v*\kappa)*0)\wedge\tilde{\mathscr{G}}(0)\\ &=\tilde{\mathscr{G}}(v*\kappa)\wedge\tilde{\mathscr{G}}(0)\\ &=\tilde{\mathscr{G}}(v*\kappa).\end{aligned}$$

(⇐) Let $\tilde{\mathscr{G}}$ be an *IVmPF* ideal such that $\tilde{\mathscr{G}}(\nu * (\kappa * (\kappa * \nu))) \geq \tilde{\mathscr{G}}(\nu * \kappa)$ for all $\nu, \kappa \in X$. By assumption, we have for all $\nu, \kappa, \hbar \in X$

$$\tilde{\mathscr{G}}(\nu * \kappa) \geq \tilde{\mathscr{G}}((\nu * \kappa) * \hbar) \wedge \tilde{\mathscr{G}}(\hbar).$$

Therefore, $\tilde{\mathscr{G}}(\nu * (\kappa * (\kappa * \nu))) \geq \tilde{\mathscr{G}}((\nu * \kappa) * \hbar) \wedge \tilde{\mathscr{G}}(\hbar)$, as required.

Theorem 5.5. *Let X be a commutative BCK-algebra. Then every IVmPF ideal of X is an IVmPF commutative ideal.*

Proof. Suppose that $\tilde{\mathscr{G}}$ is an *IVmPF* ideal of *X*. Then for all $\nu, \kappa, \hbar \in X$,

$$\begin{aligned}&((\nu * (\kappa * (\kappa * \nu))) * ((\nu * \kappa) * \hbar)) * \hbar \\ &= ((\nu * (\kappa * (\kappa * \nu))) * \hbar) * ((\nu * \kappa) * \hbar) \\ &\leq (\nu * (\kappa * (\kappa * \nu))) * (\nu * \kappa) \\ &= (\nu * (\nu * \kappa)) * (\kappa * (\kappa * \nu)) \\ &= 0\end{aligned}$$

It follows that $((\nu * (\kappa * (\kappa * \nu))) * ((\nu * \kappa) * \hbar)) \leq \hbar$. As $\tilde{\mathscr{G}}$ is an *IVmPF* ideal of *X*, then by Lemma 4.3, $\tilde{\mathscr{G}}(\nu * (\kappa * (\kappa * \nu))) \geq \tilde{\mathscr{G}}((\nu * \kappa) * \hbar) \wedge \tilde{\mathscr{G}}(\hbar)$.

6. CONCLUSION

In this paper, by applying the theory of IVmPF on *BCK/BCI*-algebra, the notions of interval-valued m-polar fuzzy subalgebras, interval-valued m-polar fuzzy ideals and interval-valued m-polar fuzzy commutative ideals are introduced and some essential properties are discussed. Characterizations of interval-valued m-polar fuzzy subalgebras and interval-valued m-polar fuzzy ideals are considered. Moreover, the relations among interval-valued m-polar fuzzy subalgebras, interval-valued m-polar fuzzy ideals and interval-valued m-polar fuzzy commutative ideals are obtained. This work can be a basis for further analysis of the interval-valued m-polar fuzzy structures in related algebraic structures. For future study, this concept may be applied to study some application fields like decision-making, knowledge base system, data analysis, and so on. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as subtraction algebras, B-algebras, MV-algebras, d-algebras, Q-algebras, and so on.

CONFLICTS OF INTEREST

Authors declare that they have no conflicts of interest.

AUTHORS' CONTRIBUTIONS

All authors have contributed to the manuscript equally.

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