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Research Article

A Hybrid Decision-Making Approach Under Complex Pythagorean Fuzzy *N*-Soft Sets

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ABSTRACT

The main objectives of this article include the formal statement of a new mathematical model of uncertain knowledge and the presentation of its potential applications. The novel hybrid model is called complex Pythagorean fuzzy *N*-soft set (CPFNSS) because it enjoys both the parametric structure of *N*-soft sets and the most prominent features of complex Pythagorean fuzzy sets in order to capture the nuances of two-dimensional inexact information. We demonstrate that this model serves as a competent tool for ranking-based modeling of parameterized fuzzy data. We propose some basic set-theoretical operations on CPFNSSs and explore some of their practical properties. Furthermore, we elaborate the Einstein and algebraic operations on complex Pythagorean fuzzy *N*-soft values (CPFNSVs). We interpret its relationships with contemporary theories to vindicate the versatility of the proposed model. Moreover, we develop three algorithms to unfold the application of proposed theory in multi-criteria decision-making (MCDM). Some illustrative applications give a practical justification for these strategies. Finally, we conduct a comparative analysis of the performance of these algorithms with existing MCDM techniques, namely, choice values and *D*-choice values of Pythagorean fuzzy *N*-soft set (PFNSS), which validates the effectiveness of the proposed techniques.

1. INTRODUCTION

Decision-making (DM) can be interpreted as a systematic process of solving real-world problems which provides an optimal solution after examining the feasible set of alternatives. DM approaches have gained popularity as they are widely used in various disciplines, including medical sciences, engineering, economics and many other areas of science and technology. Recently, the DM process has become more complex due to the presence of uncertainty and ambiguity in collected information which can bother the decision-makers to decide smoothly. The traditional DM strategies were powerless to deal with such uncertain and vague data. The pioneering solution to such problems was provided by Zadeh [53] by setting the foundations of fuzzy set (FS) theory in which each element is assigned by a membership degree lying between 0 and 1. Atanassov [11] generalized the idea of FS into intuitionistic fuzzy set (IFS) by adding nonmembership degree (λ) to the membership degree (μ) of FS, with the condition $\mu + \lambda \leq 1$. Yager [49,50] established the Pythagorean fuzzy set (PFS) theory, which relaxes the aforementioned condition of IFS to $\mu^2 + \lambda^2 \leq 1$. No doubt Pythagorean fuzzy expressions are raising the interest of many scholars, also in terms of their applications to DM. For example, Huang *et al.* [23] introduced a Pythagorean fuzzy MULTIMOORA method that uses a novel distance measure and a score function. They implemented this method for the evaluation of disk productions and energy projects. Akram *et al.* [6] have investigated risk evaluation in failure modes and effects analysis (FMEA) with the help of hybrid TOPSIS and ELECTRE I approaches under Pythagorean fuzzy information. Zhang and Xu [55] developed the TOPSIS method under Pythagorean fuzzy environment and implemented this method to evaluate the quality of private airline services. Zhou and Chen [56] presented the PF-TOPSIS method by utilizing a novel distance measure and illustrated its application for the commercialization of technology enterprises. Wang and Li [46] proposed a multi attribute DM method and power Bonferroni mean (PBM) operator for PF-information and applied this method to select the best online payment service providers. Wang and Chen [44] introduced the Pythagorean fuzzy LINMAP-based compromising method and elaborated the MCGDM problem for the selection of railway projects. Lin *et al.* [27] extended the TOPSIS method under the linguistic Pythagorean fuzzy environment by using a novel correlation coefficient and an entropy measure. They also highlighted some potential applications. Lin *et al.* [25] extended the TOPSIS and VIKOR methods for the probabilistic linguistic term information by using score function based on concentration degree. Lin *et al.* [26] put forward the directional correlation coefficient measures for Pythagorean fuzzy information and illustrated their potential applications in medical diagnosis and cluster analysis.

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The theory of FSs and its generalizations were applied and explored by many researchers as they overcame the limitations of crisp set theory and served as precise tools to deal with specific types of imprecise data. However, these theories were not able to handle the inconsistent data of periodic nature until Ramot *et al.* [37] generalized the notion of FS by establishing a new theory of complex fuzzy sets (CFSs). In this theory the range of the membership function was extended from the unit interval to the unit disc of the complex plane. To overcome some deficiencies of CFS, Alkouri and Salleh [10] proposed the notion of complex intuitionistic fuzzy set (CIFS) by introducing complex-valued membership ($\mu e^{i\alpha}$) and nonmembership ($\lambda e^{i\beta}$) degrees, in which the amplitude and phase terms are restricted by the conditions $0 \leq \mu + \lambda \leq 1$ and $0 \leq \frac{\alpha}{2\pi} + \frac{\beta}{2\pi} \leq 1$, respectively. Ullah *et al.* [43] broadened the space of CIFS by presenting the idea of complex Pythagorean fuzzy set (CPFS) and introduced new constraint conditions $0 \leq \mu^2 + \lambda^2 \leq 1$ and $0 \leq \left(\frac{\alpha}{2\pi}\right)^2 + \left(\frac{\beta}{2\pi}\right)^2 \leq 1$. The CPFS is an effective tool to capture the inconsistent data of periodic nature and it excels due to its relaxed conditions, excellent features and advantageous properties. This remarkable concept is extensively used in various territories including DM problems, networking, image processing, clustering, pattern recognition, engineering and many other fields of computer science. Lin *et al.* [29] introduced the linguistic q-rung orthopair FSs and their interactional partitioned Heronian mean aggregation operators. Wang and Garg [45] presented the multiple attribute DM approach by utilizing the interactive Archimedean norm operations and elaborated this method with the help of practical application.

The existing models such as FS [53], rough sets [34], CFS [37], CIFS [10] and CPFS [43] and many others, have been developed to capture various types of uncertainties and vagueness embedded in a system in an efficient way. However, all these theories have their own structure, impact, properties and inherent limitations. One major limitation of these models is the neglection of parametrization associated with these theories. To fix this problem, Molodtsov [33] established an entirely new theory of soft set (SS) which provides a parameterized mathematical framework relaxed from the above-stated limitations. The literature on soft set theory was further extended after observing its rich potential for practical applications in various directions. This theory soon attracted the attention of many researchers because it has been implemented in many areas of uncertainty such as mathematical analysis, forecasting, optimization theory, algebraic structures, information systems, data analysis and in DM applications. Yang *et al.* [52] introduced new fusion techniques under continuous interval-valued q-rung orthopair fuzzy environment and demonstrated its application for the selection of suitable SmartWatch design. Lin *et al.* [28] proposed integrated probabilistic linguistic multi-criteria decision-making (MCDM) method for the assessment of internet-of-things (IoT) programs. Rodríguez *et al.* [40] presented a novel consensus reaching process (CRP) model to capture the large-scale DM problems and applied it for intelligent CRP support system.

The development of soft set theory fascinated the researchers to amalgamate the soft set with other mathematical models. Peng *et al.* [35] proposed a generalization of soft set, namely, Pythagorean fuzzy soft set (PFSS) by fusing the soft set (SS) with PFS and interpreted this concept by some potential applications. Thirunavukarasu *et al.* [42] contributed to the literature by presenting the notion of complex fuzzy soft set (CFSS) and highlighting its applications. Kumar and Bajaj [24] put forward the complex intuitionistic fuzzy soft set (CIFSS) and discussed some entropies and distance measures. Lin *et al.* [30] proposed a new probability density based ordered weighted averaging (PDOWA) operator and demonstrated its application for the assessment of smart phones. Liu *et al.* [31,32] introduced hybrid models of long-term intertemporal hesitant fuzzy soft sets (LIT-HFSS) and hesitant linguistic expression soft sets (HLESS) along with significant group DM algorithms. Rodríguez *et al.* [39] proposed the comprehensive minimum cost models based on both distance measure and consensus degree to address the consistent fuzzy preference relations. Chen *et al.* [12] identified and prioritized the factors affecting the comfort of in-cabin passenger on fast-moving rail in China by using fuzzy linguistic group DM method.

In view of the theories stated above, it is clear that maximum research focus on the advanced models stimulated by soft sets was put on either its original binary interpretation (only 0 or 1 are allowed) or else real numbers within the unit interval [0, 1]. Despite of this, several daily life DM problems contain multinary but discrete type structured data. These multinary evaluations are frequently used in rating or ranking-based systems. In such systems, it is observed that ranking of alternatives such as hotels, websites, dramas, movies, music, games, magazines and books can be represented by a number of stars, dots, check marks, hearts, natural numbers and even by icons [9,16].

Obviously, this multinary parameterized data cannot be tackled by the above-mentioned soft set inspired models. Therefore, a different model was required in order to deal with such type of data. Motivated by all these facts, Fatimah *et al.* [16] developed the N-soft set (NSS) theory along with DM algorithms that emphasize the importance of ordered grades in practical examples. Later, Akram *et al.* [1,2] launched the advanced hybrid theories of hesitant N-soft set (HNSS) and fuzzy N-soft set (FNSS) by merging the concept of N-soft set with additional mathematical traits such as hesitancy and fuzziness of set, respectively. Akram *et al.* [3] combines all these features into hesitant fuzzy N-soft sets, and Akram *et al.* [4] introduced another hybrid model, namely, intuitionistic fuzzy N-soft set (IFNSS) by the suitable combination of N-soft sets with intuitionistic fuzzy expressions. Akram *et al.* [7] presented hesitant fuzzy N-soft ELECTRE-II model. Zhang *et al.* [54] developed the hybrid theory of Pythagorean fuzzy N-soft sets (PFNSSs) that merges the concepts of N-soft set and PFS. And Fatimah and Alcantud [15] have just presented the multi-fuzzy N-soft set model with applications to DM.

To summarize, the motivation of this article boils down to the following elements:

- The idea of NSS captures the characterization of the universe of objects in a multinary parametric manner. Although this concept improves upon soft set theory, still it has no potential to handle the fuzziness of parameterized characterizations.
- The CPFS theory tackles the uncertainty and periodicity of data at the same time, but it also has some deficiencies due to the inadequacy of parametrization.

- Although the PFNSS model deals with imprecision in terms of multinary parameterized descriptions of the universe of objects, this tool is limited to model the fuzziness in just one dimension because of the lack of a phase term.
- The concept of complex intuitionistic fuzzy N-soft set (CIFNSS) provides an effective model with a large ability to capture the vagueness of parameterized data. However it is restricted by the conditions $0 \leq \mu + \lambda \leq 1$ and $0 \leq \frac{\alpha}{2\pi} + \frac{\beta}{2\pi} \leq 1$.
- The complex Pythagorean fuzzy soft set (CPFSS) theory presents a binary parameterized tool that copes with uncertainty and ambiguity of data with great generality. But this theory is unable to model the multinary framework of evaluations.

Motivated by all these concerns, this article introduces a novel mathematical hybrid model, namely, complex PFNSS. It merges the remarkable features of both CPFS and NSS theories. The proposed model is especially designed for ranking-based evaluations of two-dimensional parameterized DM problems. Moreover, the fundamental set-theoretic operations and significant properties of the proposed model are discussed. The relationships between the introduced model and existing models are brought to light. Then, Einstein operators and some other operations for complex Pythagorean fuzzy N-soft values (CPFNSVs) are designed and properly interpreted. With these techniques, three advanced DM algorithms are developed. Then, we illustrate their suitability with the help of some potential applications. Finally, a comparative analysis with existing methodologies is conducted in order to justify the reliability of the proposed strategies of solution.

In a nutshell, the main contributions of this article are:

- This research article introduces a modern, productive and most general model abbreviated as complex Pythagorean fuzzy N-soft set (CPFNSS). It enables us to tackle the imprecision and periodicity of parameterized data having multinary but discrete structure.
- The rationality and feasibility of the appropriate DM algorithms that benefit from this new framework is demonstrated with the help of some potential applications.
- A comparative study with existing MCDM techniques, namely, choice values of PFNSS and D-choice values of PFNSS, validates the authenticity of the proposed techniques and justifies the consistency of the results.

This research article is structured as follows: Section 2 presents the preliminaries which include the fundamental definitions related to the CPFS, SS, NSS and PFNSS set. Section 3 introduces our new competent model and its basic set-theoretic operations. Further, it explores the relationships of proposed model with existing models. Section 4 investigates the Einstein and algebraic operations on CPFNSVs. Section 5 establishes three DMalgorithms and throws light on their application by the means of explanatory numerical examples for the selection of best laptop and plant location, respectively. Furthermore, Section 6 presents a comparison of our proposed techniques with existing MCDM techniques, namely, choice values of PFNSS and D-choice values of PFNSS. Finally, the purpose of Section 7 is to conclude this research article.

2. PRELIMINARIES

In this section, we present some elementary definitions required for the advanced developments.

Definition 2.1. [43] Let Z be a universe of discourse. A *CPFS* $\mathfrak{Q}$ on Z can be characterized as

$$\mathfrak{Q} = \left\{ \left(z, \mu_{\mathfrak{Q}}(z)e^{i\alpha_{\mathfrak{Q}}(z)}, \lambda_{\mathfrak{Q}}(z)e^{i\beta_{\mathfrak{Q}}(z)}\right) \middle| \, z \in Z \right\},$$

where $i = \sqrt{-1}$, the amplitude terms $\mu_{\mathfrak{Q}}(z), \lambda_{\mathfrak{Q}}(z) \in [0,1]$ and phase terms $\alpha_{\mathfrak{Q}}(z), \beta_{\mathfrak{Q}}(z) \in [0, 2\pi]$ satisfy the conditions $0 \leq \mu^2_{\mathfrak{Q}}(z) + \lambda^2_{\mathfrak{Q}}(z) \leq 1$ and $0 \leq \left(\frac{\alpha_{\Omega}(z)}{2\pi}\right)^2 + \left(\frac{\beta_{\mathfrak{Q}}(z)}{2\pi}\right)^2 \leq 1$. For all $z \in Z$, $\chi_{\mathfrak{Q}}(z) = \sqrt{1 - \mu^2_{\mathfrak{Q}}(z) - \lambda^2_{\mathfrak{Q}}(z)}\, e^{i2\pi\sqrt{1-\left(\frac{\alpha_{\mathfrak{Q}}(z)}{2\pi}\right)^2-\left(\frac{\beta_{\mathfrak{Q}}(z)}{2\pi}\right)^2}}$ represents the degree of indeterminacy. The pair of membership and nonmembership degree $\left(\mu_{\mathfrak{Q}}(z)e^{i\alpha_{\mathfrak{Q}}(z)}, \lambda_{\mathfrak{Q}}(z)e^{i\beta_{\mathfrak{Q}}(z)}\right)$ is called a complex Pythagorean fuzzy number (CPFN).

Definition 2.2. [33] Let Z be a universe of discourse and K be a set of parameters, $S \subseteq K$. A pair $\mathfrak{S} = (F, S)$ is said to be a *soft set* over Z if $F : S \rightarrow \mathcal{P}(Z)$, where $\mathcal{P}\,(Z)$ is the family of all subsets of Z. A soft set $\mathfrak{S}$ over the universe Z can be represented as follows:

$$\mathfrak{S} = \left\{ \left(s_q, F\left(s_q\right)\right) \middle| \, s_q \in S, F\left(s_q\right) \in \mathcal{P}(Z) \right\}.$$

Definition 2.3. [16] Let Z be a universe of discourse and K be a set of parameters. Let $S \subseteq K$ and $D = \{0, 1, \ldots, N-1\}$ be a set of ordered grades with $N \in \{2, 3, \ldots\}$. A triplet $\mathfrak{L} = (F, S, N)$ is called set on *N-soft set* on Z if $F : S \rightarrow 2^{Z \times D}$ with the property that for each $s \in S$ and $z \in Z$ there exist a unique $\left(z, d_s\right) \in Z \times D$ such that $\left(z, d_s\right) \in F(s), z \in Z, d_s \in D$. The N-soft set $\mathfrak{L}$ over the universe Z can be represented as follows:

$$\mathfrak{L} = \left\{ \left(s_q, F\left(s_q\right)\right) \middle| \, s_q \in S, F\left(s_q\right) \in 2^{Z \times D} \right\}.$$

Definition 2.4. [54] Let Z be a universe of discourse and K be a set of parameters. Let $S \subseteq K$ and $D = \{0, 1, \dots, N-1\}$ be a set of ordered grades with $N \in \{2, 3, \dots\}$. A triple (f, L, N) is called a PFNSS on Z if $L = (F, S, N)$ is NSS on Z and $f : S \to \mathcal{PF}^{Z\times D}$, where $\mathcal{PF}^{Z\times D}$ is the collection of all PFSs over $Z \times D$. In other words, PFNSS (f,L,N) is defined as follows:

$$(f, L, N) = \left\{ \left(s_q, f\left(s_q\right)\right) \middle| s_q \in S, f\left(s_q\right) \in \mathcal{PF}^{Z\times D} \right\},$$

where $f\left(s_q\right) = \left\{ \left(\left(z_p, d_{pq}\right), \mu_{pq}\left(z_p, d_{pq}\right), \lambda_{pq}\left(z_p, d_{pq}\right)\right) \middle| \left(z_p, d_{pq}\right) \in Z \times D \right\}$ represents the PFS over $Z \times D$. The membership and non-membership degrees $\mu_{pq}\left(z_p, d_{pq}\right), \lambda_{pq}\left(z_p, d_{pq}\right) \in [0, 1]$ satisfy the condition $0 \le \mu^2_{pq}\left(z_p, d_{pq}\right) + \lambda^2_{pq}\left(z_p, d_{pq}\right) \le 1$. For all $\left(z_p, d_{pq}\right) \in Z \times D$, $\chi_{pq}\left(z_p, d_{pq}\right) = \sqrt{1 - \mu^2_{pq}\left(z_p, d_{pq}\right) - \lambda^2_{pq}\left(z_p, d_{pq}\right)}$ represents the degree of indeterminacy.

Definition 2.5. [55] Let $\mathfrak{Y} = \left(\mu_{\mathfrak{Y}}, \lambda_{\mathfrak{Y}}\right)$ be a Pythagorean fuzzy number (PFN) over the universe of discourse Z. The *score function and accuracy function* of $\mathfrak{Y}$ are defined as follows:

$$\mathbb{S}(\mathfrak{Y}) = \mu^2_{\mathfrak{Y}} - \lambda^2_{\mathfrak{Y}} \quad \text{and} \quad \mathbb{A}(\mathfrak{Y}) = \mu^2_{\mathfrak{Y}} + \lambda^2_{\mathfrak{Y}}, \tag{1}$$

where $\mathbb{S}(\mathfrak{Y})$ belongs to the closed interval $[-1, 1]$ and $\mathbb{A}(\mathfrak{Y})$ belongs to the closed interval $[0, 1]$.

For other terminologies and applications, the readers are referred to [5, 8, 13, 14, 17, 18, 19, 20, 21, 22, 36, 38, 41, 47, 48, 51].

3. COMPLEX PYTHAGOREAN FUZZY *N*-SOFT SET

Definition 3.1. Let Z be a universe of discourse and K be a set of parameters. Let $S \subseteq K$ and $D = \{0, 1, \dots, N-1\}$ be a set of ordered grades with $N \in \{2, 3, \dots\}$. A triplet $\mathfrak{Y} = (h, L, N)$ is called a complex *PFNSS* on Z if $L = (H, S, N)$ is NSS on Z and $h : S \to \mathcal{CPF}^{Z\times D}$, where $\mathcal{CPF}^{Z\times D}$ is the collection of all CPFSs over $Z \times D$. The CPFNSS(h, L, N) can be defined as follows:

$$\mathfrak{Y} = \left\{ \left\langle s_q, h\left(s_q\right)\right\rangle \middle| s_q \in S, h\left(s_q\right) \in \mathcal{CPF}^{Z\times D} \right\},$$

where $h\left(s_q\right) = \left\{ \left(\left(z_p, d_{pq}\right), \mu_{pq}\left(z_p, d_{pq}\right) e^{i\alpha_{pq}\left(z_p, d_{pq}\right)}, \lambda_{pq}\left(z_p, d_{pq}\right) e^{i\beta_{pq}\left(z_p, d_{pq}\right)}\right) \middle| \left(z_p, d_{pq}\right) \in Z \times D \right\}$ represents the CPFS over $Z \times D$. The amplitude terms $\mu_{pq}(z_p, d_{pq}), \lambda_{pq}(z_p, d_{pq}) \in [0, 1]$ and phase returns $\alpha_{pq}(z_p, d_{pq}), \beta_{pq}(z_p, d_{pq}) \in [0, 2\pi]$ satisfy the conditions

$$0 \le \mu^2_{pq}\left(z_p, d_{pq}\right) + \lambda^2_{pq}\left(z_p, d_{pq}\right) \le 1,$$

$$0 \le \left(\frac{\alpha_{pq}\left(z_p, d_{pq}\right)}{2\pi}\right)^2 + \left(\frac{\beta_{pq}\left(z_p, d_{pq}\right)}{2\pi}\right)^2 \le 1,$$

where $i = \sqrt{-1}$. For all $(z_p, d_{pq}) \in Z \times D$, the degree of indeterminacy is represented by $\chi_{pq}\left(z_p, d_{pq}\right) = \sqrt{1 - \mu^2_{pq}\left(z_p, d_{pq}\right) - \lambda^2_{pq}\left(z_p, d_{pq}\right)}\, e^{i2\pi\sqrt{1-\left(\frac{\alpha_{pq}\left(z_p, d_{pq}\right)}{2\pi}\right)^2-\left(\frac{\beta_{pq}\left(z_p, d_{pq}\right)}{2\pi}\right)^2}}$.

In other words, CPFNSS is a parameterized family of CPFSs of $Z \times D$, that is, for each parameter $s_q \in S$, we can interpret that $h(s_q)$ is the CPFS of $Z \times D$.

Now, we give tabular representation of complex PFNSS:

Let Z be a set of r objects and S be a set of w attributes, then tabular representation of CPFNSS is given in Table 1.

Table 1 Tabular representation of CPFNSS.

(h, L, N)	s_1	s_2	...	s^{ω}
z_1	$\langle d_{11}, (\mu_{11} e^{i\alpha\, 11}, \lambda_{11} e^{i\beta\, 11})\rangle$	$\langle d_{12}, (\mu_{12} e^{i\alpha\, 12}, \lambda_{12} e^{i\beta\, 12})\rangle$	...	$\langle d_{1\omega}, (\mu_{1\omega} e^{i\alpha\, 1\omega}, \lambda_{1\omega} e^{i\beta\, 1\omega})\rangle$
z_2	$\langle d_{21}, (\mu_{21} e^{i\alpha\, 21}, \lambda_{21} e^{i\beta\, 21})\rangle$	$\langle d_{22}, (\mu_{22} e^{i\alpha\, 22}, \lambda_{22} e^{i\beta\, 22})\rangle$	...	$\langle d_{2\omega}, (\mu_{2\omega} e^{i\alpha\, 2\omega}, \lambda_{2\omega} e^{i\beta\, 2\omega})\rangle$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
z_r	$\langle d_{r1}, (\mu_{r1} e^{i\alpha\, r1}, \lambda_{r1} e^{i\beta\, r1})\rangle$	$\langle d_{r2}, (\mu_{r2} e^{i\alpha\, r2}, \lambda_{r2} e^{i\beta\, r2})\rangle$	...	$\langle d_{r\omega}, (\mu_{r\omega} e^{i\alpha\, r\omega}, \lambda_{r\omega} e^{i\beta\, r\omega})\rangle$

Definition 3.2. CPFNSS can be regarded as $r \times w$ table, where $r = |Z|$, $w = |S|$ whose *pqth* element is called a *complex Pythagorean fuzzy N-soft value* (CPFNSV) and it has the form, $\tau_{pq} = \langle d_{pq}, (\mu_{pq} e^{i\alpha pq}, \lambda_{pq}{}^{i\beta pq})\rangle$, where $p = 1, 2, \ldots, r$; $q = 1, 2, \ldots, w$.

Definition 3.3. The *score function* for CPFNSV, $\tau_{pq} = \langle d_{pq}, (\mu_{pq} e^{i\alpha pq}, \lambda e^{i\beta pq})\rangle$ is represented by $\mathfrak{S}$ and defined as follows:

$$\mathfrak{S}\left(\tau_{pq}\right) = \left(\frac{d_{pq}}{N-1}\right)^2 + \left(\mu_{pq}^2 - \lambda_{pq}^2\right) + \frac{1}{4\pi^2}\left(\alpha_{pq}^2 - \beta_{pq}^2\right), \tag{2}$$

where $\mathfrak{S}\left(\tau_{pq}\right) \in [-2, 3]$.

Definition 3.4. The *accuracy function* for CPFNSV, $\tau_{pq} = \langle d_{pq}, (\mu_{pq} e^{i\alpha pq}, \lambda_{pq}{}^{i\beta pq})\rangle$ is represented by $\mathfrak{A}$ and defined as follows:

$$\mathfrak{A}\left(\tau_{pq}\right) = \left(\frac{d_{pq}}{N-1}\right)^2 + \left(\mu_{pq}^2 + \lambda_{pq}^2\right) + \frac{1}{4\pi^2}\left(\alpha_{pq}^2 + \beta_{pq}^2\right), \tag{3}$$

where $\mathfrak{A}(\tau_{pq}) \in [0, 3]$.

Definition 3.5. Let $\tau_{11} = \langle d_{11}, (\mu_{11} e^{i\alpha 11}, \lambda_{11}{}^{i\beta 11})\rangle$ and $\tau_{12} = \langle d_{12}, (\mu_{12} e^{i\alpha 12}, \lambda_{12}{}^{i\beta 12})\rangle$ be any two CPFNSVs, then the comparison between τ_{11} and τ_{12} is defined as follows:

(1) If $\mathfrak{S}\left(\tau_{11}\right) < \mathfrak{S}\left(\tau_{12}\right)$, then$\tau_{11} \prec \tau_{12}$ (τ_{11} is inferior to τ_{12});

(2) If $\mathfrak{S}\left(\tau_{11}\right) > \mathfrak{S}\left(\tau_{12}\right)$, then $\tau_{11} \succ \tau_{12}$ (τ_{11} is superior to τ_{12});

(3) If $\mathfrak{S}\left(\tau_{11}\right) = \mathfrak{S}\left(\tau_{12}\right)$, then
- If $\mathfrak{A}\left(\tau_{11}\right) < \mathfrak{A}\left(\tau_{12}\right)$, then $\tau_{11} \prec \tau_{12}$ (τ_{11} is inferior to τ_{12});
- If $\mathfrak{A}\left(\tau_{11}\right) > \mathfrak{A}\left(\tau_{12}\right)$, then $\tau_{11} \succ \tau_{12}$ (τ_{11} is superior to τ_{12});
- If $\mathfrak{A}\left(\tau_{11}\right) = \mathfrak{A}\left(\tau_{12}\right)$, then $\tau_{11} \sim \tau_{12}$ (τ_{11} is equivalent to τ_{12}).

The significance and motivation of this proposed model is illustrated by the following numerical example. In order to develop intuition for this new concept, this example has been kept relatively simple. This simple example elaborates the difficulties of the existing Pythagorean fuzzy N-soft model, proposed by Zhang *et al.* [54] and demonstrates the competency of the proposed model in real-world DM problems.

Example 3.6.

A company wants to invest with another company to increase income. The selection of that company is determined by star rankings and ratings awarded by the experts of the selection panel. These ratings are based on the performance of companies in the last 12 months. Let $Z = \{z_1, z_2, z_3, z_4\}$ be a collection of four companies and $S = \{s_1$ = growth rate, s_2 = export, s_3 = interest rates$\}$ be a set of attributes which are used to assign ratings to companies.

A 5-soft set can be identified from Table 2, where

- Four stars represent “Excellent,”
- Three stars represent “Very Good,”
- Two stars represent “Good,”
- One star represents “Normal,”
- Big dot represents “Poor.”

The numbers $D = \{0, 1, 2, 3, 4\}$ can easily be associated with the graded evaluation carried out by stars as follows:

- 0 stands for “ • ,”
- 1 stands for “ ⋆, ”
- 2 stands for “ ⋆ ⋆,”
- 3 stands for “ ⋆ ⋆ ⋆ ,”
- 4 stands for “ ⋆ ⋆ ⋆ ⋆.”

The rating of companies provided by the selection panel is given in Table 2.

Now, the tabular form of its corresponding 5-soft set is presented in Table 3.

The selection panel thoroughly analyzes the companies to determine their rankings based on membership (μ) and nonmembership degrees (λ). So, we obtain Pythagorean fuzzy 5-soft set PF5SS, with the grading criteria followed by score degree of Pythagorean fuzzy number $\mathbb{S}\left(\mathfrak{Y}_{pq}\right)$, where $\mathfrak{Y}_{pq} = \left(\mu_{pq}, \lambda_{pq}\right)$, $p = 1, 2, 3, 4$, $q = 1, 2, 3$. This grading criteria is defined as follows:

$$\begin{aligned} -1.0 \leq \mathbb{S}\left(\mathfrak{Y}_{pq}\right) &< -0.6 \text{ when } d_{pq} = 0, \\ -0.6 \leq \mathbb{S}\left(\mathfrak{Y}_{pq}\right) &< -0.2 \text{ when } d_{pq} = 1, \\ -0.2 \leq \mathbb{S}\left(\mathfrak{Y}_{pq}\right) &< 0.2 \quad \text{ when } d_{pq} = 2, \\ 0.2 \leq \mathbb{S}\left(\mathfrak{Y}_{pq}\right) &< 0.6 \quad \text{ when } d_{pq} = 3, \\ 0.6 \leq \mathbb{S}\left(\mathfrak{Y}_{pq}\right) &\leq 1.0 \quad \text{ when } d_{pq} = 4. \end{aligned}$$

According to above criteria, we can obtain Table 4 of grading criteria.

Then, PF5SS is given as follows:

$$\left(h_{\mathfrak{P}}, L, 5\right) = \left\{\left(s_1, h\left(s_1\right)\right), \left(s_2, h\left(s_2\right)\right), \left(s_3, h\left(s_3\right)\right)\right\}, \text{ where}$$

$$\begin{aligned} h\left(s_1\right) &= \left\{\left\langle\left(z_1, 3\right), 0.7, 0.3\right\rangle, \left\langle\left(z_2, 0\right), 0.1, 0.9\right\rangle, \left\langle\left(z_3, 1\right), 0.3, 0.7\right\rangle, \left\langle\left(z_4, 4\right), 0.9, 0.1\right\rangle\right\}, \\ h\left(s_2\right) &= \left\{\left\langle\left(z_1, 4\right), 0.9, 0.1\right\rangle, \left\langle\left(z_2, 3\right), 0.6, 0.4\right\rangle, \left\langle\left(z_3, 0\right), 0.1, 0.9\right\rangle, \left\langle\left(z_4, 2\right), 0.4, 0.6\right\rangle\right\}, \\ h\left(s_3\right) &= \left\{\left\langle\left(z_1, 0\right), 0.1, 0.9\right\rangle, \left\langle\left(z_2, 1\right), 0.3, 0.7\right\rangle, \left\langle\left(z_3, 4\right), 0.8, 0.2\right\rangle, \left\langle\left(z_4, 3\right), 0.7, 0.3\right\rangle\right\}. \end{aligned}$$

The tabular form of PF5SS is presented in Table 5.

The experts of the selection panel evaluate all the companies and assign membership and non-membership degrees $\mu(z, d)$, $\lambda(z, d)$, respectively, to each company. Now, suppose an expert observes that "The growth rate of a company z_1 is high in the first 8 months but slightly declines in the last 4 months." Then, the values of $\mu(z_1, 3) = 0.7$, and $\lambda(z_1, 3) = 0.3$ are ambiguous and all the information regarding the time frame of reference would be lost. Thus $\mu(z_1, 3)$, $\lambda(z_1, 3)$ can be assigned by complex values which incorporate all of the information provided by the expert. Hence, we establish complex Pythagorean fuzzy 5-soft set (CPF5SS) instead of Pythagorean fuzzy 5-soft set for the evaluation of such difficulty.

Table 2 Rating of companies.

Z/S	s_1	s_2	s_3
z_1	⋆⋆⋆	⋆⋆⋆⋆	•
z_2	•	⋆⋆⋆	⋆
z_3	⋆	•	⋆⋆⋆⋆
z_4	⋆⋆⋆⋆	⋆⋆	⋆⋆⋆

Table 3 5-soft set.

$(H, S, 5)$	s_1	s_2	s_3
z_1	3	4	0
z_2	0	3	1
z_3	1	0	4
z_4	4	2	3

Table 4 Grading criteria.

D	μ_{pq}	λ_{pq}
0	[0, 0.2)	(0.8, 1]
1	[0.2, 0.4)	(0.6, 0.8]
2	[0.4, 0.6)	(0.4, 0.6]
3	[0.6, 0.8)	(0.2, 0.4]
4	[0.8, 1]	[0, 0.2]

Now, the following value may, attributed to $\mu(z_1, 3)$ and $\lambda(z_1, 3)$:

$$\mu\left(z_1, 3\right) = 0.7e^{i\frac{8}{12}\cdot 2\pi} = 0.7e^{i1.3\pi},$$
$$\lambda\left(z_1, 3\right) = 0.3e^{i\frac{4}{12}\cdot 2\pi} = 0.3e^{i0.6\pi}.$$

Here, phase term represents the information regarding the time frame of reference under consideration. Therefore, CPF5SS $(h, L, 5)$ is introduced by integrating CPFS with the 5-soft set for the two-dimensional graded evaluation of companies. Now, we redefine the grading criteria followed by score degree of CPF5SVs $\mathfrak{S}\left(\tau_{pq}\right)$, where $\tau_{pq} = \langle d_{pq}, (\mu_{pq}e^{i\alpha pq}, \lambda_{pq}e^{i\beta pq})\rangle$, $p = 1, 2, 3, 4$, $q = 1, 2, 3$ as follows:

$$\begin{aligned} -2.0 \le \mathfrak{S}\left(\tau_{pq}\right) &< -1.0 \quad \text{when } d_{pq} = 0,\\ -1.0 \le \mathfrak{S}\left(\tau_{pq}\right) &< 0.0 \quad \text{when } d_{pq} = 1,\\ 0.0 \le \mathfrak{S}\left(\tau_{pq}\right) &< 1.0 \quad \text{when } d_{pq} = 2,\\ 1.0 \le \mathfrak{S}\left(\tau_{pq}\right) &< 2.0 \quad \text{when } d_{pq} = 3,\\ 2.0 \le \mathfrak{S}\left(\tau_{pq}\right) &\le 3.0 \quad \text{when } d_{pq} = 4. \end{aligned}$$

According to above criteria, we can obtain Table 6 of corresponding redefined grading criteria.

Finally, CPF5SS is defined as follows:

$$(h, L, 5) = \left\{\left(s_1, h\left(s_1\right)\right), \left(s_2, h\left(s_2\right)\right), \left(s_3, h\left(s_3\right)\right)\right\}, \text{ where}$$

$$h\left(s_1\right) = \left\{\left\langle\left(z_1, 3\right), 0.7e^{i1.3\pi}, 0.3e^{i0.6\pi}\right\rangle, \left\langle\left(z_2, 0\right), 0.1e^{i0.1\pi}, 0.9e^{i1.9\pi}\right\rangle, \left\langle\left(z_3, 1\right), 0.3e^{i0.7\pi}, 0.7e^{i1.4\pi}\right\rangle, \left\langle\left(z_4, 4\right), 0.9e^{i1.7\pi}, 0.1e^{i0.3\pi}\right\rangle\right\},$$

$$h\left(s_2\right) = \left\{\left\langle\left(z_1, 4\right), 0.9e^{i1.7\pi}, 0.1e^{i0.3\pi}\right\rangle, \left\langle\left(z_2, 3\right), 0.6e^{i1.2\pi}, 0.4e^{i0.8\pi}\right\rangle, \left\langle\left(z_3, 0\right), 0.1e^{i0.3\pi}, 0.9e^{i1.8\pi}\right\rangle, \left\langle\left(z_4, 2\right), 0.4e^{i0.8\pi}, 0.6e^{i1.2\pi}\right\rangle\right\},$$

$$h\left(s_3\right) = \left\{\left\langle\left(z_1, 0\right), 0.1e^{i0.2\pi}, 0.9e^{i1.9\pi}\right\rangle, \left\langle\left(z_2, 1\right), 0.3e^{i0.5\pi}, 0.7e^{i1.3\pi}\right\rangle, \left\langle\left(z_3, 4\right), 0.8e^{i1.6\pi}, 0.2e^{i0.4\pi}\right\rangle, \left\langle\left(z_4, 3\right), 0.7e^{i1.3\pi}, 0.3e^{i0.7\pi}\right\rangle\right\}.$$

Clearly, CPF5SS can be represented in tabular form by Table 7.

Table 5 Pythagorean fuzzy 5-soft set.

$(h_{\mathfrak{P}}, L, 5)$	s_1	s_2	o_3
z_1	⟨3, (0.7, 0.3)⟩	⟨4, (0.9, 0.1)⟩	⟨0, (0.1, 0.9)⟩
z_2	⟨0, (0.1, 0.9)⟩	⟨3, (0.6, 0.4)⟩	⟨1, (0.3, 0.7)⟩
z_3	⟨1, (0.3, 0.7)⟩	⟨0, (0.1, 0.9)⟩	⟨4, (0.8, 0.2)⟩
z_4	⟨4, (0.9, 0.1)⟩	⟨2, (0.4, 0.6)⟩	⟨3, (0.7, 0.3)⟩

Table 6 Redefined grading criteria.

Grades	**Amplitude Terms**		**Phase Terms**	
D	μ_{pq}	λ_{pq}	α_{pq}	β_{pq}
0	[0, 0.2)	(0.8, 1]	$[0\pi, 0.4\pi)$	$(1.6\pi, 2\pi]$
1	[0.2, 0.4)	(0.6, 0.8]	$[0.4\pi, 0.8\pi)$	$(1.2\pi, 1.6\pi]$
2	[0.4, 0.6)	(0.4, 0.6]	$[0.8\pi, 1.2\pi)$	$(0.8\pi, 1.2\pi]$
3	[0.6, 0.8)	(0.2, 0.4]	$[1.2\pi, 1.6\pi)$	$(0.4\pi, 0.8\pi]$
4	[0.8, 1]	[0, 0.2]	$[1.6\pi, 2\pi]$	$[0\pi, 0.4\pi]$

Table 7 Tabular representation of CPF5SS.

$(h, L, 5)$	s_1	s_2	s_3
z_1	$\langle 3, (0.7e^{i1.3\pi}, 0.3e^{i0.6\pi})\rangle$	$\langle 4, (0.9e^{i1.7\pi}, 0.1e^{i0.3\pi})\rangle$	$\langle 0, (0.1e^{i0.2\pi}, 0.9e^{i1.9\pi})\rangle$
z_2	$\langle 0, (0.1e^{i0.1\pi}, 0.9e^{i1.9\pi})\rangle$	$\langle 3, (0.6e^{i1.2\pi}, 0.4e^{i0.8\pi})\rangle$	$\langle 1, (0.3e^{i0.5\pi}, 0.7e^{i1.3\pi})\rangle$
z_3	$\langle 1, (0.3e^{i0.7\pi}, 0.7e^{i1.4\pi})\rangle$	$\langle 0, (0.1e^{i0.3\pi}, 0.9e^{i1.8\pi})\rangle$	$\langle 4, (0.8e^{i1.6\pi}, 0.2e^{i0.4\pi})\rangle$
z_4	$\langle 4, (0.9e^{i1.7\pi}, 0.1e^{i0.3\pi})\rangle$	$\langle 2, (0.4e^{i0.8\pi}, 0.6e^{i1.2\pi})\rangle$	$\langle 3, (0.7e^{i1.3\pi}, 0.3e^{i0.7\pi})\rangle$

Remark 3.7.

(1) Any CPF2SS $(h, L, 2)$ can be naturally identified with CPFSS. We associate CPF2SS $h : S \to 2^{Z\times\{0,1\}}$ with CPFSS $h' : S \to CPF(Z)$, which is defined by

$$h'\left(s_q\right) = \left\{ \left(z_p, \mu_{pq}\left(z_p\right) e^{i\alpha_{pq}(z_p)}, \lambda_{pq}\left(z_p\right) e^{i\beta_{pq}(z_p)}\right) \middle| \left\langle \left(\left(z_p, 1\right), \mu_{pq}\left(z_p, 1\right) e^{i\alpha_{pq}(z_p,1)}, \lambda_{pq}\left(z_p, 1\right) e^{i\beta_{pq}(z_p,1)} \right) \right\rangle \in h\left(s_q\right) \right\},$$

for every $s_q \in S$.

(2) A CPFNSS (h, L, N) on a nonempty set Z is said to be *efficient* if $\langle(z_p, N-1), \mu(z_p, N-1)e^{i\alpha(zp,N-1)}, \lambda(z_p, N-1)e^{i\beta(z_p, N-1)}\rangle \in h(s_q)$, for some $s_q \in S$, $z_p \in Z$.

(3) Grade $0 \in D$ in Definition 3.1, represents the lowest score. It does not mean that there is incomplete information or lack of assessment.

We now explore the conception of complementarity of Complex PFNSSs.

Definition 3.8. Let (h, L, N) be a CPFNSS over the universe Z, where $L = (H, S, N)$ is the NSS on Z. Then, its *Complex Pythagorean fuzzy complement* (h^c, L, N) is defined as

$$(h^c, L, N) = \left\{ \left\langle s_q, h^c\left(s_q\right)\right\rangle \middle| s_q \in S, h^c\left(s_q\right) \in \mathcal{CPF}^{Z\times D} \right\}, \ \textit{where}$$

$$h^c(s_q) = \left\{ \left\langle \left(z_p, d_{pq}\right), \lambda\left(z_p, d_{pq}\right) e^{i\beta(z_p,d_{pq})}, \mu\left(z_p, d_{pq}\right) e^{i\alpha(z_p,d_{pq})} \right\rangle \middle| \left(z_p, d_{pq}\right) \in Z \times D \right\}.$$

Definition 3.9. Let (h, L, N) be a CPFNSS over the universe Z, where $L = (H, S, N)$ be the NSS on Z. Then, its *weak complex Pythagorean fuzzy complement* (h^c, L^c, N) can be interpreted as

$$(h^c, L^c, N) = \left\{ \left\langle s_q, h^c\left(s_q\right)\right\rangle \middle| s_q \in S, h^c\left(s_q\right) \in \mathcal{CPF}^{Z\times D} \right\},$$

where $h^c\left(s_q\right) = \left\{ \left\langle \left(z_p, d_{pq}\right), \lambda\left(z_p, d_{pq}\right) e^{i\beta(z_p,d_{pq})}, \mu\left(z_p, d_{pq}\right) e^{i\alpha(z_p,d_{pq})} \right\rangle \middle| \left(z_p, d_{pq}\right) \in Z \times D \right\}$ and for all $s_q \in S$, $H^c(s_q) \cap H(s_q) = \phi$.

Example 3.10.

Consider the CPF5SS as defined in Example 3.6. Then, its complex Pythagorean fuzzy complement $(h^c, L, 5)$ and weak complex Pythagorean fuzzy complement $(h^c, L^c, 5)$ are given by Tables 8 and 9, respectively.

Table 8 Complex Pythagorean fuzzy complement.

$(h^c, L, 5)$	s_1	s_2	s_3
z_1	$\langle 3, (0.3e^{i0.7\pi}, 0.7e^{i1.3\pi})\rangle$	$\langle 4, (0.1e^{i0.3\pi}, 0.9e^{i1.7\pi})\rangle$	$\langle 0, (0.9e^{i1.9\pi}, 0.1e^{i0.2\pi})\rangle$
z_2	$\langle 0, (0.9e^{i1.9e\pi}, 0.1e^{i0.1\pi})\rangle$	$\langle 3, (0.4e^{i0.8\pi}, 0.6e^{i1.2\pi})\rangle$	$\langle 1, (0.7e^{i1.3e\pi}, 0.3e^{i0.5\pi})\rangle$
z_3	$\langle 1, (0.7e^{i1.4\pi}, 0.3e^{i0.7\pi})\rangle$	$\langle 0, (0.9e^{i1.8\pi}, 0.1e^{i0.3\pi})\rangle$	$\langle 4, (0.2e^{i0.4\pi}, 0.8e^{i1.6\pi})\rangle$
z_4	$\langle 4, (0.1e^{i0.3\pi}, 0.9e^{i1.7\pi})\rangle$	$\langle 2, (0.6e^{i1.2\pi}, 0.4e^{i0.8\pi})\rangle$	$\langle 3, (0.3e^{i0.7\pi}, 0.7e^{i1.3\pi})\rangle$

Table 9 Weak complex Pythagorean fuzzy complement.

$(h^c, L^c, 5)$	s_1	s_2	s_3
z_1	$\langle 1, (0.3e^{i0.7\pi}, 0.7e^{i1.3\pi})\rangle$	$\langle 2, (0.1e^{i0.3\pi}, 0.9e^{i1.7\pi})\rangle$	$\langle 1, (0.9e^{i1.9\pi}, 0.1e^{i0.2\pi})\rangle$
z_2	$\langle 4, (0.9e^{i1.9\pi}, 0.1e^{i0.1\pi})\rangle$	$\langle 4, (0.4e^{i0.8\pi}, 0.6e^{i1.2\pi})\rangle$	$\langle 3, (0.7e^{i1.3\pi}, 0.3e^{i0.5\pi})\rangle$
z_3	$\langle 0, (0.7e^{i1.4\pi}, 0.3e^{i0.7\pi})\rangle$	$\langle 1, (0.9e^{i1.8\pi}, 0.1e^{i0.3\pi})\rangle$	$\langle 2, (0.2e^{i0.4\pi}, 0.8e^{i1.6\pi})\rangle$
z_4	$\langle 2, (0.1e^{i0.3\pi}, 0.9e^{i1.7\pi})\rangle$	$\langle 3, (0.6e^{i1.2\pi}, 0.4e^{i0.8\pi})\rangle$	$\langle 4, (0.3e^{i0.7\pi}, 0.7e^{i1.3\pi})\rangle$

Definition 3.11. Let (h, L, N) be a CPFNSS over the universe Z, where $L = (H, S, N)$ is the NSS on Z. Then, its *top complex Pythagorean fuzzy weak complement*, denoted by (h^t, L^t, N) is defined as

$$\left(h^t, L^t, N\right) = \begin{cases} h^t\left(s_q\right) = \left\langle\left(z_p, N-1\right), \lambda\left(z_p, d_{pq}\right) e^{i\beta\left(z_p, d_{pq}\right)}, \mu\left(z_p, d_{pq}\right) e^{i\alpha\left(z_p, d_{pq}\right)}\right\rangle, & \text{if } d_{pq} < N-1 \\ h^t\left(s_q\right) = \left\langle\left(z_p, 0\right), \lambda\left(z_p, d_{pq}\right) e^{i\beta\left(z_p, d_{pq}\right)}, \mu\left(z_p, d_{pq}\right) e^{i\alpha\left(z_p, d_{pq}\right)}\right\rangle, & \text{if } d_{pq} = N-1 \end{cases} \tag{4}$$

Definition 3.12. Let (h, L, N) be a CPFNSS over the universe Z, where $L = (H, S, N)$ is the NSS on Z. Then, its *bottom complex Pythagorean fuzzy weak complement*, denoted by (h^b, L^b, N) is defined as

$$\left(h^b, L^b, N\right) = \begin{cases} h^b\left(s_q\right) = \left\langle\left(z_p, 0\right), \lambda\left(z_p, d_{pq}\right) e^{i\beta\left(z_p, d_{pq}\right)}, \mu\left(z_p, d_{pq}\right) e^{i\alpha\left(z_p, d_{pq}\right)}\right\rangle, & \text{if } d_{pq} > 0, \\ h^b\left(s_q\right) = \left\langle\left(z_p, N-1\right), \lambda\left(z_p, d_{pq}\right) e^{i\beta\left(z_p, d_{pq}\right)}, \mu\left(z_p, d_{pq}\right) e^{i\alpha\left(z_p, d_{pq}\right)}\right\rangle, & \text{if } d_{pq} = 0. \end{cases} \tag{5}$$

Example 3.13.

Consider the CPF5SS as defined in Example 3.6. Then, its top complex Pythagorean fuzzy weak complement $(h^t, L^t, 5)$ and bottom complex Pythagorean fuzzy weak complement $(h^b, L^b, 5)$ are given by Tables 10 and 11, respectively.

Definition 3.14. Let (h_1, L_1, N_1) and (h_2, L_2, N_2) be two CPFNSSs over the universe Z, where $L_1 = (H_1, A, N_1)$ and $L_2 = (H_2, B, N_2)$ are NSSs on Z. Then, their *restricted intersection* denoted by $(h_1, L_1, N_1) \cap_R (h_2, L_2, N_2) = (g, L_1 \cap_R L_2, \min(N_1, N_2))$, where $L_1 \cap_R L_2 = (G, A \cap B, \min(N_1, N_2))$ is defined as

$\forall s_q \in A \cap B, z_p \in Z, \left\langle\left(z_p, d_{pq}\right), x, y\right\rangle \in g\left(s_q\right) \Leftrightarrow d_{pq} = \min\left(d^1_{pq}, d^2_{pq}\right), x = \min\left(\mu_C\left(z_p, d^1_{pq}\right), \mu_D\left(z_p, d^2_{pq}\right)\right), e^{i\min\left(\alpha_C\left(z_p, d^1_{pq}\right), \alpha_D\left(z_p, d^2_{pq}\right)\right)}, y = \max\left(\lambda_C\left(z_p, d^1_{pq}\right), \lambda_D\left(z_p, d^2_{pq}\right)\right) e^{i\max\left(\beta_C\left(z_p, d^1_{pq}\right), \beta_D\left(z_p, d^2_{pq}\right)\right)}$, if $\left\langle\left(z_p, d^1_{pq}\right), \mu_C\left(z_p, d^1_{pq}\right) e^{i\alpha_C\left(z_p, d^1_{pq}\right)}, \lambda_C\left(z_p, d^1_{pq}\right) e^{i\beta_C\left(z_p, d^1_{pq}\right)}\right\rangle \in h_1\left(s_q\right)$ and $\left\langle\left(z_p, d^2_{pq}\right), \mu_D\left(z_p, d^2_{pq}\right) e^{i\alpha_D\left(z_p, d^2_{pq}\right)}, \lambda_D\left(z_p, d^2_{pq}\right) e^{i\beta_D\left(z_p, d^2_{pq}\right)}\right\rangle \in h_2\left(s_q\right)$, C and D are CPFSs on $H_1(s_q)$ and $H_2(s_q)$, respectively.

Definition 3.15. Let (h_1, L_1, N_1) and (h_2, L_2, N_2) be two CPFNSSs over the universe Z, where $L_1 = (H_1, A, N_1)$ and $L_2 = (H_2, B, N_2)$ are NSSs on Z. Then, their extended intersection denoted by $(h_1, L_1, N_1) \cap_\varepsilon (h_2, L_2, N_2) = (f, L_1 \cap_\varepsilon L_2, \max(N_1, N_2))$, where $L_1 \cap_E L_2 = (F, A \cup B, \max(N_1, N_2))$ is defined as

$$f\left(s_q\right) = \begin{cases} h_1\left(s_q\right), & \text{if } s_q \in A - B \\ h_2\left(s_q\right), & \text{if } s_q \in B - A \\ \left\langle\left(z_p, d_{pq}\right), x, y\right\rangle, & \text{such that } d_{pq} = \min\left(d^1_{pq}, d^2_{pq}\right) \\ & x = \min\left(\mu_C\left(z_p, d^1_{pq}\right), \mu_D\left(z_p, d^2_{pq}\right)\right) e^{i\min\left(\alpha_C\left(z_p, d^1_{pq}\right), \alpha_D\left(z_p, d^2_{pq}\right)\right)} \\ & y = \max\left(\lambda_C\left(z_p, d^1_{pq}\right), \lambda_D\left(z_p, d^2_{pq}\right)\right) e^{i}\max\left(\beta_C\left(z_p, d^1_{pq}\right), \beta_D\left(z_p, d^2_{pq}\right)\right) \\ & \text{where } \left\langle\left(z_p, d^1_{pq}\right), \mu_C\left(z_p, d^1_{pq}\right) e^{i\alpha_C\left(z_p, d^1_{pq}\right)}, \lambda_C\left(z_p, d^1_{pq}\right) e^{i\beta_C\left(z_p, d^1_{pq}\right)}\right\rangle \in h_1\left(s_q\right) \\ & \text{and } \left\langle\left(z_p, d^2_{pq}\right), \mu_D\left(z_p, d^2_{pq}\right) e^{i\alpha_D\left(z_p, d^2_{pq}\right)}, \lambda_D\left(z_p, d^2_{pq}\right) e^{i\beta_D\left(z_p, d^2_{pq}\right)}\right\rangle \in h_2\left(s_q\right), \\ & \text{C and D are CPFSs on } H_1\left(s_q\right) \text{ and } H_2\left(s_q\right), \text{ respectively.} \end{cases}$$

Table 10 Top complex Pythagorean fuzzy weak complement.

$(h^t, L^t, 5)$	s_1	s_2	s_3
z_1	$\langle 4, (0.3e^{i0.7\pi}, 0.7e^{i1.3\pi})\rangle$	$\langle 0, (0.1e^{i0.3\pi}, 0.9e^{i1.7\pi})\rangle$	$\langle 4, (0.9e^{i1.9\pi}, 0.1e^{i0.2\pi})\rangle$
z_2	$\langle 4, (0.9e^{i1.9\pi}, 0.1e^{i0.1\pi})\rangle$	$\langle 4, (0.4e^{i0.8\pi}, 0.6e^{i1.2\pi})\rangle$	$\langle 4, (0.7e^{i1.3\pi}, 0.3e^{i0.5\pi})\rangle$
z_3	$\langle 4, (0.7e^{i1.4\pi}, 0.3e^{i0.7\pi})\rangle$	$\langle 4, (0.9e^{i1.8\pi}, 0.1e^{i0.3\pi})\rangle$	$\langle 0, (0.2e^{i0.4\pi}, 0.8e^{i1.6\pi})\rangle$
z_4	$\langle 0, (0.1e^{i0.3\pi}, 0.9e^{i1.7\pi})\rangle$	$\langle 4, (0.6e^{i1.2\pi}, 0.4e^{i0.8\pi})\rangle$	$\langle 4, (0.3e^{i0.7\pi}, 0.7e^{i1.3\pi})\rangle$

Example 3.16.

Consider the tabular form of CPF6SS $(h_1, L_1, 6)$ and CPF4SS $(h_2, L_2, 4)$ as given in Tables 12 and 13, respectively, where $L_1 = (H_1, A, 6)$, $L_2 = (H_2, B, 4)$ are 6-soft set and 4-soft set over Z, respectively. Then, their restricted intersection $(g, L_1 \cap_R L_2, 4)$ and extended intersection $(f, L_1 \cap_\varepsilon L_2, 6)$ are presented in Tables 14 and 15, respectively.

Definition 3.17. Let (h_1, L_1, N_1) and (h_2, L_2, N_2) be two CPFNSSs over the universe Z, where $L_1 = (H_1, A, N_1)$ and $L_2 = (H_2, B, N_2)$ are NSSs on Z. Then, their *restricted union* denoted by $(h_1, L_1, N_1) \cup_R (h_2, L_2, N_2) = (w, L_1 \cup_R L_2, \max(N_1, N_2))$, where $L_1 \cup_R L_2 = (W, A \cap B, \max(N_1, N_2))$ is defined as

for all $s_q \in A \cap B$, $z_p \in Z$, $\langle (z_p, d_{pq}), x, y\rangle \in w(s_q) \Leftrightarrow d_{pq}$ $d_{pq} = \max\left(d^1_{pq}, d^2_{pq}\right)$, $x = \max\left(\mu_C\left(z_p, d^1_{pq}\right), \mu_D\left(z_p, d^2_{pq}\right)\right) e^{i\max\left(\alpha_C\left(z_p, d^1_{pq}\right), \alpha_D\left(z_p, d^2_{pq}\right)\right)}$, $y = \min\left(\lambda_C\left(z_p, d^1_{pq}\right), \lambda_D\left(z_p, d^2_{pq}\right)\right) e^{i\min\left(\beta_C\left(z_p, d^1_{pq}\right), \beta_D\left(z_p, d^2_{pq}\right)\right)}$, if $\left\langle \left(z_p, d^1_{pq}\right), \mu_C\left(z_p, d^1_{pq}\right) e^{i\alpha_C\left(z_p, d^1_{pq}\right)}, \lambda_C\left(z_p, d^1_{pq}\right) e^{i\beta_C\left(z_p, d^1_{pq}\right)} \right\rangle \in h_1\left(s_q\right)$ and $\left\langle \left(z_p, d^2_{pq}\right), \mu_D\left(z_p, d^2_{pq}\right) e^{i\alpha_D\left(z_p, d^2_{pq}\right)}, \lambda_D\left(z_p, d^2_{pq}\right) e^{i\beta_D\left(z_p, d^2_{pq}\right)} \right\rangle \in h_2\left(s_q\right)$ C and D are CPFSs on H_1(sq) and H_2(sq), respectively.

Table 11 Bottom complex Pythagorean fuzzy weak complement.

$(h^b, L^b, 5)$	s_1	s_2	s_3
z_1	$\langle 0, (0.3e^{i0.7\pi}, 0.7e^{i1.3\pi})\rangle$	$\langle 0, (0.1e^{i0.3\pi}, 0.9e^{i1.7\pi})\rangle$	$\langle 4, (0.9e^{i1.9\pi}, 0.1e^{i0.2\pi})\rangle$
z_2	$\langle 4, (0.9e^{i1.9\pi}, 0.1e^{i0.1\pi})\rangle$	$\langle 0, (0.4e^{i0.8\pi}, 0.6e^{i1.2\pi})\rangle$	$\langle 0, (0.7e^{i1.3\pi}, 0.3e^{i0.5\pi})\rangle$
z_3	$\langle 0, (0.7e^{i1.4\pi}, 0.3e^{i0.1\pi})\rangle$	$\langle 4, (0.9e^{i1.8\pi}, 0.1e^{i0.3\pi})\rangle$	$\langle 0, (0.2e^{i0.4\pi}, 0.8e^{i1.6\pi})\rangle$
z_4	$\langle 0, (0.1e^{i0.3\pi}, 0.9e^{i1.7\pi})\rangle$	$\langle 0, (0.6e^{i1.2\pi}, 0.4e^{i0.8\pi})\rangle$	$\langle 0, (0.3e^{i0.7\pi}, 0.7e^{i1.3\pi})\rangle$

Table 12 CPF6SS.

$(h_1, L_1, 6)$	s_1	s_2	s_3
z_1	$\langle 3, (0.59e^{i1.23\pi}, 0.42e^{i0.78\pi})\rangle$	$\langle 4, (0.76e^{i1.45\pi}, 0.32e^{i0.66\pi})\rangle$	$\langle 3, (0.61e^{i1.30\pi}, 0.48e^{i0.88\pi})\rangle$
z_2	$\langle 5, (0.95e^{i1.82\pi}, 0.14e^{i0.12\pi})\rangle$	$\langle 2, (0.43e^{i0.66\pi}, 0.64e^{i1.32\pi})\rangle$	$\langle 0, (0.15e^{i0.28\pi}, 0.85e^{i1.88\pi})\rangle$
z_3	$\langle 1, (0.26e^{i0.33\pi}, 0.82e^{i1.65\pi})\rangle$	$\langle 0, (0.12e^{i0.29\pi}, 0.92e^{i1.78\pi})\rangle$	$\langle 1, (0.28e^{i0.51\pi}, 0.81e^{i1.45\pi})\rangle$
z_4	$\langle 4, (0.73e^{i1.45\pi}, 0.32e^{i0.66\pi})\rangle$	$\langle 5, (0.85e^{i1.78\pi}, 0.12e^{i1.30\pi})\rangle$	$\langle 2, (0.41e^{i0.81\pi}, 0.62e^{i1.12\pi})\rangle$

Table 13 CPF6SS.

$(h_2, L_2, 4)$	s_1	s_2	s_4
z^1	$\langle 2, (0.65e^{i1.25\pi}, 0.41e^{i0.85\pi})\rangle$	$\langle 3, (0.85e^{i1.85\pi}, 0.35e^{i0.46\pi})\rangle$	$\langle 1, (0.35e^{i0.5\pi}, 0.61e^{i1.50\pi})\rangle$
z_2	$\langle 1, (0.41e^{i0.82\pi}, 0.71e^{i1.29\pi})\rangle$	$\langle 2, (0.55e^{i1.45\pi}, 0.15e^{i0.62\pi})\rangle$	$\langle 0, (0.21e^{i0.12\pi}, 0.85e^{i1.62\pi})\rangle$
z_3	$\langle 0, (0.91e^{i0.42\pi}, 0.82e^{i1.81\pi})\rangle$	$\langle 1, (0.45e^{i0.71\pi}, 0.65e^{i1.44\pi})\rangle$	$\langle 3, (0.92e^{i1.92\pi}, 0.12e^{i0.43\pi})\rangle$
z_4	$\langle 3, (0.81e^{i1.68\pi}, 0.21e^{i0.12\pi})\rangle$	$\langle 0, (0.14e^{i0.14\pi}, 0.98e^{i1.92\pi})\rangle$	$\langle 2, (0.71e^{i1.34\pi}, 0.41e^{i0.71\pi})\rangle$

Table 14 Restricted intersection.

$(g, L_1 \cap_R L_2, 4)$	s_1	s_2
z_1	$\langle 2, (0.59e^{i1.23\pi}, 0.42e^{i0.85\pi})\rangle$	$\langle 3, (0.76e^{i1.45\pi}, 0.35e^{i0.66\pi})\rangle$
z_2	$\langle 1, (0.41e^{i0.82\pi}, 0.71e^{i1.29\pi})\rangle$	$\langle 2, (0.43e^{i0.66\pi}, 0.64e^{i1.32\pi})\rangle$
z_3	$\langle 0, (0.91e^{i0.33\pi}, 0.82e^{i1.81\pi})\rangle$	$\langle 0, (0.12e^{i0.29\pi}, 0.92e^{i1.78\pi})\rangle$
z_4	$\langle 3, (0.73e^{i1.45\pi}, 0.32e^{i0.66\pi})\rangle$	$\langle 0, (0.14e^{i0.14\pi}, 0.98e^{i1.92\pi})\rangle$

Table 15 Extended intersection.

$(f, L_1 \cap_\varepsilon L_2, 6)$	s_1	s_2	s_3	s_4
z_1	$\langle 2, (0.59e^{i1.23\pi}, 0.42e^{i0.85\pi})\rangle$	$\langle 3, (0.76e^{i1.45\pi}, 0.35e^{i0.66\pi})\rangle$	$\langle 3, (0.61e^{i1.30\pi}, 0.48e^{i0.88\pi})\rangle$	$\langle 1, (0.35e^{i0.50\pi}, 0.61e^{i1.50\pi})\rangle$
z^2	$\langle 1, (0.41e^{i0.82\pi}, 0.71e^{i1.29\pi})\rangle$	$\langle 2, (0.43e^{i1.66\pi}, 0.64e^{i1.32\pi})\rangle$	$\langle 0, (0.15e^{i0.28\pi}, 0.85e^{i1.88\pi})\rangle$	$\langle 0, (0.21e^{i0.12\pi}, 0.85e^{i1.62\pi})\rangle$
z^3	$\langle 0, (0.19e^{i0.33\pi}, 0.82e^{i1.81\pi})\rangle$	$\langle 0, (0.12e^{i0.29\pi}, 0.92e^{i1.78\pi})\rangle$	$\langle 1, (0.28e^{i0.51\pi}, 0.81e^{i1.45\pi})\rangle$	$\langle 3, (0.92e^{i1.92\pi}, 0.12e^{i0.43\pi})\rangle$
z^4	$\langle 3, (0.73e^{i1.45\pi}, 0.32e^{i0.66\pi})\rangle$	$\langle 0, (0.14e^{i0.14\pi}, 0.98e^{i1.92\pi})\rangle$	$\langle 2, (0.41e^{i0.81\pi}, 0.62e^{i1.12\pi})\rangle$	$\langle 2, (0.71e^{i1.34\pi}, 0.41e^{i0.71\pi})\rangle$

Definition 3.18. Let (h_1, L_1, N_1) and (h_2, L_2, N_2) be two CPFNSSs over the universe Z, where $L_1 = (H_1, A, N_1)$ and $L_2 = (H_2, B, N_2)$ are NSSs on Z. Then, their extended union denoted by $(h_1, L_1, N_1) \cup_E (h_2, L_2, N_2) = (z, L_1 \cup_\varepsilon L_2, \max(N_1, N_2))$, where $L_1 \cup_\varepsilon L_2 = (Z, A \cup B, \max(N_1, N_2))$ is defined as

$$z\left(s_q\right) = \begin{cases} h_1\left(s_q\right), & \text{if } s_q \in A - B, \\ h_2\left(s_q\right), & \text{if } s_q \in B - A, \\ \left\langle \left(z_p, d_{pq}\right), x, y \right\rangle, & \text{such that } d_{pq} = \max\left(d^1_{pq}, d^2_{pq}\right), \\ & x = \max\left(\mu_C\left(z_p, d^1_{pq}\right), \mu_D\left(z_p, d^2_{pq}\right)\right) e^{i \max\left(\alpha_C\left(z_p, d^1_{pq}\right), \alpha_D\left(z_p, d^2_{pq}\right)\right)}, \\ & y = \min\left(\lambda_C\left(z_p, d^1_{pq}\right), \lambda_D\left(z_p, d^2_{pq}\right)\right) e^{i \min\left(\beta_C\left(z_p, d^1_{pq}\right), \beta_D\left(z_p, d^2_{pq}\right)\right)}, \\ & \text{where } \left\langle \left(z_p, d^1_{pq}\right), \mu_C\left(z_p, d^1_{pq}\right) e^{i\alpha_C\left(z_p, d^1_{pq}\right)}, \lambda_C\left(z_p, d^1_{pq}\right) e^{i\beta_C\left(z_p, d^1_{pq}\right)} \right\rangle \in h_1\left(s_q\right), \\ & \text{and } \left\langle \left(z_p, d^2_{pq}\right), \mu_D\left(z_p, d^2_{pq}\right) e^{i\alpha_D\left(z_p, d^2_{pq}\right)}, \lambda_D\left(z_p, d^2_{pq}\right) e^{i\beta_D\left(z_p, d^2_{pq}\right)} \right\rangle \in h_2\left(s_q\right), \\ & \text{C and D are CPFSs on } H_1\left(s_q\right) \text{ and } H_2\left(s_q\right), \text{ respectively.} \end{cases}$$

Example 3.19.

Consider the CPF6SS and CPF4SS as given in Example 3.16. Then, their restricted union $(w, L_1 \cup_R L_2, 6)$ and extended union $(z, L_1 \cup_\varepsilon L_2, 6)$ are presented in Tables 16 and 17, respectively.

Remark 3.20. The Definitions 3.14, 3.15, 3.17, and 3.18 of restricted(or extended) intersection and union will satisfy the grading criteria if and only if $N_1 = N_2$

The concept of CPFNSS can be related to extant theories including complex PFSS, PFNSS, N-soft set and soft set. We shall establish all these relationships in this section. In order to derive CPFSS and SS from CPFNSS, we use the following definitions:

Definition 3.21. Let (h, L, N) be a CPFNSS over a universe of discourse Z, where $L = (H, S, N)$ is the NSS on Z. Let $0 < T < N$ be a threshold. Then, CPFSS over Z associated with (h, L, N) and T denoted by (h^T, S) is defined as follows:

$$h^T\left(s_q\right) = \begin{cases} \left\langle \mu_{pq} e^{i\alpha_{pq}}, \lambda_{pq} e^{i\beta_{pq}} \right\rangle, & \textit{if } \left\langle d_{pq}, \left(\mu_{pq} e^{i\alpha_{pq}}, \lambda_{pq} e^{i\beta_{pq}}\right) \right\rangle \in h\left(s_q\right) \textit{ and } d_{pq} \geq T, \\ \begin{cases} \left\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi} \right\rangle, & \textit{if } \frac{d_{pq}}{N} \geq 0.5 \\ \left\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi} \right\rangle, & \textit{if } \frac{d_{pq}}{N} < 0.5. \end{cases} \end{cases}$$

Specifically, (h^1, S) is said to be bottom CPFSS and (h^{N-1}, S) is said to be top CPFSS associated with CPFNSS.

Definition 3.22. Let $0 < T < N$ and $\rho \in [-2, 2]$ be thresholds. The soft set over Z associated with (h, L, N) and (T, ρ), denoted by $(h^{(T,\rho)}, S)$ is defined by the assignment:

$$h^{(T,\rho)}\left(s_q\right) = \left\{z \in Z : \mathfrak{S}\left(h^T\left(s_q\right)\right) > \rho\right\}, \textit{ for each } s_q \in S$$

Table 16 Restricted union.

$(w, L_1 \cup_R L_2, 6)$	s_1	s_2
z_1	$\langle 3, (0.65e^{i1.25\pi}, 0.41e^{i0.78\pi})\rangle$	$\langle 4, (0.85e^{i1.85\pi}, 0.32e^{i0.46\pi})\rangle$
z_2	$\langle 5, (0.95e^{1.82\pi}, 0.14e^{i0.12\pi})\rangle$	$\langle 2, (0.55e^{1.45\pi}, 0.15e^{i0.62\pi})\rangle$
z_3	$\langle 1, (0.26e^{i0.42\pi}, 0.82e^{i1.65\pi})\rangle$	$\langle 1, (0.45e^{i0.71\pi}, 0.65e^{i1.44\pi})\rangle$
z_4	$\langle 4, (0.81e^{i1.68\pi}, 0.21e^{i0.12\pi})\rangle$	$\langle 5, (0.85e^{i1.78\pi}, 0.12e^{i0.30\pi})\rangle$

Table 17 Extended union.

$(z, L_1 \cup_\varepsilon L_2, 6)$	s_1	s_2	s_3	s_4
z_1	$\langle 3, (0.65e^{i1.25\pi}, 0.41e^{i0.78\pi})\rangle$	$\langle 4, (0.85e^{i1.85\pi}, 0.32e^{i0.46\pi})\rangle$	$\langle 3, (0.61e^{i1.30\pi}, 0.48e^{i0.88\pi})\rangle$	$\langle 1, (0.35e^{i0.50\pi}, 0.61e^{i1.50\pi})\rangle$
z_2	$\langle 5, (0.95e^{i1.82\pi}, 0.14e^{i0.12\pi})\rangle$	$\langle 2, (0.55e^{i1.45\pi}, 0.15e^{i0.62\pi})\rangle$	$\langle 0, (0.15e^{i0.28\pi}, 0.85e^{i1.88\pi})\rangle$	$\langle 0, (0.21e^{i0.12\pi}, 0.85e^{i1.62\pi})\rangle$
z_3	$\langle 1, (0.26e^{i0.42\pi}, 0.82e^{i1.65\pi})\rangle$	$\langle 1, (0.45e^{i0.71\pi}, 0.65e^{i1.44\pi})\rangle$	$\langle 1, (0.28e^{i0.51\pi}, 0.81e^{i1.45\pi})\rangle$	$\langle 3, (0.92e^{i1.92\pi}, 0.12e^{i0.43\pi})\rangle$
z_4	$\langle 4, (0.81e^{i1.68\pi}, 0.21e^{i0.12\pi})\rangle$	$\langle 5, (0.85e^{i1.78\pi}, 0.12e^{i0.30\pi})\rangle$	$\langle 2, (0.41e^{i0.81\pi}, 0.62e^{i1.12\pi})\rangle$	$\langle 2, (0.71e^{i1.34\pi}, 0.41e^{i0.71\pi})\rangle$

where $\mathfrak{S}\left(h^T\left(s_q\right)\right)$ is the score function of $h^T(s_q) = \left\langle \mu^T_{pq} e^{i\alpha^T_{pq}}, \lambda^T_{pq} e^{i\beta^T_{pq}} \right\rangle$.

Example 3.23.

Consider the CPF5SS $(h, L, 5)$, represented by Table 7. We have $0 < T < 5$, from Definition 3.21. The CPFSS associated with threshold $T = 2$ is given by Table 18. Meanwhile, while taking $(T, \rho) = (3, 0.2)$, we can obtain the soft set $(h^{(3,0.2)}, S)$ which is given in Table 19.

In view of above analysis, it is observed that CPFNSS can be converted into CPFSS and SS under certain conditions. In other words, CPFNSS is the generalization of CPFSS and SS.

Remark 3.24.

(1) Let Z be a universe of discourse, K be a set of parameters and $S \subseteq K$. Let (h, L, N) be a CPFNSS on Z. Then, the NSS associated with CPFNSS (h, L, N) is L, where $L = (H, S, N)$.
This simple assignment shows that CPFNSS generalizes NSS, therefore SS as well.

(2) Every complex intuitionistic fuzzy N-soft set (CIFNSS) is also a CPFNSS but converse is not true, because,

- If $0 \leq \mu + \lambda \leq 1$, then $0 \leq \mu^2 + \lambda^2 \leq 1$, for all $\mu, \lambda \in [0, 1]$.
- If $0 \leq \alpha + \beta \leq 2\pi$, then $0 \leq \left(\frac{\alpha}{2\pi}\right)^2 + \left(\frac{\beta}{2\pi}\right)^2 \leq 1$, for all $\alpha, \beta \in [0, 2\pi]$.

This implies that CPFNSS generalizes the CIFNSS.

(3) CPFNSS is also a PFNSS when both phase terms of membership and nonmembership degrees are zero, that is, when $\alpha_{pq} = 0 = \beta_{pq}$ in Definition 3.1, we get the definition of PFNSS which is given as follows:

$$(h, L, N) = \left\{\left\langle s_q, h\left(s_q\right)\right\rangle | s_q \in S, h\left(s_q\right) \in \mathcal{PFS}^{Z \times D}\right\},$$

where $h(s_q) = \{((z_q, d_{pq}), \mu_{pq}(z_q, d_{pq}), \lambda_{pq}(z_q, d_{pq})|(z_q, d_{pq}) \in Z \times D\}$ represents the PFS and $\mathcal{PFS}^{Z \times D}$ be the family of all PFSs over $Z \times D$ Also, it is known that PFNSSs generalize the IFNSSs, FNSSs, NSSs, SSs, PFSSs and IFSSs. Therefore, we can also claim that CPFNSSs generalize all those models.

4. OPERATIONS

We now present operations on complex Pythagorean fuzzy N-soft values.

Definition 4.1. Let $\tau_{1q} = \left\langle d_{1q}, \left(\mu_{1q} e^{i\alpha_{1q}}, \lambda_{1q} e^{i\beta_{1q}}\right)\right\rangle$ $(q = 1, 2)$, and $\tau = \langle d, (\mu e^{i\alpha}, \lambda e^{i\beta})\rangle$ be any three CPFNSVs and $\Omega > 0$ be any real number. Then, the following operations are defined over CPFNSVs as follows:

(1) $$\tau_{11} \oplus \tau_{12} = \left\langle \max\left(d_{11}, d_{12}\right), \left(\sqrt{\mu_{11}^2 + \mu_{12}^2 - \mu_{11}^2 \mu_{12}^2}\, e^{i2\pi\sqrt{\left(\frac{\alpha_{11}}{2\pi}\right)^2 + \left(\frac{\alpha_{12}}{2\pi}\right)^2 - \left(\frac{\alpha_{11}}{2\pi}\right)^2\left(\frac{\alpha_{12}}{2\pi}\right)^2}}, \lambda_{11}\lambda_{12} e^{i2\pi\left(\frac{\beta_{11}}{2\pi}\frac{\beta_{12}}{2\pi}\right)}\right)\right\rangle;$$

Table 18 CPFSS associated with $(h, L, 5)$ and threshold $T = 2$.

(h^2, S)	s_1	s_2	s_3
z_1	$\langle 0.7e^{i1.3\pi}, 0.3e^{i0.7\pi}\rangle$	$\langle 0.97e^{i1.7\pi}, 0.1e^{i0.3\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$
z_2	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.6e^{i0.0\pi}, 0.4e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$
z_3	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.8e^{i1.6\pi}, 0.2e^{i0.4\pi}\rangle$
z_4	$\langle 0.9e^{i1.7\pi}, 0.1e^{i0.3\pi}\rangle$	$\langle 0.4e^{i0.8\pi}, 0.6e^{i1.2\pi}\rangle$	$\langle 0.7e^{i1.3\pi}, 0.3e^{i0.7\pi}\rangle$

Table 19 SS associated with $(h, L, 5)$ and thresholds $T = 3$, $\rho = 0.2$.

$(h^{(3,0.2)}, S)$	s_1	s_2	s_3
z_1	1	1	0
z_2	0	1	0
z_3	0	0	1
z_4	1	0	1

(2) $\tau_{11} \otimes \tau_{12} = \left\langle \min\left(d_{11}, d_{12}\right), \left(\mu_{11}\mu_{12} e^{i2\pi\left(\frac{\alpha_{11}}{2\pi}\frac{\alpha_{12}}{2\pi}\right)}, \sqrt{\lambda_{11}^2 + \lambda_{12}^2 - \lambda_{11}^2\lambda_{12}^2}\, e^{i2\pi\sqrt{\left(\frac{\beta_{11}}{2\pi}\right)^2 + \left(\frac{\beta_{12}}{2\pi}\right)^2 - \left(\frac{\beta_{11}}{2\pi}\right)^2\left(\frac{\beta_{12}}{2\pi}\right)^2}} \right) \right\rangle;$

(3) $\Omega\tau = \left\langle d, \left(\sqrt{1-(1-\mu^2)^\Omega}\, e^{i2\pi\sqrt{1-\left(1-\left(\frac{\alpha}{2\pi}\right)^2\right)^\Omega}}, \lambda^\Omega e^{i2\pi\left(\frac{\beta}{2\pi}\right)^\Omega} \right) \right\rangle;$

(4) $\tau^\Omega = \left\langle d, \left(\mu^\Omega e^{i2\pi\left(\frac{\alpha}{2\pi}\right)^\Omega}, \sqrt{1-(1-\lambda^2)^\Omega}\, e^{i2\pi\sqrt{1-\left(1-\left(\frac{\beta}{2\pi}\right)^2\right)^\Omega}} \right) \right\rangle.$

Definition 4.2. Consider any three CPFNSVs, $\tau = \left\langle d, \left(\mu e^{i\alpha}, \lambda e^{i\beta}\right)\right\rangle, \tau_{1q} = \left\langle 1q, \left(\mu_{1q}e^{i\alpha_{1q}}, \lambda_{1q}e^{i\beta_{1q}}\right)\right\rangle$ $(q = 1, 2)$ and $\Omega > 0$ be any real number. Then, the following Einstein operations are defined over CPFNSVs as follows:

(1) $\tau_{11} \oplus_e \tau_{12} = \left\langle \max\left(d_{11}, d_{12}\right), \left(\sqrt{\frac{\mu_{11}^2+\mu_{12}^2}{1+\mu_{11}^2\mu_{12}^2}}\, e^{i2\pi\sqrt{\frac{\left(\frac{\alpha_{11}}{2\pi}\right)^2+\left(\frac{\alpha_{12}}{2\pi}\right)^2}{1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\left(\frac{\alpha_{12}}{2\pi}\right)^2}}}, \frac{\lambda_{11}\lambda_{12}}{\sqrt{1+(1-\lambda_{11}^2)(1-\lambda_{12}^2)}}\, e^{i2\pi\frac{\left(\frac{\beta_{11}}{2\pi}\right)\left(\frac{\beta_{12}}{2\pi}\right)}{\sqrt{1+\left(1-\left(\frac{\beta_{11}}{2\pi}\right)^2\right)\left(1-\left(\frac{\beta_{12}}{2\pi}\right)^2\right)}}} \right) \right\rangle;$

(2) $\tau_{11} \oplus_e \tau_{12} = \left\langle \min\left(d_{11}, d_{12}\right), \left(\frac{\mu_{11}\mu_{12}}{\sqrt{1+(1-\mu_{11}^2)(1-\mu_{12}^2)}}\, e^{i2\pi\frac{\left(\frac{\alpha_{11}}{2\pi}\right)\left(\frac{\alpha_{12}}{2\pi}\right)}{\sqrt{1+\left(1-\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)\left(1-\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)}}}, \sqrt{\frac{\lambda_{11}^2+\lambda_{12}^2}{1+\lambda_{11}^2\lambda_{12}^2}}\, e^{i2\pi\sqrt{\frac{\left(\frac{\beta_{11}}{2\pi}\right)^2+\left(\frac{\beta_{12}}{2\pi}\right)^2}{1+\left(\frac{\beta_{11}}{2\pi}\right)^2\left(\frac{\beta_{12}}{2\pi}\right)^2}}} \right) \right\rangle;$

(3) $\Omega \cdot e^{\tau} = \left\langle d, \left(\sqrt{\frac{(1+\mu^2)^\Omega-(1-\mu^2)^\Omega}{(1+\mu^2)^\Omega+(1-\mu^2)^\Omega}}\, e^{i2\pi\sqrt{\frac{\left(1+\left(\frac{\alpha}{2\pi}\right)^2\right)^\Omega-\left(1-\left(\frac{\alpha}{2\pi}\right)^2\right)^\Omega}{\left(1+\left(\frac{\alpha}{2\pi}\right)^2\right)^\Omega+\left(1-\left(\frac{\alpha}{2\pi}\right)^2\right)^\Omega}}}, \frac{\sqrt{2}\lambda^\Omega}{\sqrt{(2-\lambda^2)^\Omega+(\lambda^2)^\Omega}}\, e^{i2\pi\frac{\sqrt{2}\left(\frac{\beta}{2\pi}\right)^\Omega}{\sqrt{\left(2-\left(\frac{\beta}{2\pi}\right)^2\right)^\Omega+\left(\left(\frac{\beta}{2\pi}\right)^2\right)^\Omega}}} \right) \right\rangle;$

(4) $\tau^\Omega = \left\langle d, \left(\frac{\sqrt{2}\mu^\Omega}{\sqrt{(2-\mu^2)^\Omega+(\mu^2)^\Omega}}\, e^{i2\pi\frac{\sqrt{2}\left(\frac{\alpha}{2\pi}\right)^\Omega}{\sqrt{\left(2-\left(\frac{\alpha}{2\pi}\right)^2\right)^\Omega+\left(\left(\frac{\alpha}{2\pi}\right)^2\right)^\Omega}}}, \sqrt{\frac{(1+\lambda^2)^\Omega-(1-\lambda^2)^\Omega}{(1+\lambda^2)^\Omega+(1-\lambda^2)^\Omega}}\, e^{i2\pi\sqrt{\frac{\left(1+\left(\frac{\beta}{2\pi}\right)^2\right)^\Omega-\left(1-\left(\frac{\beta}{2\pi}\right)^2\right)^\Omega}{\left(1+\left(\frac{\beta}{2\pi}\right)^2\right)^\Omega+\left(1-\left(\frac{\beta}{2\pi}\right)^2\right)^\Omega}}} \right) \right\rangle;$

These operations developed on the basis of Einstein t-norm and t-conorm, are more advantageous in DM as they provide considerable accurate and consistent results as compared to algebraic operations.

Theorem 4.1. *Let* $\tau = \left\langle d, \left(\mu e^{i\alpha}, \lambda e^{i\beta}\right)\right\rangle, \tau_{11} = \left\langle d_{11}, \left(\mu_{11}e^{i\alpha_{11}}, \lambda_{11}e^{i\beta_{11}}\right)\right\rangle$ *and* $\tau_{12} = \left\langle d_{12}, \left(\mu_{12}e^{i\alpha_{12}}, \lambda_{12}e^{i\beta_{12}}\right)\right\rangle$ *be any three CPFNSVs and* $\Omega, \Omega_1, \Omega_2 > 0$ *be any three real numbers, then*

(1) $\tau_{11} \otimes_e \tau_{12} = \tau_{12} \otimes_e \tau_{11},$

(2) $\tau_{11} \otimes_e \tau_{12} = \tau_{12} \otimes_e \tau_{11},$

(3) $\Omega_{.e}\left(\tau_{11} \oplus_e \tau_{12}\right) = \Omega_{.e}\tau_{11} \oplus_e \Omega_{.e}\tau_{12},$

(4) $\left(\tau_{11} \otimes_e \tau_{12}\right)^\Omega = \left(\tau_{11}\right)^\Omega \otimes_e \left(\tau_{12}\right)^\Omega,$

(5) $\left(\Omega_{1.e} \oplus_e \Omega_{2.e}\right) \tau = \Omega_{1.e}\tau \oplus_e \Omega_{2.e}\tau$,

(6) $\tau^{\Omega_1} \otimes_e \tau^{\Omega_2} = \tau^{\Omega_1+\Omega_2}$.

***Proof*:**

(1)

$$\tau_{11} \oplus_e \tau_{12} = \left\langle \max\left(d_{11}, d_{12}\right), \left(\sqrt{\frac{\mu_{11}^2+\mu_{12}^2}{1+\mu_{11}^2\mu_{12}^2}}\, e^{i2\pi \sqrt{\frac{\left(\frac{\alpha_{11}}{2\pi}\right)^2+\left(\frac{\alpha_{12}}{2\pi}\right)^2}{1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\left(\frac{a_{12}}{2\pi}\right)^2}}}, \frac{\lambda_{12}\lambda_{11}}{\sqrt{1+(1-\lambda_{12}^2)(1-\lambda_{11}^2)}}\, e^{i2\pi \frac{\left(\frac{\beta_{11}}{2\pi}\right)\left(\frac{\beta_{12}}{2\pi}\right)}{\sqrt{1+\left(1-\left(\frac{\beta_{11}}{2\pi}\right)^2\right)\left(1-\left(\frac{\beta_{12}}{2\pi}\right)^2\right)}}} \right) \right\rangle$$

$$= \left\langle \max\left(d_{12}, d_{11}\right) \left(\sqrt{\frac{\mu_{12}^2+\mu_{11}^2}{1+\mu_{12}^2\mu_{11}^2}}\, e^{i2\pi \sqrt{\frac{\left(\frac{\alpha_{12}}{2\pi}\right)^2+\left(\frac{\alpha_{11}}{2\pi}\right)^2}{1+\left(\frac{\alpha_{12}}{2\pi}\right)^2\left(\frac{\alpha_{11}}{2\pi}\right)^2}}}, \frac{\lambda_{12}\lambda_{11}}{\sqrt{1+(1-\lambda_{12}^2)(1-\lambda_{11}^2)}}\, e^{i2\pi \frac{\left(\frac{\beta_{12}}{2\pi}\right)\left(\frac{\beta_{11}}{2\pi}\right)}{\sqrt{1+\left(1-\left(\frac{\beta_{12}}{2\pi|}\right)^2\right)\left(1-\left(\frac{\beta_{11}}{2\pi}\right)^2\right)}}} \right) \right\rangle$$

$$= \tau_{12} \oplus_e \tau_{11}$$

Similarly, we can prove (2).

(3) First, consider the L.H.S,

$$\tau_{11} \oplus_e \tau_{12} = \left\langle \max\left(d_{11}, d_{12}\right) \left(\sqrt{\frac{\mu_{11}^2+\mu_{12}^2}{1+\mu_{11}^2\mu_{12}^2}}\, e^{i2\pi \sqrt{\frac{\left(\frac{\alpha_{11}}{2\pi}\right)^2+\left(\frac{\alpha_{12}}{2\pi}\right)^2}{1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\left(\frac{\alpha_{12}}{2\pi}\right)^2}}}, \frac{\lambda_{11}\lambda_{12}}{\sqrt{1+\left(1-\lambda_{11}^2\right)\left(1-\lambda_{12}^2\right)}}\, e^{i2\pi \frac{\left(\frac{\beta_{11}}{2\pi}\right)\left(\frac{\beta_{12}}{2\pi}\right)}{\sqrt{1+\left(1-\left(\frac{\beta_{11}}{2\pi}\right)^2\right)\left(1-\left(\frac{\beta_{12}}{2\pi}\right)^2\right)}}} \right) \right\rangle.$$

Above equation can be converted into following equation:

$$\tau_{11} \oplus_e \tau_{12} = \left\langle \max\left(d_{11}, d_{12}\right), \left(\sqrt{\frac{(1+\mu_{11}^2)(1+\mu_{12}^2)-(1-\mu_{11}^2)(1-\mu_{12}^2)}{(1+\mu_{11}^2)(1+\mu_{12}^2)+(1-\mu_{11}^2)(1-\mu_{12}^2)}}\; e^{i2\pi \sqrt{\frac{\left(1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)\left(1+\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)-\left(1-\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)\left(1-\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)}{\left(1+\left(\frac{\alpha_{1}1}{2\pi}\right)^2\right)\left(1+\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)+\left(1-\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)\left(1-\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)}}}, \right.\right.$$

$$\left.\left. \frac{\sqrt{2}\lambda_{11}\lambda_{12}}{\sqrt{(2-\lambda_{11}^2)(2-\lambda_{12}^2)+\lambda_{11}^2\lambda_{12}^2}}\, e^{i2\pi \frac{\sqrt{2}\left(\frac{\beta_{11}}{2\pi}\right)\left(\frac{\beta_{12}}{2\pi}\right)}{\sqrt{\left(2-\left(\frac{\beta_{11}}{2\pi}\right)^2\right)\left(2-\left(\frac{\beta_{12}}{2\pi}\right)^2+\left(\frac{\beta_{11}}{2\pi}\right)^2\right)\left(\frac{\beta_{12}}{2\pi}\right)^2}}} \right) \right\rangle.$$

Let us consider,

$$a = \left(1+\mu_{11}^2\right)\left(1+\mu_{12}^2\right),\; b = \left(1-\mu_{11}^2\right)\left(1-\mu_{12}^2\right),\; c = \left(1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)\left(1+\left(\frac{\alpha_{12}}{2\pi}\right)^2\right),\; f = \left(1-\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)\left(1-\left(\frac{\alpha_{12}}{2\pi}\right)^2\right),$$

$$u = \lambda_{11}^2\lambda_{12}^2,\; v = \left(2-\lambda_{11}^2\right)\left(2-\lambda_{12}^2\right), \quad w = \left(\frac{\beta_{11}}{2\pi}\right)^2\left(\frac{\beta_{12}}{2\pi}\right)^2, \quad x = \left(2-\left(\frac{\beta_{11}}{2\pi}\right)^2\right)\left(2-\left(\frac{\beta_{12}}{2\pi}\right)^2\right).$$

Then,

$$\tau_{11} \oplus_e \tau_{12} = \left\langle \max\left(d_{11}, d_{12}\right), \left(\sqrt{\frac{a-b}{a+b}}\, e^{i2\pi\sqrt{\frac{c-f}{c+f}}}, \sqrt{\frac{2u}{v+u}}\, e^{i2\pi\sqrt{\frac{2w}{w+x}}}\right)\right\rangle.$$

$$\Omega.e\left(\tau_{11} \oplus_e \tau_{12}\right) = \left\langle \max\left(d_{11}, d_{12}\right), \left(\sqrt{\frac{\left(1+\left(\frac{a-b}{a+b}\right)\right)^{\Omega}-\left(1-\left(\frac{a-b}{a+b}\right)\right)^{\Omega}}{\left(1+\left(\frac{a-b}{a+b}\right)\right)^{\Omega}+\left(1-\left(\frac{a-b}{a+b}\right)\right)^{\Omega}}}\, e^{i2\pi\sqrt{\frac{\left(1+\left(\frac{c-f}{c+f}\right)\right)^{\Omega}-\left(1-\left(\frac{c-f}{c+f}\right)\right)^{\Omega}}{\left(1+\left(\frac{c-f}{c+f}\right)\right)^{\Omega}+\left(1-\left(\frac{c-f}{c+f}\right)\right)^{\Omega}}}}, \frac{\sqrt{2}\left(\sqrt{\frac{2u}{v+u}}\right)^{\Omega}}{\sqrt{\left(2-\frac{2u}{v+u}\right)^{\Omega}+\left(\frac{2u}{v+u}\right)^{\Omega}}}\, e^{i2\pi\frac{\sqrt{2}\left(\sqrt{\frac{2m}{x+w}}\right)^{\Omega}}{\sqrt{\left(2-\frac{2w}{x+w}\right)^{\Omega}+\left(\frac{2w}{x+w}\right)^{\Omega}}}}\right)\right\rangle$$

$$= \left\langle \max\left(d_{11}, d_{12}\right), \left(\sqrt{\frac{a^{\Omega}-b^{\Omega}}{a^{\Omega}+b^{\Omega}}}\, e^{i2\pi\sqrt{\frac{c^{\Omega}-f^{\Omega}}{c^{\Omega}+f^{\Omega}}}}, \sqrt{\frac{2u^{\Omega}}{v^{\Omega}+u^{\Omega}}}\, e^{i2\pi\sqrt{\frac{2w^{\Omega}}{x^{\Omega}+w^{\Omega}}}}\right)\right\rangle$$

$$= \left\langle \max\left(d_{11}, d_{12}\right), \left(\sqrt{\frac{(1+\mu_{11}^2)^{\Omega}(1+\mu_{12}^2)^{\Omega}-(1-\mu_{11}^2)^{\Omega}(1-\mu_{12}^2)^{\Omega}}{(1+\mu_{11}^2)^{\Omega}(1+\mu_{12}^2)^{\Omega}+(1-\mu_{11}^2)^{\Omega}(1-\mu_{12}^2)^{\Omega}}}\, e^{i2\pi\sqrt{\frac{\left(1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}\left(1+\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}-\left(1-\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}\left(1-\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}}{\left(1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}\left(1+\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}+\left(1-\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}\left(1-\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}}}},\right.\right.$$

$$\left.\left.\frac{\sqrt{2}\lambda_{11}^{\Omega}\lambda_{12}^{\Omega}}{\sqrt{(2-\lambda_{11}^2)^{\Omega}(2-\lambda_{12}^2)^{\Omega}+(\lambda_{11}^2)^{\Omega}(\lambda_{12}^2)^{\Omega}}}\, e^{i2\pi\frac{\sqrt{2}\left(\frac{\beta_{11}}{2\pi}\right)^{\Omega}\left(\frac{\beta_{12}}{2\pi}\right)^{\Omega}}{\sqrt{\left(2-\left(\frac{\beta_{11}}{2\pi}\right)^2\right)^{\Omega}\left(2-\left(\frac{\beta_{12}}{2\pi}\right)^2\right)^{\Omega}+\left(\left(\frac{\beta_{11}}{2\pi}\right)^2\right)^{\Omega}\left(\left(\frac{\beta_{12}}{2\pi}\right)^2\right)^{\Omega}}}}\right)\right\rangle.$$

Now, consider the R.H.S,

$$\Omega_{.e}\tau_{11} = \left\langle d_{11}, \left(\sqrt{\frac{(1+\mu_{11}^2)^{\Omega}-(1-\mu_{11}^2)^{\Omega}}{(1+\mu_{11}^2)^{\Omega}+(1-\mu_{11}^2)^{\Omega}}}\, e^{i2\pi\sqrt{\frac{\left(1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}-\left(1-\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}}{\left(1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}+\left(1-\left(\frac{\alpha|1}{2\pi}\right)^2\right)^{\Omega}}}}, \frac{\sqrt{2}\lambda_{11}^{\Omega}}{\sqrt{(2-\lambda_{11}^2)^{\Omega}+(\lambda_{11}^2)^{\Omega}}}\, e^{i2\pi\frac{\sqrt{2}\left(\frac{\beta_{11}}{2\pi}\right)^{\Omega}}{\sqrt{\left(2-\left(\frac{\beta_{11}}{2\pi}\right)^2\right)^{\Omega}+\left(\left(\frac{\beta_{11}}{2\pi}\right)^2\right)^{\Omega}}}}\right).\right\rangle$$

$$\Omega_{.e}\tau_{12} = \left\langle d_{12}, \left(\sqrt{\frac{(1+\mu_{12}^2)^{\Omega}-(1-\mu_{12}^2)^{\Omega}}{(1+\mu_{12}^2)^{\Omega}+(1-\mu_{12}^2)^{\Omega}}}\, e^{i2\pi\sqrt{\frac{\left(1+\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}-\left(1-\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}}{\left(1+\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}+\left(1-\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}}}}, \frac{\sqrt{2}\lambda_{12}^{\Omega}}{\sqrt{(2-\lambda_{12}^2)^{\Omega}+(\lambda_{12}^2)^{\Omega}}}\, e^{i2\pi\frac{\sqrt{2}\left(\frac{\beta_{12}}{2\pi}\right)^{\Omega}}{\sqrt{\left(2-\left(\frac{\beta_{12}}{2\pi}\right)^2\right)^{\Omega}+\left(\left(\frac{\beta_{12}}{2\pi}\right)^2\right)^{\Omega}}}}\right).\right\rangle$$

Let us consider,

$$a_1 = \left(1+\mu_{11}^2\right)^{\Omega},\ b_1 = \left(1-\mu_{11}^2\right)^{\Omega},\ c_1 = \left(1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}, f_1 = \left(1-\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega},$$

$$w_1 = \left(\lambda_{11}^2\right)^{\Omega}, \quad x_1 = \left(2-\lambda_{11}^2\right)^{\Omega},\ y_1 = \left(\left(\frac{\beta_{11}}{2\pi}\right)^2\right)^{\Omega}, \quad z_1 = \left(2-\left(\frac{\beta_{11}}{2\pi}\right)^2\right)^{\Omega}.$$

and

$$a_2 = \left(1+\mu_{12}^2\right)^{\Omega},\ b_2 = \left(1-\mu_{12}^2\right)^{\Omega},\quad c_2 = \left(1+\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega},\quad f_2 = \left(1-\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega},$$
$$w_2 = \left(\lambda_{12}^2\right)^{\Omega},\quad x_2 = \left(2-\lambda_{12}^2\right)^{\Omega},\quad y_2 = \left(\left(\frac{\beta_{12}}{2\pi}\right)^2\right)^{\Omega},\quad z_2 = \left(2-\left(\frac{\beta_{12}}{2\pi}\right)^2\right)^{\Omega}.$$

Then, the above equations can be rewritten as follows:

$$\Omega_{.e}\tau_{11} = \left\langle d_{11}\cdot\left(\sqrt{\frac{a_1-b_1}{a_1+b_1}}e^{i2\pi\sqrt{\frac{c_1-f_1}{c_1+f_1}}},\sqrt{\frac{2w_1}{x_1+w_1}}e^{i2\pi\sqrt{\frac{2y_1}{z_1+y_1}}}\right)\right\rangle.$$
$$\Omega_{.e}\tau_{12} = \left\langle d_{12},\left(\sqrt{\frac{a_2-b_2}{a_2+b_2}}e^{i2\pi\sqrt{\frac{c_2-f_2}{c_2+f_2}}},\sqrt{\frac{2w_2}{x_2+w_2}}e^{12\pi\sqrt{\frac{2y_2}{z_2+y_2}}}\right)\right\rangle.$$

Now,

$$\begin{aligned}
\Omega_{.e}\tau_{11}\oplus_e\Omega_{.e}\tau_{12} &= \left\langle \max\left(d_{11},d_{12}\right),\left(\sqrt{\frac{\frac{a_1-b_1}{a_1+b_1}+\frac{a_2-b_2}{a_2+b_2}}{1+\left(\frac{a_1-b_1}{a_1+b_1}\right)\left(\frac{a_2-b_2}{a_2+b_2}\right)}}e^{i2\pi\sqrt{\frac{\frac{c_1-f_1}{c_1+f_1}+\frac{c_2-f_2}{c_2+f_2}}{1+\left(\frac{c_1-f_1}{c_1+f_1}\right)\left(\frac{c_2-f_2}{c_2+f_2}\right)}}},\sqrt{\frac{\frac{4w_1w_2}{(x_1+w_1)(x_2+w_2)}}{1+\left(1-\frac{2w_1}{x_1+w_1}\right)\left(1-\frac{2w_2}{x_2+w_2}\right)}}e^{i2\pi\sqrt{\frac{\frac{4y_1y_2}{(z_1+y_1)(z_2+y_2)}}{1+\left(1-\frac{2y_1}{z_1+y_1}\right)\left(1-\frac{2y_2}{z_2+y_2}\right)}}}\right)\right\rangle \\
&= \left\langle \max\left(d_{11},d_{12}\right),\left(\sqrt{\frac{a_1a_2-b_1b_2}{a_1a_2+b_1b_2}}e^{i2\pi\sqrt{\frac{c_1c_2-f_1f_2}{c_1c_2+f_1f_2}}},\sqrt{\frac{2w_1w_2}{x_1x_2+w_1w_2}}e^{i2\pi\sqrt{\frac{2y_1y_2}{z_1z_2+y_1y_2}}}\right)\right\rangle \\
&= \left\langle \max\left(d_{11},d_{12}\right),\left(\sqrt{\frac{\left(1+\mu_{11}^2\right)^{\Omega}\left(1+\mu_{12}^2\right)^{\Omega}-\left(1-\mu_{11}^2\right)^{\Omega}\left(1-\mu_{12}^2\right)^{\Omega}}{\left(1+\mu_{11}^2\right)^{\Omega}\left(1+\mu_{12}^2\right)^{\Omega}+\left(1-\mu_{11}^2\right)^{\Omega}\left(1-\mu_{12}^2\right)^{\Omega}}}e^{i2\pi\sqrt{\frac{\left(1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}\left(1+\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}-\left(1-\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}\left(1-\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}}{\left(1+\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}\left(1+\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}+\left(1-\left(\frac{\alpha_{11}}{2\pi}\right)^2\right)^{\Omega}\left(1-\left(\frac{\alpha_{12}}{2\pi}\right)^2\right)^{\Omega}}}},\right.\right. \\
&\qquad\left.\left.\frac{\sqrt{2}\lambda_{11}^{\Omega}\lambda_{12}^{\Omega}}{\sqrt{\left(2-\lambda_{11}^2\right)^{\Omega}\left(2-\lambda_{12}^2\right)^{\Omega}+\left(\lambda_{11}^2\right)^{\Omega}\left(\lambda_{12}^2\right)^{\Omega}}}e^{i2\pi\frac{\sqrt{2}\left(\frac{\beta_{11}}{2\pi}\right)^{\Omega}\left(\frac{\beta_{12}}{2\pi}\right)^{\Omega}}{\sqrt{\left(2-\left(\frac{\beta_{11}}{2\pi}\right)^2\right)^{\Omega}\left(2-\left(\frac{\beta_{12}}{2\pi}\right)^2\right)^{\Omega}+\left(\left(\frac{\beta_{11}}{2\pi}\right)^2\right)^{\Omega}\left(\left(\frac{\beta_{12}}{2\pi}\right)^2\right)^{\Omega}}}}\right)\right\rangle.
\end{aligned}$$

Hence, proved that $\Omega_{.e}(\tau_{11}\oplus_e \tau 12) = \Omega_{.e}\tau_{11}\oplus_e\Omega_{.e}\tau_{12}$.
Similarly, we can prove (4).

(5) Consider $\Omega_1, \Omega_2 > 0$,

$$\left(\Omega_{1.e}\oplus_e\Omega_{2.e}\right)\tau = \left\langle d,\left(\sqrt{\frac{(1+\mu^2)^{\Omega_1+\Omega_2}-(1-\mu^2)^{\Omega_1+\Omega_2}}{(1+\mu^2)^{\Omega_1+\Omega_2}+(1-\mu^2)^{\Omega_1+\Omega_2}}}e^{i2\pi\sqrt{\frac{\left(1+\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_1+\Omega_2}-\left(1-\left(\frac{\alpha_{-}}{2\pi}\right)^2\right)^{\Omega_1+\Omega_2}}{\left(1+\left(\frac{\alpha_1}{2\pi}\right)^2\right)^{\Omega_1+\Omega_2}+\left(1-\left(\frac{\alpha_1}{2\pi}\right)^2\right)^{\Omega_1+\Omega_2}}}},\frac{\sqrt{2}\lambda^{\Omega_1+\Omega_2}}{\sqrt{(2-\lambda^2)^{\Omega_1+\Omega_2}+(\lambda^2)^{\Omega_1+\Omega_2}}}e^{i2\pi\frac{\sqrt{2}\left(\frac{\beta}{2\pi}\right)^{\Omega_1+\Omega_2}}{\sqrt{\left(2-\left(\frac{\beta}{2\pi}\right)^2\right)^{\Omega_1+\Omega_2}+\left(\left(\frac{\beta}{2\pi}\right)^2\right)^{\Omega_1+\Omega_2}}}}\right)\right\rangle.$$

Now, Consider $R.H.S$, we have

$$\Omega_{1,e}\tau = \left\langle d, \left(\sqrt{\frac{(1+\mu^2)^{\Omega_1}-(1-\mu^2)^{\Omega_1}}{(1+\mu^2)^{\Omega_1}+(1-\mu^2)^{\Omega_1}}}\, e^{i2\pi \sqrt{\frac{\left(1+\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_1} - \left(1-\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_1}}{\left(1+\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_1} + \left(1-\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_1}}}}, \frac{\sqrt{2}\lambda^{\Omega_1}}{\sqrt{(2-\lambda^2)^{\Omega_1}+(\lambda^2)^{\Omega_1}}}\, e^{i2\pi \frac{\sqrt{2}\left(\frac{\beta}{2\pi}\right)^{\Omega_1}}{\sqrt{\left(2-\left(\frac{\beta}{2\pi}\right)^2\right)^{\Omega_1} + \left(\left(\frac{\beta}{2\pi}\right)^2\right)^{\Omega_1}}}} \right) \right\rangle.$$

$$\Omega_{2\cdot e}\tau = \left\langle d, \left(\sqrt{\frac{(1+\mu^2)^{\Omega_2}-(1-\mu^2)^{\Omega_2}}{(1+\mu^2)^{\Omega_2}+(1-\mu^2)^{\Omega_2}}}\, e^{i2\pi \sqrt{\frac{\left(1+\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_2} - \left(1-\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega 2}}{\left(1+\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_2} + \left(1-\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_2}}}}, \frac{\sqrt{2}\lambda^{\Omega_2}}{\sqrt{(2-\lambda^2)^{\Omega_2}+(\lambda^2)^{\Omega_2}}}\, e^{i2\pi \frac{\sqrt{2}\left(\frac{\beta}{2\pi}\right)^{\Omega_2}}{\sqrt{\left(2-\left(\frac{\beta}{2\pi}\right)^2\right)^{\Omega_2} + \left(\left(\frac{\beta}{2\pi}\right)^2\right)^{\Omega_2}}}} \right) \right\rangle.$$

Let us consider, for $k = 1, 2$

$$a_k = \left(1+\mu^2\right)^{\Omega_k}, \quad b_k = \left(1-\mu^2\right)^{\Omega_k}, \quad c_k = \left(1+\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_k}, \quad f_k = \left(1-\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_k},$$
$$w_k = \left(\lambda^2\right)^{\Omega_k}, \quad x_k = \left(2-\lambda^2\right)^{\Omega_k}, \quad y_k = \left(\left(\frac{\beta}{2\pi}\right)^2\right)^{\Omega_k}, \quad z_k = \left(2-\left(\frac{\beta}{2\pi}\right)^2\right)^{\Omega_k}.$$

$$\begin{aligned}
\Omega_{1.e}\tau \oplus_e \Omega_{2.e}\tau &= \left\langle \max\left(d_{11}, d_{12}\right), \left(\sqrt{\frac{\frac{a_1-b_1}{a_1+b_1}+\frac{a_2-b_2}{a_2+b_2}}{1+\left(\frac{a_1-b_1}{a_1+b_1}\right)\left(\frac{a_2-b_2}{a_2+b_2}\right)}}\, e^{i2\pi\sqrt{\frac{\frac{c_1-f_1}{c_1+f_1}+\frac{c_2-f_2}{c_2+f_2}}{1+\left(\frac{c_1-f_1}{c_1+f_1}\right)\left(\frac{c_2-f_2}{c_2+f_2}\right)}}}, \sqrt{\frac{\frac{4w_1w_2}{(x_1+w_1)(x_2+w_2)}}{1+\left(1-\frac{2w_1}{x_1+w_1}\right)\left(1-\frac{2w_2}{x_2+w_2}\right)}}\, e^{i2\pi\sqrt{\frac{\frac{4y_1y_2}{(z_1+y_1)(z_2+y_2)}}{1+\left(1-\frac{2y_1}{z_1+y_1}\right)\left(1-\frac{2y_2}{z_2+y_2}\right)}}} \right) \right\rangle \\
&= \left\langle \max\left(d_{11}, d_{12}\right), \left(\sqrt{\frac{a_1a_2-b_1b_2}{a_1a_2+b_1b_2}}\, e^{i2\pi\sqrt{\frac{c_1c_2-f_1f_2}{c_1c_2+f_1f_2}}}, \sqrt{\frac{2w_1w_2}{x_1x_2+w_1w_2}}\, e^{i2\pi\sqrt{\frac{2y_1y_2}{z_1z_2+y_1y_2}}} \right) \right\rangle \\
&= \left\langle d, \left(\sqrt{\frac{(1+\mu^2)^{\Omega_1+\Omega_2}-(1-\mu^2)^{\Omega_1+\Omega_2}}{(1+\mu^2)^{\Omega_1+\Omega_2}+(1-\mu^2)^{\Omega_1+\Omega_2}}}\, e^{i2\pi\sqrt{\frac{\left(1+\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_1+\Omega_2} - \left(1-\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_1+\Omega_2}}{\left(1+\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_1+\Omega_2} + \left(1-\left(\frac{\alpha}{2\pi}\right)^2\right)^{\Omega_1+\Omega_2}}}}, \frac{\sqrt{2}\lambda^{\Omega_1+\Omega_2}}{\sqrt{(2-\lambda^2)^{\Omega_1+\Omega_2}+(\lambda^2)^{\Omega_1+\Omega_2}}}\, e^{i2\pi\frac{\sqrt{2}\left(\frac{\beta}{2\pi}\right)^{\Omega_1+\Omega_2}}{\sqrt{\left(2-\left(\frac{\beta}{2\pi}\right)^2\right)^{\Omega_1}+\Omega^2+\left(\left(\frac{\beta}{2\pi}\right)^2\right)^{\Omega_1+\Omega_2}}}} \right) \right\rangle
\end{aligned}$$

Hence, proved that $(\Omega_{1.e} \oplus_e \Omega_{2.e})\tau = \Omega_{1.e}\tau \oplus_e \Omega_{1.e}\tau$.
Similarly, we can prove (6).□

5. APPLICATIONS

In this section, we establish specific DM strategies that operate on the proposed CPFNSS model. These algorithms are designed for identification of the best alternative of MCDM problems from a rational set of alternatives. Further, in order to prove their validity and effectiveness, we apply them to real DM situations. The step-by-step procedure of DM Algorithms 1, 2 and 3 are presented as follows.

The algorithm of choice values of CPFNSS

Algorithm 1

(1) Input $Z = \{z_1, z_2, \ldots, z_r\}$ as a universe of objects.

(2) Input $S = \{s_1, s_2, \ldots, s_w\}$ as a set of attributes.

(3) Input the N-soft set (H, S, N) with $D = \{0, 1, \ldots, N-1\}$, where $N \in \{2, 3, \ldots\}$ Then, for each $z_p \in Z$, $s_q \in S$, there exist unique $d_{pq} \in D$.

(4) Input CPFNSS (h, L, N), where $L = (H, S, N)$.

(5) Compute $\xi_p = \sum_{q=1}^{w} \tau_{pq}$, where $\tau_{pq} = \left\langle d_{pq}, \left(\mu_{pq} e^{i\alpha_{pq}}, \lambda_{pq} e^{i\beta_{pq}}\right)\right\rangle$ and Einstein addition of two τ_{pq} is defined as follows:

$$\tau_{11} \oplus_e \tau_{12} = \left\langle \max\left(d_{11}, d_{12}\right), \left(\sqrt{\frac{\mu_{11}^2+\mu_{12}^2}{1+\mu_{11}^2\mu_{12}^2}}\, e^{i2\pi\sqrt{\frac{\left(\frac{\alpha_{11}}{2\pi}\right)^2+\left(\frac{\alpha_{12}}{2\pi}\right)^2}{1+\left(\frac{a11}{2\pi}\right)^2\left(\frac{212}{2\pi}\right)^2}}}, \frac{\lambda_{11}\lambda_{12}}{\sqrt{1+(1-\lambda_{11}^2)(1-\lambda_{12}^2)}}\, e^{i2\pi\frac{\left(\frac{\beta_{11}}{2\pi}\right)\left(\frac{\beta_{12}}{2\pi}\right)}{\sqrt{1+\left(1-\left(\frac{\beta 11}{2\pi}\right)^2\right)\left(1-\left(\frac{\beta 12}{2\pi}\right)^2\right)}}} \right) \right\rangle.$$

(6) For each z_p, compute its choice value $\mathfrak{S}_p\left(\xi_p\right) = \left(\frac{d_p}{N-1}\right)^2 + \left(\mu_p^2 - \lambda_p^2\right) + \frac{1}{4\pi^2}\left(\alpha_p^2 - \beta_p^2\right)$, for each $p = 1, 2, \ldots, r$.

(7) Compute all the indices k for which $\mathfrak{S}_k = \max_p \mathfrak{S}_p$.

(8) Any of the alternatives for which $\mathfrak{S}_k = \max_p \mathfrak{S}_p$ can be chosen.

The algorithm of T-choice values of CPFNSS

Algorithm 2

(1) Input $Z = \{z_1, z_2, \ldots, z_r\}$ as a universe of objects.

(2) Input $S = \{s_1, s_2, \ldots, s_w\}$ as a set of attributes.

(3) Input the NSS (H, S, N) with $D = \{0, 1, \ldots, N-1\}$, where $N \in \{2, 3, \ldots\}$. Then, for each $z_p \in Z$, $s_q \in S$, there exist unique $d_{pq} \in D$.

(4) Input CPFNSS (h, L, N), where $L = (H, S, N)$.

(5) Input T threshold.

(6) Compute$h^T\left(s_q\right)\left(z_p\right) = \begin{cases} \left\langle \mu_{pq} e^{i\alpha_{pq}}, \lambda_{pq} e^{i\beta_{pq}} \right\rangle, & \text{if } \left\langle d_{pq}, \left(\mu_{pq} e^{i\alpha_{pq}}, \lambda_{pq} e^{i\beta_{pq}}\right)\right\rangle \in h\left(s_q\right) \text{ and } d_{pq} \geq T, \\ \begin{cases} \left\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\right\rangle, & \text{if } \frac{d_{pq}}{N} \geq 0.5 \\ \left\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\right\rangle, & \text{if } \frac{d_{pq}}{N} < 0.5 \end{cases} \end{cases}$

(7) Compute $\xi_p^T = \sum_{q=1}^{w} \tau_{pq}^T$, where $\tau_{pq}^T = \left\langle \mu_{pq}^T e^{i\alpha_{pq}^T}, \lambda_{pq}^T e^{i\beta_{pq}^T} \right\rangle$ and Einstein addition of two τ_{pq}^T is defined as follows:

$$\tau_{11}^T \oplus_e \tau_{12}^T = \left\langle \sqrt{\frac{\left(\mu_{11}^T\right)^2+\left(\mu_{12}^T\right)^2}{1+\left(\mu_{11}^T\right)^2\left(\mu_{12}^T\right)^2}}\, e^{i2\pi\sqrt{\frac{\left(\frac{\alpha_{11}^T}{2\pi}\right)^2+\left(\frac{\alpha_{12}^T}{2\pi}\right)^2}{1+\left(\frac{\alpha_{11}^T}{2\pi}\right)^2\left(\frac{\alpha_{12}^T}{2\pi}\right)^2}}}, \frac{\lambda_{11}^T\lambda_{12}^T}{\sqrt{1+\left(1-\left(\lambda_{11}^T\right)^2\right)\left(1-\left(\lambda_{12}^T\right)^2\right)}}\, e^{i2\pi\frac{\left(\frac{\beta_{11}^T}{2\pi}\right)\left(\frac{\beta_{12}^T}{2\pi}\right)}{\sqrt{1+\left(1-\left(\frac{\beta_{11}^T}{2\pi}\right)^2\right)\left(1-\left(\frac{\beta_{12}^T}{2\pi}\right)^2\right)}}} \right\rangle.$$

(8) For each z_p, compute its T-choice value $\mathfrak{S}_p^T\left(\xi_p\right) = \left(\left(\mu_p^T\right)^2 - \left(\lambda_p^T\right)^2\right) + \frac{1}{4\pi^2}\left(\left(\alpha_p^T\right)^2 - \left(\beta_p^T\right)^2\right)$, for each $p = 1, 2, \ldots, r$.

(9) Compute all the indices k for which $\mathfrak{S}_k^T = \max_p \mathfrak{S}_p^T$.

(10) Any of the alternatives for which $\mathfrak{S}_k^T = \max_p \mathfrak{S}_p^T$ can be chosen.

The algorithm of comparison table of CPFNSS

Algorithm 3

(1) Input $Z = \{z_1, z_2, \ldots, z_r\}$ as a universe of objects.

(2) Input $S = \{s_1, s_2, \ldots, s_w\}$ as a set of attributes.

(3) Consider a CPFNSS in tabular form.

(4) Compute the comparison table for score degree of CPFNSVs.

(5) Compute the final outcome.

(6) Find the maximum final outcome, then we will choose the alternative z_p, if it appears in p-th row, $1 \leq p \leq r$.

5.1. Selection of the Best Laptop

Suppose that the laptop of a research scholar is not working properly and his research work is badly affected by malfunctioning of his laptop. In order to fix this problem, he wants to purchase the best laptop of the year. Productive selection of laptops can only be done by the ideal matching of his requirements. This effective selection will ensure the high-quality performance of the laptop and he will do his research work effectively. Consider the set of five laptops $Z = \{z_1, z_2, z_3, z_4, z_5\}$ under consideration and let $S = \{s_1 = \text{Price}, s_2 = \text{Graphics}, s_3 = \text{Storage},$

s_4 = Battery life} be the set of parameters related to laptops in Z, which are used for the grading of laptops. A 6-soft set can be identified from Table 20, where

- Five check marks represent "Superb,"
- Four check marks represent "Excellent,"
- Three check marks represent "Very Good,"
- Two check marks represent "Good,"
- One check mark represents "Normal,"
- Big dot represents "Poor."

The numbers $D = \{0, 1, 2, 3, 4, 5\}$ can easily be associated with this graded evaluation followed by check marks as follows:

- 0 stands for "•,"
- 1 stands for "✓,"
- 2 stands for "✓✓,"
- 3 stands for "✓✓✓,"
- 4 stands for "✓✓✓✓,"
- 5 stands for "✓✓✓✓✓,"

The rating of laptops is given in Table 20.

Now, the tabular form of its corresponding 6-soft set is presented in Table 21.

In this respect, the grading criteria based on the parameters of laptops is given below:

$$\begin{aligned}
-2.0 \leq \mathfrak{S}\left(\tau_{pq}\right) < -1.17 &\quad \text{when } d_{pq} = 0\\
-1.17 \leq \mathfrak{S}\left(\tau_{pq}\right) < -0.34 &\quad \text{when } d_{pq} = 1\\
-0.34 \leq \mathfrak{S}\left(\tau_{pq}\right) < 0.50 &\quad \text{when } d_{pq} = 2\\
0.50 \leq \mathfrak{S}\left(\tau_{pq}\right) < 1.33 &\quad \text{when } d_{pq} = 3\\
1.33 \leq \mathfrak{S}\left(\tau_{pq}\right) < 2.17 &\quad \text{when } d_{pq} = 4\\
2.17 \leq \mathfrak{S}\left(\tau_{pq}\right) \leq 3.0 &\quad \text{when } d_{pq} = 5
\end{aligned}$$

where $\mathfrak{S}\left(\tau_{pq}\right) = \left(\frac{d_{pq}}{N-1}\right)^2 + \left(\mu_{pq}^2 - \lambda_{pq}^2\right) + \frac{1}{4\pi^2}\left(\alpha_{pq}^2 - \beta_{pq}^2\right)$ and $\tau_{pq} = \left\langle d_{pq}, \left(\mu_{pq}e^{i\alpha_{pq}}, \lambda_{pq}e^{i\beta_{pq}}\right)\right\rangle$; $p = 1, 2, 3, 4, 5$, $q = 1, 2, 3, 4$.

Table 20 Rating of laptops.

Z/S	s_1	s_2	s_3	s_4
z_1	✓✓✓	✓✓✓✓	•	✓✓✓✓✓
z_2	•	✓	✓✓	✓✓✓
z_3	✓✓✓✓	✓✓✓✓✓	✓✓✓	✓✓✓✓✓
z_4	✓✓✓✓✓	✓✓	•	✓
z_5	✓	✓✓✓	✓✓	✓✓✓✓

Table 21 Associated 6-soft set.

$(h, s, 6)$	s_1	s_2	s_3	s_4
z_1	3	4	0	5
z_2	0	1	2	3
z_3	4	5	3	5
z_4	5	2	0	1
z_5	1	3	2	4

Therefore, by Definition 3.1, CPF6SS is defined as

$$(h, L, 6) = \{(s_1, h(s_1)), (s_2, h(s_2)), (s_3, h(s_3)), (s_4, h(s_4))\}, \text{ where}$$

$$\begin{aligned}
h(s_1) = \{&\langle(z_1,3), 0.58e^{i1.22\pi}, 0.41e^{i0.77\pi}\rangle, \langle(z_2,0), 0.05e^{i0.22\pi}, 0.92e^{i1.77\pi}\rangle, \langle(z_3,4), 0.82e^{i1.55\pi}, 0.31e^{i0.55\pi}\rangle,\\
&\langle(z_4,5), 0.86e^{i1.76\pi}, 0.14e^{i0.28\pi}\rangle, \langle(z_5,1), 0.22e^{i0.58\pi}, 0.73e^{i1.52\pi}\rangle\},\\
h(s_2) = \{&\langle(z_1,4), 0.78e^{i1.42\pi}, 0.27e^{i0.44\pi}\rangle, \langle(z_2,1), 0.31e^{i0.44\pi}, 0.78e^{i1.42\pi}\rangle, \langle(z_3,5), 0.92e^{i1.75\pi}, 0.08e^{i0.22\pi}\rangle,\\
&\langle(z_4,2), 0.38e^{i0.72\pi}, 0.61e^{i1.03\pi}\rangle, \langle(z_5,3), 0.63e^{i1.28\pi}, 0.46e^{i0.84\pi}\rangle\},\\
h(s_3) = \{&\langle(z_1,0), 0.12e^{i0.24\pi}, 0.83e^{i1.68\pi}\rangle, \langle(z_2,2), 0.42e^{i0.68\pi}, 0.55e^{i1.22\pi}\rangle, \langle(z_3,3), 0.61e^{i1.05\pi}, 0.43e^{i0.79\pi}\rangle,\\
&\langle(z_4,0), 0.13e^{i0.11\pi}, 0.89e^{i1.92\pi}\rangle, \langle(z_5,2), 0.45e^{i0.92\pi}, 0.63e^{i1.13\pi}\rangle\},\\
h(s_4) = \{&\langle(z_1,5), 0.88e^{i1.78\pi}, 0.12e^{i0.22\pi}\rangle, \langle(z_2,3), 0.58e^{i1.12\pi}, 0.44e^{i0.88\pi}\rangle, \langle(z_3,5), 0.98e^{i1.78\pi}, 0.09e^{i0.25\pi}\rangle,\\
&\langle(z_4,1), 0.24e^{i0.52\pi}, 0.82e^{i1.55\pi}\rangle, \langle(z_5,4), 0.72e^{i1.60\pi}, 0.25e^{i0.63\pi}\rangle\}.
\end{aligned}$$

Now, CPF6SS can be represented in tabular form by Table 22.

Choice values of CPF6SS

The calculated choice values of CPF6SS for the selection of best laptop is given in Table 23.

According to choice values of CPF6SS, the laptops can be ranked in following order:

$$z_3 \succ z_1 \succ z_4 \succ z_5 \succ z_2.$$

Hence, we conclude that z_3 is the best laptop of the year having maximum choice value $\mathfrak{S}_p$.

T-choice values of CPF6SS

The calculated T-choice values of CPF6SS is given in Table 24, where we choose T= 5.

Table 22 Tabular representation of CPF6SS.

(h, L, 6)	s_1	s_2	s_3	s_4
z_1	$\langle 3, (0.58e^{i1.22\pi}, 0.41e^{i0.77\pi})\rangle$	$\langle 4, (0.78e^{i1.42\pi}, 0.27e^{i0.44\pi})\rangle$	$\langle 0, (0.12e^{i0.24\pi}, 0.83e^{i1.68\pi})\rangle$	$\langle 5, (0.88e^{i1.78\pi}, 0.12e^{i0.22\pi})\rangle$
z_2	$\langle 0, (0.05e^{i0.22\pi}, 0.92e^{i1.77\pi})\rangle$	$\langle 1, (0.31e^{i0.44\pi}, 0.78e^{i0.42\pi})\rangle$	$\langle 2, (0.42e^{i0.68\pi}, 0.55e^{i1.22\pi})\rangle$	$\langle 3, (0.58e^{i1.12\pi}, 0.44e^{i0.88\pi})\rangle$
z_3	$\langle 4, (0.82e^{i1.55\pi}, 0.31e^{i0.55\pi})\rangle$	$\langle 5, (0.92e^{i1.75\pi}, 0.08e^{i0.22\pi})\rangle$	$\langle 3, (0.61e^{i1.05\pi}, 0.43e^{i0.79\pi})\rangle$	$\langle 5, (0.98e^{i1.78\pi}, 0.09e^{i0.25\pi})\rangle$
z_4	$\langle 5, (0.86e^{i1.76\pi}, 0.14e^{i0.28\pi})\rangle$	$\langle 2, (0.38e^{i0.72\pi}, 0.61e^{i1.03\pi})\rangle$	$\langle 0, (0.13e^{i0.11\pi}, 0.89e^{i1.92\pi})\rangle$	$\langle 1, (0.24e^{i0.52\pi}, 0.82e^{i1.55\pi})\rangle$
z_5	$\langle 1, (0.22e^{i0.58\pi}, 0.73e^{i1.52\pi})\rangle$	$\langle 3, (0.63e^{i1.28\pi}, 0.46\ ^{i0.84\pi})\rangle$	$\langle 2, (0.45e^{i0.92\pi}, 0.63e^{i1.13\pi})\rangle$	$\langle 4, (0.72e^{i1.60\pi}, 0.25e^{i0.63\pi})\rangle$

Table 23 Choice values of CPF6SS.

(h, L, 6)	s_1	s_2	s_3	s_4	ξ_p	$\mathfrak{S}_p$
z_1	$\langle 3, (0.58e^{i1.22\pi}, 0.41e^{i0.77\pi})\rangle$	$\langle 4, (0.78e^{i1.42\pi}, 0.27e^{i0.44\pi})\rangle$	$\langle 0, (0.12e^{i0.24\pi}, 0.83e^{i1.68\pi})\rangle$	$\langle 5, (0.88e^{i1.78\pi}, 0.12e^{i0.22\pi})\rangle$	$\langle 5, (0.9851e^{i1.9662\pi}, 0.0051e^{i0.0072\pi})\rangle$	2.937
z_2	$\langle 0, (0.05e^{i0.22\pi}, 0.92e^{i1.77\pi})\rangle$	$\langle 1, (0.31e^{i0.44\pi}, 0.78e^{i1.42\pi})\rangle$	$\langle 2, (0.42e^{i0.68\pi}, 0.55e^{i1.22\pi})\rangle$	$\langle 3, (0.58e^{i1.12\pi}, 0.44e^{i0.88\pi})\rangle$	$\langle 3, (0.7457e^{i1.3609\pi}, 0.1103e^{i0.2056\pi})\rangle$	1.356
z_3	$\langle 4, (0.82e^{i1.55\pi}, 0.31e^{i0.55\pi})\rangle$	$\langle 5, (0.92e^{i1.75\pi}, 0.08e^{i0.22\pi})\rangle$	$\langle 3, (0.61e^{i1.05\pi}, 0.43e^{i0.79\pi})\rangle$	$\langle 5, (0.98e^{i1.78\pi}, 0.09e^{i0.25\pi})\rangle$	$\langle 5, (0.9998e^{i1.9956\pi}, 0.0003e^{i0.0011\pi})\rangle$	2.995
z_4	$\langle 5, (0.86e^{i1.76\pi}, 0.14e^{i0.28\pi})\rangle$	$\langle 2, (0.38e^{i0.72\pi}, 0.61e^{i1.03\pi})\rangle$	$\langle 0, (0.13e^{i0.11\pi}, 0.89e^{i1.92\pi})\rangle$	$\langle 1, (0.24e^{i0.52\pi}, 0.82e^{i1.55\pi})\rangle$	$\langle 5, (0.9078e^{i1.8365\pi}, 0.0387e^{i0.0666\pi})\rangle$	2.664
z_5	$\langle 1, (0.22e^{i0.58\pi}, 0.73e^{i1.52\pi})\rangle$	$\langle 3, (0.63e^{i1.28\pi}, 0.46e^{i0.84\pi})\rangle$	$\langle 2, (0.45e^{i0.92\pi}, 0.63e^{i1.13\pi})\rangle$	$\langle 4, (0.72e^{i1.60\pi}, 0.25e^{i0.63\pi})\rangle$	$\langle 4, (0.9206e^{i1.9013\pi}, 0.0261e^{i0.0558\pi})\rangle$	2.389

According to Table 24, the ranking of laptops is given as follows:

$$z_3 > z_1 > z_4 > z_5 > z_2.$$

Hence, we infer that z_3 is selected as best laptop of the year having maximum 5-choice $\mathfrak{S}_p^5$.

Comparison table of CPF6SS

A comparison table is a square table whose both horizontal and vertical entries are tagged by the universe of objects $z_1, z_2, \ldots z_r$. The number of parameters for which score degree of z_p exceeds or equal to the score degree of z_q specify the entry e_{pq} of comparison table. The tabular representation of score degrees of CPF6SVs of Table 22 is given by Table 25. The comparison table of score degrees is presented in Table 26. The final outcome for each laptop is calculated by subtracting the column sum from the row sum of Table 27.

From Table 27, we infer that the laptops can be ranked in following order:

$$z_3 \succ z_1 \succ z_5 \succ z_4 \succ z_2.$$

Hence, we conclude that z_3 is the best laptop of the year having maximum final outcome.

Table 24 5-choice values of CPF6SS.

(h^5, S)	s^1	s^2	s_3	s_4	ξ_p^5	$\mathfrak{S}_p^5$
z_1	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.88e^{i1.78\pi}, 0.12e^{i0.22\pi}\rangle$	$\langle 0.88e^{i1.78\pi}, 0.01e^{i0.03\pi}\rangle$	1.5661
z_2	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	−0.5
z_3	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	$\langle 0.92e^{i1.75\pi}, 0.08e^{i0.22\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	$\langle 0.98e^{i1.78\pi}, 0.09e^{i0.25\pi}\rangle$	$\langle 0.99e^{i1.96\pi}, 0.0006e^{i0.002\pi}\rangle$	1.9404
z_4	$\langle 0.86e^{i1.76\pi}, 0.14e^{i0.28\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.86e^{i1.76\pi}, 0.14e^{i0.28\pi}\rangle$	1.4748
z_5	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 0.2e^{i0.4\pi}\rangle$	−0.08

Table 25 Score degrees of CPF6SVs.

$\mathfrak{S}_{pq}$	s_1	s_2	s_3	s_4
z_1	0.7521	1.6312	−1.3657	2.5400
z_2	−1.6150	−0.9280	−0.2226	0.6228
z_3	1.7413	2.5935	0.6668	2.7287
z_4	2.4748	−0.2033	−1.6937	−1.1078
z_5	−0.9380	0.7785	−0.1420	1.6366

Table 26 Comparison table of score degrees.

.	z_1	z_2	z_3	z_4	z_5
z_1	4	3	0	3	3
z_2	1	4	0	2	0
z_3	4	4	4	3	4
z_4	1	2	1	4	1
z_5	1	4	0	3	4

Table 27 Final outcome with grades.

	Grade sum(h)	Row sum(r)	Column sum(c)	Final outcome $(r-c)$
z_1	12	13	11	2
z_2	6	7	17	−10
z_3	17	19	5	14
z_4	8	9	15	−6
z_5	10	12	12	0

5.2. Selection of the Plant Location

Plant location is one of the most important managerial DM problems as it strongly influences the profitability and future expansion of business. It has a great importance for both newly and already established firms. Suppose that a multinational enterprise wants to start up a new manufacturing plant to serve their increased set of demands. Consider the universe of five locations $Z = \{z_1, z_2, z_3, z_4, z_5$ under consideration and let $S = \{s_1, s_2, s_3, s_4\}$ be the set of factors affecting the selection of plant location, where

s_1 : Cost of production: The total cost of production is the most important factor in plant location. This criteria includes all the expenditures required for land, raw material, transportation, distribution and labor. Plant location with minimum cost of production is selected.

s_2 : Supply of labor: The adequate number of laborers with specific skills is essential for the evaluation of plant location. Plant location nearer to the source of manpower should be preferable.

s_3 : Infrastructure: The infrastructure facilities are the backbone of all industries. This includes telecommunication, transport, support services and regular supply of fuel.

s_4 : Proximity to markets: Proximity to markets is mandatory for the location of plants. It reduces the cost of transportation of furnished goods and offers quick services to customers.

The purpose of this study is to determine the best location for the establishment of a new plant. An ideal selection of plant location will minimize the cost of production and distribution to a large extent which in turn optimize the profit of enterprise. Now, A 5-soft set can be obtained from Table 28, where

- Four check marks represent "Superb,"
- Three check marks represent "Excellent,"
- Two check marks represent "Good,"
- One check mark represents "Normal,"
- Big dot represents 'Poor'.

This graded evaluation by stars can easily be identified with numbers $D = \{0, 1, 2, 3, 4\}$, where

- 0 stands for " •,"
- 1 stands for " ⋆,"
- 2 stands for " ⋆ ⋆,"
- 3 stands for " ⋆ ⋆ ⋆,"
- 4 stands for " ⋆ ⋆ ⋆ ⋆,"

Table 28 Rating of plant locations.

Z/S	s_1	s_2	s_3	s_4
z_1	⋆ ⋆ ⋆	⋆ ⋆ ⋆⋆	⋆ ⋆ ⋆	⋆ ⋆ ⋆⋆
z_2	⋆⋆	⋆	⋆ ⋆ ⋆	⋆
z_3	•	⋆⋆	⋆⋆	⋆
z_4	⋆ ⋆ ⋆	⋆ ⋆ ⋆⋆	⋆ ⋆ ⋆⋆	•
z_5	⋆ ⋆ ⋆⋆	•	⋆ ⋆ ⋆	⋆⋆

Table 29 Associated 5-soft set.

$(H, S, 5)$	s_1	s_2	s_3	s_4
z_1	3	4	3	4
z_2	2	1	3	1
z_3	0	2	2	1
z_4	3	4	4	0
z_5	4	0	3	2

The rating of plant locations is given in Table 28.

Now, the tabular form of its corresponding 5-soft set is presented in Table 29.

In this respect, the grading criteria based on the parameters of plant locations is given below:

$$\begin{array}{ll} -2.0 \le \mathfrak{S}\left(\tau_{pq}\right) < -1.0 & \text{when } d_{pq} = 0, \\ -1.0 \le \mathfrak{S}\left(\tau_{pq}\right) < 0.0 & \text{when } d_{pq} = 1, \\ 0.0 \le \mathfrak{S}\left(\tau_{pq}\right) < 1.0 & \text{when } d_{pq} = 2, \\ 1.0 \le \mathfrak{S}\left(\tau_{pq}\right) < 2.0 & \text{when } d_{pq} = 3, \\ 2.0 \le \mathfrak{S}\left(\tau_{pq}\right) \le 3.0 & \text{when } d_{pq} = 4, \end{array}$$

where $\mathfrak{S}\left(\tau_{pq}\right) = \left(\frac{d_{pq}}{N-1}\right)^2 + \left(\mu^2_{pq} - \lambda^2_{pq}\right) + \frac{1}{4\pi^2}\left(\alpha^2_{pq} - \beta^2_{pq}\right)$ and $\tau_{pq} = \left\langle d_{pq}, \left(\mu_{pq}e^{i\alpha_{pq}}, \lambda_{pq}e^{i\beta_{pq}}\right)\right\rangle$; $p = 1,2,3,4,5$, $q = 1,2,3,4$.

Therefore, by Definition 3.1, CPF5SS is defined as

$$(h, L, 5) = \left\{\left(s_1, h\left(s_1\right)\right), \left(s_2, h\left(s_2\right)\right)), \left(s_3, h\left(s_3\right)\right), \left(s_4, h\left(s_4\right)\right)\right\}, \text{ where}$$

$$\begin{aligned} h\left(s_1\right) = \Big\{ & \left\langle\left(z_1, 3\right), 0.75e^{i1.25\pi}, 0.38e^{i0.45\pi}\right\rangle, \left\langle\left(z_2, 2\right), 0.55e^{i0.83\pi}, 0.58e^{i0.99\pi}\right\rangle, \left\langle\left(z_3, 0\right), 0.15e^{i0.37\pi}, 0.99e^{i1.65\pi}\right\rangle, \\ & \left\langle\left(z_4, 3\right), 0.65e^{i1.58\pi}, 0.35e^{i0.52\pi}\right\rangle, \left\langle\left(z_5, 4\right), 0.95e^{i1.92\pi}, 0.13e^{i0.28\pi}\right\rangle\Big\}, \\ h\left(s_2\right) = \Big\{ & \left\langle\left(z_1, 4\right), 0.99e^{i1.76\pi}, 0.13e^{i0.23\pi}\right\rangle, \left\langle\left(z_2, 1\right), 0.38e^{i0.45\pi}, 0.78e^{i1.22\pi}\right\rangle, \left\langle\left(z_3, 2\right), 0.48e^{i0.93\pi}, 0.51e^{i0.84\pi}\right\rangle, \\ & \Big\{\left\langle\left(z_4, 4\right), 0.88e^{i1.75\pi}, 0.15e^{i0.32\pi}\right\rangle, \left\langle\left(z_5, 0\right), 0.09e^{i0.35\pi}, 0.93e^{i1.92\pi}\right\rangle, \\ h\left(s_3\right) = \Big\{ & \left\langle\left(z_1, 3\right), 0.68e^{i1.34\pi}, 0.33e^{i0.75\pi}\right\rangle, \left\langle\left(z_2, 3\right), 0.77e^{i1.44\pi}, 0.28e^{i0.62\pi}\right\rangle, \left\langle\left(z_3, 2\right), 0.53e^{i0.85\pi}, 0.48e^{i0.91\pi}\right\rangle, \\ & \Big\{\left\langle\left(z_4, 4\right), 0.89e^{i1.83\pi}, 0.99e^{i0.22\pi}\right\rangle, \left\langle\left(z_5, 3\right), 0.68e^{i1.42\pi}, 0.29e^{i0.66\pi}\right\rangle, \\ h\left(s_4\right) = \Big\{ & \left\langle\left(z_1, 4\right), 0.85e^{i1.88\pi}, 0.17e^{i0.34\pi}\right\rangle, \left\langle\left(z_2, 1\right), 0.29e^{i0.73\pi}, 0.75e^{i1.53\pi}\right\rangle, \left\langle\left(z_3, 1\right), 0.34e^{i0.62\pi}, 0.76e^{i1.37\pi}\right\rangle, \\ & \Big\{\left\langle\left(z_4, 0\right), 0.19e^{i0.28\pi}, 0.83e^{i1.88\pi}\right\rangle, \left\langle\left(z_5, 2\right), 0.44e^{i0.92\pi}, 0.57e^{i0.87\pi}\right\rangle. \end{aligned}$$

Now, CPF5SS can be represented in tabular form by Table 30.

Choice values of CPF5SS

The calculated choice values of CPF5SS for the selection of best plant location is given in Table 31.

According to choice values of CPF5SS, the ranking of locations of plant is given as follows:

$$z_1 \succ z_4 \succ z_5 \succ z_2 \succ z_3.$$

Thus, we infer that z_1 is the best location of the plant having maximum choice value $\mathfrak{G}_{\mathrm{P}}$.

Table 30 Tabular representation of the CPF5SS.

(*H*, *S*, 5)	s_1	s_2	s_3	s_4
z_1	$\langle 3,(0.75e^{i1.25\pi},0.38e^{i0.45\pi})\rangle$	$\langle 4,(0.99e^{i1.76\pi},0.13e^{i0.23\pi})\rangle$	$\langle 3,(0.68e^{i1.34\pi},0.33e^{i0.75\pi})\rangle$	$\langle 4,(0.85e^{i1.88\pi},0.17e^{i0.34\pi})\rangle$
z_2	$\langle 2,(0.55e^{i0.83\pi},0.58e^{i0.99\pi})\rangle$	$\langle 1,(0.38e^{i0.45\pi},0.78e^{i1.22\pi})\rangle$	$\langle 3,(0.77e^{i1.44\pi},0.28e^{i0.62\pi})\rangle$	$\langle 1,(0.29e^{i0.73\pi},0.75e^{i1.53\pi})\rangle$
z_3	$\langle 0,(0.15e^{i0.37\pi},0.99e^{i1.65\pi})\rangle$	$\langle 2,(0.48e^{i0.93\pi},0.51e^{i0.84\pi})\rangle$	$\langle 2,(0.53e^{i0.85\pi},0.48e^{i0.91\pi})\rangle$	$\langle 1,(0.34e^{i0.62\pi},0.76e^{i1.37\pi})\rangle$
z_4	$\langle 3,(0.65e^{i1.92\pi},0.35e^{i0.28\pi})\rangle$	$\langle 4,(0.88e^{i0.35\pi},0.15e^{i1.92\pi})\rangle$	$\langle 4,(0.89e^{i1.42\pi},0.99e^{i0.66\pi})\rangle$	$\langle 0,(0.19e^{i0.92\pi},0.83e^{i0.87\pi})\rangle$
z_5	$\langle 4,(0.95e^{i1.92\pi},0.13e^{i0.28\pi})\rangle$	$\langle 0,(0.09e^{i0.35\pi},0.93\ e^{i1.92\pi})\rangle$	$\langle 3,(0.68e^{i1.42\pi},0.29e^{i0.66\pi})\rangle$	$\langle 2,(0.44e^{i0.22\pi},0.57e^{i0.87\pi})\rangle$

T-choice values of CPF5SS

The calculated T-choice values of CPF5SS is given in Table 32, where we choose $T = 4$.

According to Table 32, the locations of plant can be ranked in following order:

$$z_1 \succ z_4 \succ z_5 \succ z_2 \succ z_3.$$

Thus, we infer that z_1 is selected as a suitable location of plant having maximum 4-choice value $\mathfrak{S}^4_p$.

Comparison table of CPF5SS

The tabular representation of score degrees of CPF5SVs of Table 30 is given by Table 33. The comparison table of score degrees is presented in Table 34. The final outcome for each plant location is calculated by subtracting the column sum from the row sum of Table 35.

From Table 35, we infer that the locations of plant can be ranked as follows:

$$z_1 \succ z_4 \succ z_5 \succ z_2 \succ z_3.$$

Table 31 Tabular representation of the CPF5SS.

$(H, S, 5)$	s_1	s_2	s_3	s_4	ξ_p	$\mathfrak{G}_p$
z_1	$\langle 3,(0.75e^{i1.25\pi}, 0.38e^{i0.45\pi})\rangle$	$\langle 4,(0.99e^{i1.76\pi}, 0.13e^{i0.23\pi})\rangle$	$\langle 3,(0.68\langle e^{i1.34\pi}, 0.33e^{i0.75\pi})\rangle$	$\langle 4,(0.85e^{i1.88\pi}, e^{i0.34\pi})\rangle$	$\langle 4,(0.99e^{i1.99\pi}, 0.0008e^{i0.0009\pi})\rangle$	2.9701
z_2	$\langle 2,(0.55e^{i0.83\pi}, 0.58e^{i0.99\pi})\rangle$	$\langle 2,(0.55e^{i0.83\pi}, 0.58e^{i0.99\pi})\rangle$	$\langle 3,(0.77e^{i1.44\pi}, 0.28e^{i0.62\pi})\rangle$	$\langle 4,(0.85e^{i1.88\pi}, 0.17e^{i0.34\pi})\rangle$	$\langle 4,(0.99e^{i1.99\pi}, 0.0008e^{i0.0009\pi})\rangle$	2.3054
z_3	$\langle 0,(0.15e^{i0.37\pi}, 0.99e^{i1.65\pi})\rangle$	$\langle 2,(0.48e^{i0.93\pi}, 0.51e^{i0.84\pi})\rangle$	$\langle 2,(0.53e^{i0.85\pi}, 0.48e^{i0.91\pi})\rangle$	$\langle 1,(0.34e^{i0.62\pi}, 0.76e^{i1.37\pi})\rangle$	$\langle 2,(0.75e^{i1.39\pi}, 0.11e^{i0.11\pi})\rangle$	1.5304
z_4	$\langle 3,(0.65e^{i1.58\pi}, 0.35e^{i0.52\pi})\rangle$	$\langle 4,(0.88e^{i1.75\pi}, 0.15e^{i0.32\pi})\rangle$	$\langle 4,(0.89e^{i1.83\pi}, 0.99e^{i0.22\pi})\rangle$	$\langle 0,(0.19e^{i0.28\pi}, 0.83e^{i1.88\pi})\rangle$	$\langle 4,(0.99e^{i1.99\pi}, 0.02e^{i0.004\pi})\rangle$	2.9697
z_5	$\langle 4,(0.95e^{i1.92\pi}, 0.13e^{i0.28\pi})\rangle$	$\langle 0,(0.09e^{i0.35\pi}, 0.93e^{i1.92\pi})\rangle$	$\langle 3,(0.68e^{i1.42\pi}, 0.29e^{i0.66\pi})\rangle$	$\langle 2,(0.44e^{i0.92\pi}, 0.57e^{i0.87\pi})\rangle$	$\langle 4,(0.98e^{i1.74\pi}, 0.009e^{i0.01\pi})\rangle$	2.7171

Table 32 4-choice values of CPF5SS.

(h^4, S)	s_1	s_2	s_3	s_4	ξ^4_p	$\mathfrak{S}^4_p$
z_1	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	$\langle 0.99e^{i1.76\pi}, 0.13e^{i0.23\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	$\langle 0.85e^{i1.88\pi}, 0.17e^{i0.34\pi}\rangle$	$\langle 0.99e^{i1.98\pi}, 0.0016e^{i0.0033\pi}\rangle$	1.9601
z_2	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	−0.5000
z_3	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	−2.000
z_4	$\langle 0.0e^{i0.5\pi}, 1.0e^{i1.0\pi}\rangle$	$\langle 0.88e^{i1.75\pi}, 0.15e^{i0.32\pi}\rangle$	$\langle 0.89e^{i1.83\pi}, 0.99e^{i0.22\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.98e^{i1.97\pi}, 0.04e^{i0.009\pi}\rangle$	1.9290
z_5	$\langle 0.95e^{i1.92\pi}, 0.13e^{i0.28\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 0.5e^{i1.0\pi}\rangle$	$\langle 0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}\rangle$	$\langle 0.95e^{i1.92\pi}, 0.04e^{i0.106\pi}\rangle$	1.8196

Table 33 Score degrees of CPF5SVs.

$\mathfrak{S}_{pq}$	s_1	s_2	s_3	s_4
z_1	1.3206	2.7243	1.2242	2.5483
z_2	0.4133	−0.7229	1.4993	−0.8679
z_3	−1.6040	0.26010	0.2741	−0.7726
z_4	1.4190	2.4919	1.6371	−1.5168
z_5	2.7876	−1.7477	1.3360	0.1440

Table 34 Comparison table of score degrees.

.	z_1	z_2	z_3	z_4	z_5
z_1	4	3	4	2	2
z_2	1	4	2	1	2
z_3	0	2	4	1	1
z_4	2	3	3	4	2
z_5	2	2	3	2	4

Hence, we conclude that z_1 will be the best location of a manufacturing plant having maximum final outcome.

6. COMPARATIVE ANALYSIS

In this section, a comparative analysis of our proposed techniques with existing MCDM techniques, namely, choice values of PFNSS and *D*-choice values of PFNSS is presented. We illustrate the authenticity and validity of our presented methods by evaluating the numerical example named "Selection of plant location" using choice values of PFNSS and *D*-choice values of PFNSS techniques.

Choice values of PFNSS method

We now solve the numerical example 5.2 with the technique, namely, choice values of PFNSS that was proposed by Zhang *et al.* [54].

First, construct the Pythagorean fuzzy 5-soft set (PF5SS) from Table 30 by taking phase terms of all their CPFNSVs equal to zero to apply the "Choice values of PFNSS" technique. The constructed PF5SS $(h_{\mathfrak{P}}, S, 5)$ is given in Table 36.

Further, compute the choice values ζ_p of each z_p as follows:

$$\zeta_p = \left\langle \sum_{q=1}^{4} d_{pq}, \sum_{q=1}^{4} R^{\star}_{pq} \right\rangle, \tag{6}$$

where $R^{\star}_{pq} = \frac{1}{2} + r_{pq}\left(\frac{1}{2} - \frac{2\theta_{pq}}{\pi}\right)$, $r_{pq} = \sqrt{\mu^2_{pq} + \lambda^2_{pq}}$ be the commitment strength and $\theta_{pq} = tan^{-1}\left(\frac{\lambda_{pq}}{\mu_{pq}}\right)$ be the angle between commitment strength r_{pq} and membership degree μ_{pq}. These computed choice values ζ_p are given in Table 37.

According to choice values of PF5SS, the ranking of locations of plant is given as follows:

$$z_1 \succ z_4 \succ z_5 \succ z_2 \succ z_3.$$

Hence, we conclude that z_1 is the best location of plant having maximum choice value ζ_p.

Table 35 Final outcome with grades.

.	**Grade sum(*h*)**	**Row sum(*r*)**	**column sum(*c*)**	**Final outcome(*r* – *c*)**
z_1	14	15	9	6
z_2	7	10	14	−4
z_3	5	8	16	−8
z_4	11	14	10	4
z_5	9	13	11	2

Table 36 Tabular representation of the PF5SS.

$(h_{\mathfrak{P}}, S, 5)$	s_1	s_2	s_3	s_4
z_1	⟨3,(0.75, 0.38)⟩	⟨4, (0.99, 0.13)⟩	⟨3, (0.68, 0.33)⟩	⟨4, (0.85, 0.17)⟩
z_2	⟨2, (0.55, 0.58)⟩	⟨1, (0.38, 0.78)⟩	⟨3, (0.77, 0.28)⟩	⟨1, (0.29, 0.75)⟩
z_3	⟨0, (0.15, 0.99)⟩	⟨2, (0.48, 0.51)⟩	⟨2, (0.53, 0.48)⟩	⟨1, (0.34, 0.76)⟩
z_4	⟨3, (0.65, 0.35)⟩	⟨4, (0.88, 0.15)⟩	⟨4, (0.89, 0.99)⟩	⟨0, (0.19, 0.83)⟩
z_5	⟨4, (0.95, 0.13)⟩	⟨0, (0.09, 0.93)⟩	⟨3, (0.68, 0.29)⟩	⟨2, (0.44, 0.57)⟩

Table 37 Choice value of PF5SS.

$(h_{\mathfrak{P}}, S, 5)$	s_1	s_2	s_3	s_4	ζ_p
z_1	⟨3, (0.75, 0.38)⟩	⟨4, (0.99, 0.13)⟩	⟨3, (0.68, 0.33)⟩	⟨4, (0.85, 0.17)⟩	⟨14, 3.0704⟩
z_2	⟨2, (0.55, 0.58)⟩	⟨1, (0.38, 0.78)⟩	⟨3, (0.77, 0.28)⟩	⟨1, (0.29, 0.75)⟩	⟨7, 1.8174⟩
z_3	⟨0, (0.15, 0.99)⟩	⟨2, (0.48, 0.51)⟩	⟨2, (0.53, 0.48)⟩	⟨1, (0.34, 0.76)⟩	⟨5, 1.4107⟩
z_4	⟨3, (0.65, 0.35)⟩	⟨4, (0.88, 0.15)⟩	⟨4, (0.89, 0.99)⟩	⟨0, (0.19, 0.83)⟩	⟨11, 2.1383⟩
z_5	⟨4, (0.95, 0.13)⟩	⟨0, (0.09, 0.93)⟩	⟨3, (0.68, 0.29)⟩	⟨2, (0.44, 0.57)⟩	⟨9, 2.1077⟩

D-choice values of PFNSS

We now solve the numerical example 5.2 with the technique, namely, D-choice values of PFNSS that was proposed by Zhang *et al.* [54]. Construct Pythagorean fuzzy 5-soft set from Table 30 by taking phase terms of all their CPFNSVs equal to zero to apply the "D-Choice values of PFNSS" technique. The constructed PF5SS is given in Table 36.

Further, compute the D-choice value ζ_p^D of each z_p as follows:

$$\zeta_p^D = \frac{\sum_{q=1}^{4} \mathfrak{S}\left(h_{pq}^D\right)}{4}, \text{ where } \mathfrak{S}\left(h_{pq}^D\right) = \mu_{pq}^2 - \lambda_{pq}^2, \tag{7}$$

$$h_{pq}^D = \begin{cases} \left\langle \mu_{pq}, \lambda_{pq} \right\rangle, & \text{if } \left\langle d_{pq}, \left(\mu_{pq}, \lambda_{pq}\right)\right\rangle \in h\left(s_q\right) \text{ and } d_{pq} \geq D, \\ \begin{cases} \langle 0.0, 0.5 \rangle, & \text{if } \frac{d_{pq}}{N} \geq 0.5, \\ \langle 0.0, 1.0 \rangle, & \text{if } \frac{d_{pq}}{N} < 0.5. \end{cases} \end{cases} \tag{8}$$

The calculated D-choice values of PF5SS is given in Table 38, where we choose T = 4.

According to Table 38, the locations of plant can be ranked in following order:

$$z_1 \succ z_4 \succ z_5 \succ z_2 \succ z_3.$$

Thus, we infer that z_1 is selected as a suitable location of plant having maximum 4-choice value ζ_p^4.

6.1. Discussion

- We conduct a comparison with existing MCDM techniques, namely, choice values of PFNSS and D-choice values of PFNSS to demonstrate the proficiency of proposed techniques. The results of compared methods are presented in Table 39.
- All the compared and proposed techniques prioritize z_1 as the most profitable location for the start-up of a new manufacturing plant of an enterprise. Also, ranking of plant locations remains the same in all these techniques which depicts the authenticity and validity of our proposed techniques in MADM problems.
- The comparison among the results of proposed and existing DM techniques is displayed in Figure 1 by plotting an illustrative bar chart among locations of plant and their order of ranking which demonstrate the reliability and proficiency of the proposed techniques.
- Our presented strategies are most generalized and flexible methods as they integrate existing MADM techniques, namely, choice values of PFNSS and D-choice values of PFNSS by capturing both aspects of two-dimensional vague information.

Table 38 4-choice values of PF5SS.

$\left(h_{\mathfrak{P}}^4, S, 5\right)$	s_1	s_2	s_3	s_4	ζ_p^4
z_1	⟨0.0, 0.5⟩	⟨0.99, 0.13⟩	⟨0.0, 0.5⟩	⟨0.85, 0.17⟩	0.2892
z_2	⟨0.0, 1.0⟩	⟨0.0, 1.0⟩	⟨0.0, 0.5⟩	⟨0.0, 1.0⟩	−0.8125
z_3	⟨0.0, 1.0⟩	⟨0.0, 1.0⟩	⟨0.0, 1.0⟩	⟨0.0, 1.0⟩	−1.0000
z_4	⟨0.0, 0.5⟩	⟨0.88, 0.15⟩	⟨0.89, 0.99⟩	⟨0.0, 1.0⟩	−0.1715
z_5	⟨0.95, 0.13⟩	⟨0.0, 1.0⟩	⟨0.0, 0.5⟩	⟨0.0, 1.0⟩	−0.3411

Table 39 Comparative analysis.

Methods	Ranking	Best Location
Proposed choice values of CPFNSS method	$z_1 \succ z_4 \succ z_5 \succ z_2 \succ z_3$	z_1
Proposed T-choice values CPFNSS method	$z_1 \succ z_4 \succ z_5 \succ z_2 \succ z_3$	z_1
Proposed comparison table method	$z_1 \succ z_4 \succ z_5 \succ z_2 \succ z_3$	z_1
Choice values of PFNSS method [54]	$z_1 \succ z_4 \succ z_5 \succ z_2 \succ z_3$	z_1
D-choice values of PFNSS method [54]	$z_1 \succ z_4 \succ z_5 \succ z_2 \succ z_3$	z_1

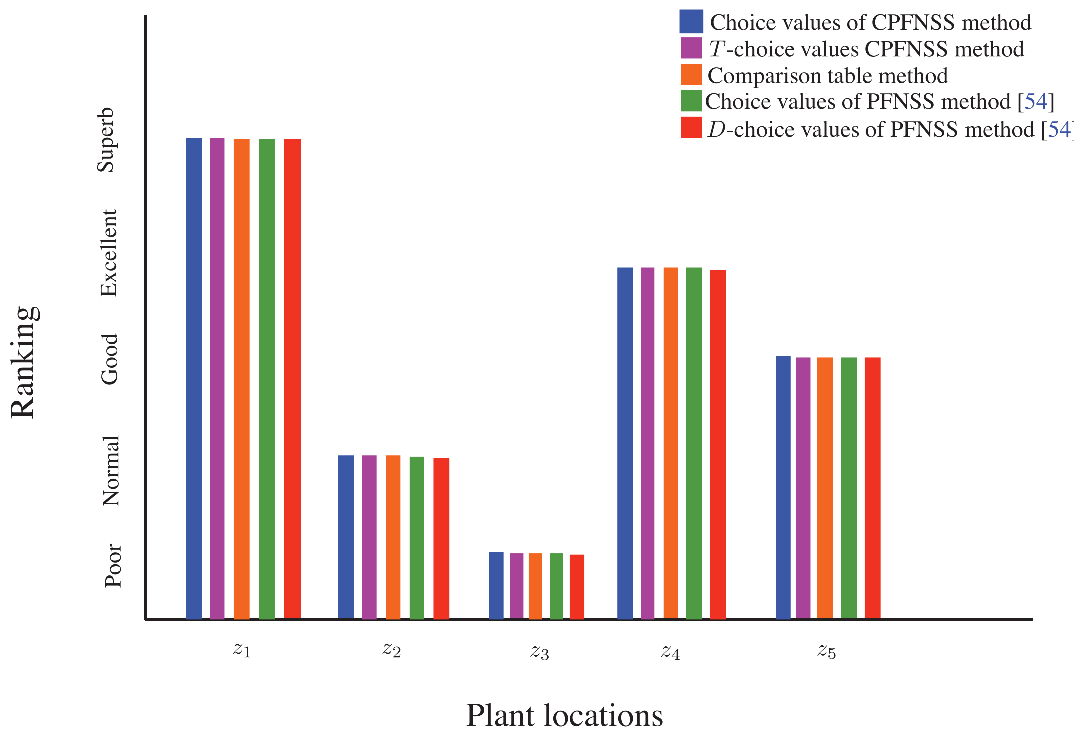

Figure 1 | Comparative analysis.

- Our proposed techniques have potential to deal with Pythagorean fuzzy N-soft information by taking phase terms equal to zero. Also, they provide the same results as provided by compared techniques. However, compared methods are not capable enough to deal with complex Pythagorean fuzzy N-soft information because of inadequacy of phase terms and they are restricted to capture only one dimensional information. This special trait models our presented techniques superior and stronger than existing DM techniques.
- The proposed CPFNSS provides a parameterized mathematical framework for the modeling of fuzziness and vagueness of data. It also has the potential to deal with the obscurity of two-dimensional information, that is, it can capture the uncertainty and periodicity of data at the same time. It is specially designed for the ranking and rating based evaluations of inconsistent and imprecise data.

7. CONCLUSION

In this research article, a novel theory has been established by the fusion of complex PFS theory with N-soft sets. The powerful and generalized model that arises, abbreviated as CPFNSS, has been introduced to provide a parameterized mathematical tool for the modeling of two-dimensional vague information. It has extrapolated the existing models due to its multinary but discrete evaluations of imprecise parameterized data having complex membership and nonmembership grades. Firstly, we have put forward the formal definition of CPFNSS along with its elementary operations. We have argued that the proposed model has an edge over the existing techniques as it overcomes deficiencies of NSS, PFNSS, CPFS and CIFNSS. Then, we have presented the fundamental Einstein and some algebraic operations of CPFNSVs. Another major contribution of this study is the development of three MCDM approaches for the selection of favorable alternative under CPFNS environment. We have put into practice the presented methodologies on two real life applications to exhibit the significance of our model. Finally, their validity has been demonstrated by comparative analyses with existing MCDM techniques. Admittedly, our presented theory has some limitations because the space of the proposed model is bounded by some constraint conditions. Due to this, the proposed DM methods are restricted to address the MCDM problems within the confined space of the proposed model. Also, the calculations performed by these methods are quite massive and cumbersome. Thus in the future, we are planning to extend our research to the more generalized mathematical theories covering a broader range of space and reduce the lengthy calculations of these DMtechniques by computer programing. We also intend to develop more MCDM strategies, including TOPSIS method, VIKOR method and ELECTRE I method under such a rich and flexible environment of CPFNSS.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

AUTHORS' CONTRIBUTIONS

Muhammad Akram proposed the methodology; Faiza Wasim investigated the results; Ahmad N. Al-Kenani reviewed the results. All authors read and approved the final manuscript.

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