International Journal of Computational Intelligence Systems
Vol. **14(1)**, 2021, *pp.* 1403–1418
DOI: https://doi.org/10.2991/ijcis.d.210409.001; ISSN: 1875-6891; eISSN: 1875-6883
https://www.atlantis-press.com/journals/ijcis/

Research Article

Some New Classes of Preinvex Fuzzy-Interval-Valued Functions and Inequalities

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ARTICLE INFO

Article History

Received 24 Dec 2020
Accepted 18 Mar 2021

Keywords

Fuzzy-interval-valued functions
Fuzzy Riemann integrals
$(\hbar_1, \hbar_2)$-preinvex fuzzy-interval-valued functions
Hermite–Hadamard inequalities

ABSTRACT

It is well known that convexity and nonconvexity develop a strong relationship with different types of integral inequalities. Due to the importance of the concept of nonconvexity and integral inequality, in this paper, we present some new classes of preinvex fuzzy-interval-valued functions involving two arbitrary auxiliary functions is known as $(\hbar_1, \hbar_2)$-preinvex fuzzy-interval-valued functions ($(\hbar_1, \hbar_2)$-preinvex fuzzy-IVFs). With the help of these classes, we derive some new Hermite–Hadamard inequalities (*HH*-inequalities) by means of fuzzy order relation on fuzzy-interval space and verify with the support of some nontrivial examples. This fuzzy order relation is defined level-wise through Kulisch–Miranker order relation defined on fuzzy-interval space. Moreover, several new and previously known results are also discussed which can be deducted from our main results. These results and different approaches may open new directions for fuzzy optimization problems, modeling, and interval-valued functions.

1. INTRODUCTION

In the development of pure and applied mathematics [1–3] convexity has played a key role. Convex sets and convex functions have been generalized and expanded in many mathematical directions due to their robustness; see [4–10]. In particular, several inequalities can be obtained in literature through convexity theory. In linear programing, combinatory, orthogonal polynomials, quantum theory, number theory, optimization theory, dynamics, and in the theory of relativity, integral inequalities [11–16] have various applications. This subject has gained substantial attention from researchers [17–19] and is therefore considered to be an integrative topic between economics, mathematics, physics, and statistics [1,2,20]. To the best of understanding, the *HH*-inequality is a familiar, supreme, and broadly useful inequality [2,21–27]. This inequality has fundamental significance [21,22] due to other classical inequalities such as the Oslen and Gagliardo–Nirenberg, Hardy, Oslen, Opial, Young, Linger, Arithmetic's-Geometric, Ostrowski, levison, Minkowski, Beckenbach-Dresher, Ky-fan and Holer inequality, which are closely linked to the classical *HH*-inequality. It can be stated as follows:

Let $\mathcal{F} : K \to \mathbb{R}$ be a convex function on a convex set K and $u, \vartheta \in K$ with $u \le \vartheta$. Then,

$$\mathcal{F}\left(\frac{u+\vartheta}{2}\right) \le \frac{1}{\vartheta - u}\int_u^\vartheta \mathcal{F}(x)\,dx \le \frac{\mathcal{F}(u)+\mathcal{F}(\vartheta)}{2}. \tag{1}$$

Fejér considered the major generalizations of *HH*-inequality in [18] which is known as *HH*-Fejér inequality.

Let $\mathcal{F} : K \to \mathbb{R}$ be a convex function on a convex set K and $u, \vartheta \in K$ with $u \le \vartheta$. Then,

$$\begin{aligned}\mathcal{F}\left(\frac{u+\vartheta}{2}\right) &\le \frac{1}{\int_u^\vartheta \Omega(x)\,dx}\int_u^\vartheta \mathcal{F}(x)\,\Omega(x)\,dx \\ &\le \frac{\mathcal{F}(u)+\mathcal{F}(\vartheta)}{2}\int_u^\vartheta \Omega(x))dx\end{aligned} \tag{2}$$

where $\Omega : [u, \vartheta] \to \mathbb{R}$ with $\Omega(x) \ge 0$, is a symmetric function with respect to $\frac{u+\vartheta}{2}$, and $\int_u^\vartheta \Omega(x)dx > 0$. If $\Omega(x) = 1$ then, we obtain (1) from (2). With the assistance of inequality (1), many classical inequalities can be obtained through special convex function. In addition, these inequalities have a very significant role for convex functions in both pure and applied mathematics.

Hanson [28] initiated to introduce a generalized class of convexity which is known as an invex function. The invex function played a significant role in mathematical programing. A step forward, the invex set and preinvex function were introduced and studied by Israel and Mond [10]. Also, Noor [29] examined the optimality conditions of differentiable preinvex functions and proved that variational-like inequalities would characterize the minimum. Many generalizations and extensions of classical convexity have been investigated by several authors. In 2007, Noor [26] introduced

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$\hbar$-preinvex functions and with the help of this concept, presented the following HH-inequality for $\hbar$-preinvex function:

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right) \le \frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}(x)\,dx \le [\mathcal{F}(u)+\mathcal{F}(\vartheta)]\int_0^1 \hbar(\tau)d\tau, \tag{3}$$

where $\partial : [u,\vartheta]\times[u,\vartheta] \to [u,\vartheta]$ and $\mathcal{F} : K \to \mathbb{R}^+$ is a preinvex function on the invex set $K = [u, u+\partial(\vartheta,u)]$ with $u < u+\partial(\vartheta,u)$ and $\hbar : [0,1] \to \mathbb{R}^+$ with $\hbar\left(\frac{1}{2}\right) \ne 0$. A step forward, Marian Matloka [30] constructed HH-Fejér inequalities for $\hbar$-preinvex function and investigated some different properties of differentiable preinvex function. Recently, Noor *et al.* [31] new classes of preinvex functions involving two arbitrary auxiliary functions and derived some following HH-inequalities for these classes of preinvex functions:

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right) \le \frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}(x)\,dx \le [\mathcal{F}(u)+\mathcal{F}(\vartheta)]\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\,d\tau. \tag{4}$$

where $\mathcal{F} : K \to \mathbb{R}^+$ is a $(\hbar_1,\hbar_2)$-preinvex function on the invex set $K = [u, u+\partial(\vartheta,u)]$ with $u < u+\partial(\vartheta,u)$ and $\hbar_1,\hbar_2 : [0,1] \to \mathbb{R}^+$ with $\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right) \ne 0$.

From the other side, because of the absence of implementations of the theory of interval analysis in other sciences, this theory fell into oblivion for a long time. Moore [32] and Kulish and Miranker [33] suggested and investigated the idea of interval analysis. In numerical analysis, it is first used to evaluate the error bounds of a finite state machine's numerical solutions. We refer the readers [34–36] and the references therein for basic information and applications.

Recently, Zhao *et al.* [37] introduced $\hbar$-convex interval-valued functions ($\hbar$-convex interval-valued functions (IVFs), in short) and proved the HH-type inequalities and Jensen HH-type inequalities for $\hbar$-convex IVFs. Besides, An *et al.* [38] defined the class of $(\hbar_1,\hbar_2)$-convex IVFs and established following interval-valued HH-type inequality for $(\hbar_1,\hbar_2)$-convex IVFs.

Let $\mathcal{F} : [u,\vartheta] \subset \mathbb{R} \to \mathcal{K}_C^+$ be an $(\hbar_1,\hbar_2)$-convex IVF given by $\mathcal{F}(x) = [\mathcal{F}_*(x), \mathcal{F}^*(x)]$ for all $x \in [u,\vartheta]$, with $\hbar_1,\hbar_2 : [0,1] \to \mathbb{R}^+$ with $\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right) \ne 0$, where $\mathcal{F}_*(x)$ and $\mathcal{F}^*(x)$ both are $(\hbar_1,\hbar_2)$-convex function. If $\mathcal{F}$ is Riemann integrable (in sort, IR-integrable) then,

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}\left(\frac{u+\vartheta}{2}\right) \supseteq \frac{1}{\vartheta-u}(IR)\int_u^{\vartheta}\mathcal{F}(x)\,dx \supseteq [\mathcal{F}(u)+\mathcal{F}(\vartheta)]\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\,d\tau. \tag{5}$$

For further review of the literature on the applications and properties of generalized convex functions and Hermite–Hadamrd inequalities, see [39–42] and the references therein.

Similarly, the notions of convexity and generalized convexity play a vital role in optimization under fuzzy domain because during characterization of the optimality condition of convexity, we obtain fuzzy variational inequalities so variational inequality theory and fuzzy complementary problem theory established powerful mechanism of the mathematical problems and they have a friendly relationship. Many authors contributed to this fascinating and interesting field. In 1989, Nanda and Kar [43] initiated to introduce convex fuzzy mappings from convex set to the set of fuzzy numbers and characterized the notion of convex fuzzy mapping through the idea of epigraph. A step forward, Furukawa [44] and Syau [45] proposed and examined fuzzy mapping from space $\mathbb{R}^n$ to the set of fuzzy numbers, fuzzy valued Lipschitz continuity, logarithmic convex fuzzy mappings, and quasi-convex fuzzy mappings. Besides, Chang [1] discussed the idea of convex fuzzy mapping and find its optimality condition with the support of fuzzy variational inequality. Generalization and extension of fuzzy convexity play a vital and significant implementation in diverse directions. So let's note that, one of the most considered classes of nonconvex fuzzy mapping is preinvex fuzzy mapping. Noor [46] introduced this idea and proved some results that distinguish the fuzzy optimality condition of differentiable preinvex fuzzy mappings by fuzzy variational-like inequality. Fuzzy variational inequality theory and complementary problem theory established a strong relationship with mathematical problems. For more useful details about the applications and properties of variational inequalities and generalized convex fuzzy mappings, see [47–52] and the references therein.

The fuzzy mappings are fuzzy-IVFs. There are some integrals to deal with fuzzy-IVFs, where the integrands are fuzzy-IVFs. For instance, Oseuna-Gomez *et al.* [53] and Costa *et al.* [54] constructed Jensen's integral inequality for fuzzy-IVFs. By using same approach Costa and Floures also presented Minkowski and Beckenbach's inequalities, where the integrands are fuzzy-IVFs. Motivated by [41,42,53,54] and especially by [55] because Costa *et al* established relation between elements of fuzzy-interval space and interval space, and introduced level-wise fuzzy order relation on fuzzy-interval space through Kulisch–Miranker order relation defined on interval space. By using this concept on fuzzy-interval space, we generalize integral inequality (1) by constructing fuzzy-interval integral inequalities for convex fuzzy-IVFs, where the integrands are convex fuzzy-IVFs.

This study is organized as follows: Section 2 presents preliminary and new concepts and results in interval space, the space of fuzzy intervals, and fuzzy convex analysis. Section 3 obtains fuzzy-interval HH-inequalities and HH-Fejér inequalities via $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVFs. In addition, some interesting examples are also given to verify our results. Section 4 gives conclusions and future plans.

2. PRELIMINARIES

In this section, we recall some basic preliminary notions, definitions, and results. With the help of these results, some new basic definitions and results are also discussed.

We begin by recalling basic notations and definitions. We define interval,

$$[\omega_*, \omega^*] = \{x \in \mathbb{R} : \omega_* \le x \le \omega^* \text{ and } \omega_*, \omega^* \in \mathbb{R}\},$$

where $\omega_* \leq \omega^*$.

We write len $[\omega_*, \omega^*] = \omega^* - \omega_*$, If len $[\omega_*, \omega^*] = 0$ then, $[\omega_*, \omega^*]$ is called degenerate. In this article, all intervals will be nondegenerate intervals. The collection of all closed and bounded intervals of $\mathbb{R}$ is denoted defined as $\mathscr{K}_C = \{[\omega_*, \omega^*] : \omega_*, \omega^* \in \mathbb{R} \text{ and } \omega_* \leq \omega^*\}$. If $\omega_* \geq 0$ then, $[\omega_*, \omega^*]$ is called positive interval. The set of all positive interval is denoted by $\mathscr{K}_C{}^+$ and defined as $\mathscr{K}_C{}^+ = \{[\omega_*, \omega^*] : [\omega_*, \omega^*] \in \mathscr{K}_C \text{ and } \omega_* \geq 0\}$.

We now discuss some properties of intervals under the arithmetic operations addition, multiplication, and scalar multiplication. If $[\mu_*, \mu^*], [\omega_*, \omega^*] \in \mathscr{K}_C$ and $\rho \in \mathbb{R}$ then, arithmetic operations are defined by

$$[\mu_*, \mu^*] + [\omega_*, \omega^*] = [\mu_* + \omega_*, \mu^* + \omega^*],$$

$$[\mu_*, \mu^*] \times [\omega_*, \omega^*] = \left[min \left\{ \begin{array}{c} \mu_*\omega_*, \mu^*\omega_*, \\ \mu_*\omega^*, \mu^*\omega^* \end{array} \right\}, max \left\{ \begin{array}{c} \mu_*\omega_*, \mu^*\omega_*, \\ \mu_*\omega^*, \mu^*\omega^* \end{array} \right\} \right],$$

$$\rho. [\mu_*, \mu^*] = \begin{cases} [\rho\mu_*, \rho\mu^*] & \text{if } \rho \geq 0, \\ [\rho\mu^*, \rho\mu_*] & \text{if } \rho < 0. \end{cases}$$

For $[\mu_*, \mu^*], [\omega_*, \omega^*] \in \mathscr{K}_C$, the inclusion "$\subseteq$" is defined by

$[\mu_*, \mu^*] \subseteq [\omega_*, \omega^*]$, if and only if $\omega_* \leq \mu_*$, $\mu^* \leq \omega^*$.

Remark 2.1. The relation "$\leq_I$" defined on $\mathscr{K}_C$ by

$[\mu_*, \mu^*] \leq_I [\omega_*, \omega^*]$ if and only if $\mu_* \leq \omega_*$, $\mu^* \leq \omega^*$,

for all $[\mu_*, \mu^*], [\omega_*, \omega^*] \in \mathscr{K}_C$, it is an order relation, see [33]. For given $[\mu_*, \mu^*], [\omega_*, \omega^*] \in \mathscr{K}_C$, we say that $[\mu_*, \mu^*] \leq_I [\omega_*, \omega^*]$ if and only if $\mu_* \leq \omega_*$, $\mu^* \leq \omega^*$ or $\mu_* \leq \omega_*$, $\mu^* < \omega^*$.

The concept of Riemann integral for IVF first introduced by Moore [32] is defined as follows:

Theorem 2.2 *If $\mathcal{F} : [c, d] \subset \mathbb{R} \to \mathscr{K}_C$ is an IVF such that $[\mathcal{F}_*, \mathcal{F}^*]$ then, $\mathcal{F}$ is Riemann integrable over: $[c, d]$ if and only if, $\mathcal{F}_*$ and $\mathcal{F}^*$ both are Riemann integrable over: $[c, d]$ such that*

$$(IR) \int_c^d \mathcal{F}(x)\, dx = \left[(R) \int_c^d \mathcal{F}_*(u)\, dx, (R) \int_c^d \mathcal{F}^*(u)\, dx \right]$$

The collection of all Riemann integrable real valued functions and Riemann integrable IVFs is denoted by $\mathcal{R}_{[c,d]}$ and $\mathcal{IR}_{[c,d]}$, respectively.

Let $\mathbb{R}$ be the set of real numbers. A fuzzy subset set $\mathscr{A}$ of $\mathbb{R}$ is distinguished by a function $\varphi : \mathbb{R} \to [0, 1]$ called the membership function. In this study this depiction is approved. Moreover, the collection of all fuzzy subsets of $\mathbb{R}$ is denoted by $\mathbb{F}(\mathbb{R})$.

A real fuzzy-interval φ is a fuzzy set in $\mathbb{R}$ with the following properties:

(1) φ is normal, i.e., there exists $x \in \mathbb{R}$ such that $\varphi(x) = 1$;

(2) φ is upper semi continuous, i.e., for given $x \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\varphi(x) - \varphi(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|x - y| < \delta$.

(3) φ is fuzzy convex, i.e., $\varphi((1-\tau)x + \tau y) \geq min(\varphi(x), \varphi(y))$, $\forall x, y \in \mathbb{R}$ and $\tau \in [0, 1]$;

(4) φ is compactly supported i.e., $cl\{x \in \mathbb{R} | \varphi(x) > 0\}$ is compact.

The collection of all real fuzzy-intervals is denoted by $\mathbb{F}_C(\mathbb{R})$.

Let $\varphi \in \mathbb{F}_C(\mathbb{R})$ be real fuzzy-interval, if and only if, γ-levels $[\varphi]^\gamma$ is a nonempty compact convex set of $\mathbb{R}$. This is represented by

$$[\varphi]^\gamma = \{x \in \mathbb{R} | \varphi(x) \geq \gamma\},$$

from these definitions, we have

$$[\varphi]^\gamma = [\varphi_*(\gamma), \varphi^*(\gamma)],$$

where

$$\varphi_*(\gamma) = inf\{x \in \mathbb{R} | \varphi(x) \geq \gamma\},$$
$$\varphi^*(\gamma) = sup\{x \in \mathbb{R} | \varphi(x) \geq \gamma\}.$$

Thus a real fuzzy-interval φ can be identified by a parametrized triples

$$\{(\varphi_*(\gamma), \varphi^*(\gamma), \gamma) : \gamma \in [0, 1]\}.$$

This leads the following characterization of a real fuzzy-interval in terms of the two end point functions $\varphi_*(\gamma)$ and $\varphi^*(\gamma)$.

Theorem 2.3 *[9] Suppose that $\varphi_*(\gamma) : [0, 1] \to \mathbb{R}$ and $\varphi^*(\gamma) : [0, 1] \to \mathbb{R}$ satisfy the following conditions:*

(1) $\varphi_*(\gamma)$ is a nondecreasing function.

(2) $\varphi^*(\gamma)$ is a nonincreasing function.

(3) $\varphi_*(1) \leq \varphi^*(1)$.

(4) $\varphi_*(\gamma)$ and $\varphi^*(\gamma)$ are bounded and left continuous on $(0, 1]$ and right continuous at $\gamma = 0$.

Moreover, If $\varphi : \mathbb{R} \to [0, 1]$ is a real fuzzy-interval given by $[\varphi_*(\gamma), \varphi^*(\gamma)]$ then, function $\varphi_*(\gamma)$ and $\varphi^*(\gamma)$ find the conditions (1–4).

Proposition 2.4. [55] Let $\varphi, \phi \in \mathbb{F}_C(\mathbb{R})$. Then, fuzzy order relation "$\preccurlyeq$" given on $\mathbb{F}_C(\mathbb{R})$ by

$\varphi \preccurlyeq \phi$ if and only if, $[\varphi]^\gamma \leq_I [\phi]^\gamma$ for all $\gamma \in [0, 1]$,

it is partial order relation.

We now discuss some properties of real fuzzy-intervals under addition, scalar multiplication, multiplication, and division. If $\varphi, \phi \in \mathbb{F}_C(\mathbb{R})$ and $\rho \in \mathbb{R}$ then, arithmetic operations are defined by

$$[\varphi \tilde{+} \phi]^\gamma = [\varphi]^\gamma + [\phi]^\gamma, \quad (6)$$

$$[\varphi \tilde{\times} \phi]^\gamma = [\varphi]^\gamma \times [\phi]^\gamma, \quad (7)$$

$$[\rho.\varphi]^\gamma = \rho.[\varphi]^\gamma \tag{8}$$

For $\psi \in \mathbb{F}_C(\mathbb{R})$ such that $\varphi = \phi \tilde{+} \psi$ then, by this result we have existence of Hukuhara difference of φ and ϕ, and we say that ψ is the H-difference of φ and ϕ, and denoted by $\varphi \tilde{-} \phi$. If H-difference exists then,

$$[\psi]^\gamma = [\varphi \tilde{-} \phi]^\gamma = [\varphi]^\gamma - [\phi]^\gamma, \tag{9}$$

where $(\psi)^*(\gamma) = (\varphi \tilde{-} \phi)^*(\gamma) = \varphi^*(\gamma) - \phi^*(\gamma)$, $(\psi)_*(\gamma) = (\varphi \tilde{-} \phi)_*(\gamma) = \varphi_*(\gamma) - \phi_*(\gamma)$.

Remark 2.5. Obviously, $\mathbb{F}_C(\mathbb{R})$ is closed under addition and nonnegative scaler multiplication and above defined properties on $\mathbb{F}_C(\mathbb{R})$ are equivalent to those derived from the usual extension principle. Furthermore, for each scalar number $\rho \in \mathbb{R}$,

$$[\rho \tilde{+} \varphi]^\gamma = \rho + [\varphi]^\gamma. \tag{10}$$

Theorem 2.6. *[17] The space $\mathbb{F}_C(\mathbb{R})$ dealing with a supremum metric i.e, for $\psi, \phi \in \mathbb{F}_C(\mathbb{R})$*

$$D(\psi, \phi) = \sup_{0 \le \gamma \le 1} H([\psi]^\gamma, [\phi]^\gamma),$$

it is a complete metric space, where H denote the well-known Hausdorff metric on space of intervals.

Definition 2.7. [55] A fuzzy-interval-valued map $\mathcal{F} : K \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ is called fuzzy-IVF. For each $\gamma \in [0,1]$, whose γ-levels define the family of IVFs $\mathcal{F}_\gamma : K \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\mathcal{F}_\gamma(x) = [\mathcal{F}_*(x,\gamma), \mathcal{F}^*(x,\gamma)]$ for all $x \in K$. Here, for each $\gamma \in [0,1]$, the end point real functions $\mathcal{F}_*(.,\gamma), \mathcal{F}^*(.,\gamma) : K \to \mathbb{R}$ are called lower and upper functions of $\mathcal{F}$.

Remark 2.8. Let $\mathcal{F} : K \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF. Then, $\mathcal{F}(x)$ is said to be continuous at $x \in K$, if for each $\gamma \in [0,1]$, both end point functions $\mathcal{F}_*(x,\gamma)$ and $\mathcal{F}^*(x,\gamma)$ are continuous at $x \in K$.

From above literature review, following results can be concluded, see [33,49,50,55]:

Definition 2.9. Let $\mathcal{F} : [c,d] \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ is called fuzzy-IVF. The fuzzy integral of $\mathcal{F}$ over $[c,d]$, denoted by $(FR)\int_c^d \mathcal{F}(x)\,dx$, it is defined level-wise by

$$\left[(FR)\int_c^d \mathcal{F}(x)\,dx\right]^\gamma = (IR)\int_c^d \mathcal{F}_\gamma(x)\,dx = \left\{\int_c^d \mathcal{F}(x,\gamma)\,dx : \mathcal{F}(x,\gamma) \in \mathcal{R}_{[c,d]}\right\}, \tag{11}$$

for all $\gamma \in [0,1]$, where $\mathcal{R}_{[c,d]}$ is the collection of end point functions of IVFs. $\mathcal{F}$ is (FR)-integrable over $[c,d]$ if $(FR)\int_c^d \mathcal{F}(x)\,dx \in \mathbb{F}_C(\mathbb{R})$. Note that, if both end point functions are Lebesgue-integrable then, $\mathcal{F}$ is fuzzy Aumann-integrable, see [33,49,50,55].

Theorem 2.10. *Let $\mathcal{F} : [c,d] \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF, whose γ-levels define the family of IVFs $\mathcal{F}_\gamma : [c,d] \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\mathcal{F}_\gamma(x) = [\mathcal{F}_*(x,\gamma), \mathcal{F}^*(x,\gamma)]$ for all $x \in [c,d]$ and for all $\gamma \in [0,1]$. Then, $\mathcal{F}$ is (FR)-integrable over $[c,d]$ if and only if, $\mathcal{F}_*(x,\gamma)$ and $\mathcal{F}^*(x,\gamma)$ both are R-integrable over $[c,d]$. Moreover, if $\mathcal{F}$ is (FR)-integrable over $[c,d]$ then,*

$$\left[(FR)\int_c^d \mathcal{F}(x)\,dx\right]^\gamma = \left[(R)\int_c^d \mathcal{F}_*(x,\gamma)\,dx, (R)\int_c^d \mathcal{F}^*(x,\gamma)\,dx\right] = (IR)\int_c^d \mathcal{F}_\gamma(x)\,dx, \tag{12}$$

for all $\gamma \in [0,1]$.

The family of all (FR)-integrable fuzzy-IVFs and R-integrable functions over $[c,d]$ are denoted by $\mathcal{FR}_{([c,d],\gamma)}$ and $\mathcal{R}_{([c,d],\gamma)}$, for all $\gamma \in [0,1]$.

Definition 2.11. [46] Let K be an invex set. Then, fuzzy-IVF $\mathcal{F} : K \to \mathbb{F}_C(\mathbb{R})$ is said to be preinvex on K with respect to ∂ if

$$\mathcal{F}(x + (1-\tau)\partial(y,x)) \preccurlyeq \tau\mathcal{F}(x) \tilde{+} (1-\tau)\mathcal{F}(y),$$

for all $x, y \in K, \tau \in [0,1]$, where $\mathcal{F}(x) \succcurlyeq \tilde{0}$, $\partial : K \times K \to \mathbb{R}$. If $\mathcal{F}$ is preconcave fuzzy-IVF then, $-\mathcal{F}$ is preinvex fuzzy-IVF.

Definition 2.13. Let K be an invex set and $\hbar_1, \hbar_2 : [0,1] \subseteq K \to \mathbb{R}$ such that $\hbar_1, \hbar_2 \not\equiv 0$. Then, fuzzy-IVF $\mathcal{F} : K \to \mathbb{F}_C(\mathbb{R})$ is said to be

- $(\hbar_1, \hbar_2)$-preinvex on K if

$$\mathcal{F}(x + (1-\tau)\partial(y,x)) \preccurlyeq \hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}(x) \tilde{+} \hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}(y), \tag{13}$$

  for all $x, y \in K, \tau \in [0,1]$, where $\mathcal{F}(x) \succcurlyeq \tilde{0}$ and $\partial : K \times K \to \mathbb{R}$.

- $(\hbar_1, \hbar_2)$-preconcave on K if inequality (13) is reversed.
- affine $(\hbar_1, \hbar_2)$-preinvex on K if

$$\mathcal{F}(x + (1-\tau)\partial(y,x)) = \hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}(x) \tilde{+} \hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}(y), \tag{14}$$

for all $x, y \in K, \tau \in [0,1]$, where $\mathcal{F}(x) \succcurlyeq \tilde{0}$ and $\partial : K \times K \to \mathbb{R}$.

Remark 2.14. If $\hbar_2(\tau) \equiv 1$ then, $(\hbar_1, \hbar_2)$-preinvex fuzzy-IVF becomes $\hbar_1$-preinvex fuzzy-IVF, i.e.,

$$\mathcal{F}(x + (1-\tau)\partial(y,x)) \preccurlyeq \hbar_1(\tau)\mathcal{F}(x) \tilde{+} \hbar_1(1-\tau)\mathcal{F}(y), \forall x, y \in K, \tau \in [0,1].$$

If $\hbar_1(\tau) = \tau^s$, $\hbar_2(\tau) \equiv 1$ then, $(\hbar_1, \hbar_2)$-preinvex fuzzy-IVF becomes s-preinvex fuzzy-IVF, i.e.,

$$\mathcal{F}(x + (1-\tau)\partial(y,x)) \preccurlyeq \tau^s\mathcal{F}(x) \tilde{+} (1-\tau)^s\mathcal{F}(y), \forall x, y \in K, \tau \in [0,1].$$

If $\hbar_1(\tau) = \tau, \hbar_2(\tau) \equiv 1$ then, $(\hbar_1, \hbar_2)$-preinvex fuzzy-IVF becomes preinvex fuzzy-IVF, i.e.,

$$\mathcal{F}\left(x + (1-\tau)\partial\left(y,x\right)\right) \preccurlyeq \mathcal{F}(x) \tilde{+} (1-\tau)\mathcal{F}\left(y\right), \forall x, y \in K, \tau \in [0,1].$$

If $\hbar_1(\tau) = \hbar_2(\tau) \equiv 1$ then, $(\hbar_1, \hbar_2)$-preinvex fuzzy-IVF becomes P-preinvex fuzzy-IVF, i.e.,

$$\mathcal{F}\left(x + (1-\tau)\partial\left(y,x\right)\right) \preccurlyeq \mathcal{F}(x) \tilde{+} \mathcal{F}\left(y\right), \forall x, y \in K, \tau \in [0,1].$$

Theorem 2.15. *Let K be an invex set and non-negative real-valued function $\hbar_1, \hbar_2 : [0,1] \subseteq K \to \mathbb{R}$ such that $\hbar_1, \hbar_2 \not\equiv 0$. Let $\mathcal{F} : K \to \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF with $\mathcal{F}(x) \succcurlyeq \tilde{0}$, whose γ-levels define the family of IVFs $\mathcal{F}_\gamma : [c,d] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by*

$$\mathcal{F}_\gamma(x) = \left[\mathcal{F}_*(x,\gamma), \mathcal{F}^*(x,\gamma)\right], \forall x \in K. \tag{15}$$

for all $x \in [c,d]$ and for all $\gamma \in [0,1]$. Then, $\mathcal{F}$ is $(\hbar_1, \hbar_2)$-preinvex fuzzy-IVF on K, if and only if, for all $\gamma \in [0,1], \mathcal{F}^*(x,\gamma)$ and $\mathcal{F}^*(x,\gamma)$ both are $(\hbar_1, \hbar_2)$-preinvex functions. (16)

Proof. Assume that for each $\gamma \in [0,1], \mathcal{F}^*(x,\gamma)$ and $\mathcal{F}^*(x,\gamma)$ are $(\hbar_1, \hbar_2)$-preinvex on K. Then, from (13), we have

$$\mathcal{F}_*\left(x + (1-\tau)\partial\left(y,x\right), \gamma\right) \le \hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}_*(x,\gamma) + \hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}_*\left(y,\gamma\right)$$

for all $x, y \in K, \tau \in [0,1]$,

And

$$\mathcal{F}^*\left(x + (1-\tau)\partial\left(y,x\right), \gamma\right) \le \hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}^*(x,\gamma) + \hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}^*\left(y,\gamma\right),$$

for all $x, y \in K, \tau \in [0,1]$.

Then, by (15), (5) and (7), we obtain

$$\begin{aligned} &\mathcal{F}_\gamma\left(x + (1-\tau)\partial\left(y,x\right)\right) \\ &= \left[\mathcal{F}_*\left(x + (1-\tau)\partial(y,x), \gamma\right),\ \mathcal{F}^*\left(x + (1-\tau)\partial(y,x), \gamma\right)\right], \\ &\le \left[\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}_*(x,\gamma),\ \hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}^*(x,\gamma)\right] \\ &\quad + \left[\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}_*\left(y,\gamma\right),\ \hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}^*\left(y,\gamma\right)\right], \end{aligned}$$

i.e.,

$$\mathcal{F}\left(x + (1-\tau)\partial\left(y,x\right)\right) \preccurlyeq \hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}(x) \tilde{+} \hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}\left(y\right), \forall x, y \in K, \tau \in [0,1].$$

Hence, $\mathcal{F}$ is $(\hbar_1, \hbar_2)$-preinvex fuzzy-IVF on K.

Conversely, let $\mathcal{F}$ is $(\hbar_1, \hbar_2)$-preinvex fuzzy-IVF on K. Then, for all $x, y \in K$ and $\tau \in [0,1]$, we have $\mathcal{F}\left(x + (1-\tau)\partial(y,x)\right) \preccurlyeq \hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}(x) \tilde{+} \hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}\left(y\right)$. Therefore, from (15), we have

$$\begin{aligned} &\mathcal{F}_\gamma\left(x + (1-\tau)\partial(y,x)\right) \\ &= \left[\mathcal{F}_*\left(x + (1-\tau)\partial(y,x), \gamma\right), \mathcal{F}^*\left(x + (1-\tau)\partial(y,x), \gamma\right)\right]. \end{aligned}$$

Again, from (15), (5) and (7), we obtain

$$\begin{aligned} &\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}_\gamma(x) \tilde{+} \hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}_\gamma(x) \\ &= \left[\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}_*(x,\gamma), \hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}^*(x,\gamma)\right] \\ &\quad + \left[\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}_*\left(y,\gamma\right), \hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}^*\left(y,\gamma\right)\right], \end{aligned}$$

for all $x, y \in K$ and $\tau \in [0,1]$. Then, by $(\hbar_1, \hbar_2)$-preinvexity of $\mathcal{F}$, we have for all $x, y \in K$ and $\tau \in [0,1]$ such that

$$\begin{aligned} &\mathcal{F}_*\left(x + (1-\tau)\partial\left(y,x\right), \gamma\right) \\ &\le \hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}_*(x,\gamma) + \hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}_*\left(y,\gamma\right), \end{aligned}$$

and

$$\begin{aligned} &\mathcal{F}^*\left(x + (1-\tau)\partial\left(y,x\right), \gamma\right) \\ &\le \hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}^*(x,\gamma) + \hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}^*\left(y,\gamma\right), \end{aligned}$$

for each $\gamma \in [0,1]$. Hence, the result follows.

Theorem 2.16. *Let K be an invex set and non-negative real-valued function $\hbar_1, \hbar_2 : [0,1] \subseteq K \to \mathbb{R}$ such that $\hbar_1, \hbar_2 \not\equiv 0$. Let $\mathcal{F} : K \to \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF with $\mathcal{F}(x) \succcurlyeq \tilde{0}$, whose γ-levels define the family of IVFs $\mathcal{F}_\gamma : [c,d] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by*

$$\mathcal{F}_\gamma(x) = \left[\mathcal{F}_*(x,\gamma), \mathcal{F}^*(x,\gamma)\right], \forall x \in K. \tag{17}$$

for all $x \in [c,d]$ and for all $\gamma \in [0,1]$. Then, $\mathcal{F}$ is $(\hbar_1, \hbar_2)$-preconcave fuzzy-INF on K, if and only if, for all $\gamma \in [0,1], \mathcal{F}_*(x,\gamma)$ and $\mathcal{F}^*(x,\gamma)$ both are $(\hbar_1, \hbar_2)$-preconcave function. (18)

Proof. The demonstration is analogous to the demonstration of Theorem 2.15.

3. MAIN RESULTS

In this section, we put forward some HH-inequalities for $(\hbar_1, \hbar_2)$-preinvex fuzzy-IVFs via fuzzy Riemann integrals.

Theorem 3.1. *Let $\mathcal{F} : [u, u + \partial(\vartheta, u)] \to \mathbb{F}_C(\mathbb{R})$ be a $(\hbar_1, \hbar_2)$-preinvex fuzzy-IVF with non-negative real-valued functions $\hbar_1, \hbar_2 : [0,1] \to \mathbb{R}$ and $\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right) \ne 0$, whose γ-levels define the family of IVFs $\mathcal{F}_\gamma : [u, u + \partial(\vartheta, u)] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{F}_\gamma(x) = \left[\mathcal{F}_*(x,\gamma), \mathcal{F}^*(x,\gamma)\right]$ for all $x \in [u, u + \partial(\vartheta, u)]$ and for all $\gamma \in [0,1]$. If $\mathcal{F} \in \mathcal{FR}_{([u, u+\partial(\vartheta,u)], \gamma)}$ then,*

$$\begin{aligned} &\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}\left(\frac{2u + \partial(\vartheta,u)}{2}\right) \preccurlyeq \frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)\,dx \\ &\preccurlyeq \left[\mathcal{F}(u) \tilde{+} \mathcal{F}(\vartheta)\right]\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\,d\tau. \end{aligned} \tag{19}$$

Proof. Let $\mathcal{F} : [u, u+\partial(\vartheta,u)] \to \mathbb{F}_C(\mathbb{R})$, $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVF. Then, by hypothesis, we have

$$\frac{1}{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right)$$
$$\preccurlyeq \mathcal{F}(u+(1-\tau)\partial(\vartheta,u))\tilde{+}\mathcal{F}(u+\tau\partial(\vartheta,u)).$$

Therefore, for every $\gamma \in [0,1]$, we have

$$\frac{1}{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right) \le \mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma) + \mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma),$$
$$\frac{1}{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right) \le \mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma) + \mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma).$$

Then,

$$\frac{1}{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\int_0^1 \mathcal{F}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)d\tau$$
$$\le \int_0^1 \mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\,d\tau + \int_0^1 \mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma)\,d\tau,$$
$$\frac{1}{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\int_0^1 \mathcal{F}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)d\tau$$
$$\le \int_0^1 \mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\,d\tau + \int_0^1 \mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma)\,d\tau.$$

It follows that

$$\frac{1}{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right) \le \frac{2}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)} \mathcal{F}_*(x,\gamma)\,dx,$$
$$\frac{1}{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right) \le \frac{2}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)} \mathcal{F}^*(x,\gamma)\,dx.$$

That is,

$$\frac{1}{\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\left[\mathcal{F}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right),\mathcal{F}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)\right]$$
$$\le_I \frac{2}{\partial(\vartheta,u)}\left[\int_u^{u+\partial(\vartheta,u)} \mathcal{F}_*(x,\gamma)\,dx, \int_u^{u+\partial(\vartheta,u)} \mathcal{F}^*(x,\gamma)\,dx\right].$$

Thus,

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right) \preccurlyeq \frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)\,dx. \tag{20}$$

In a similar way as above, we have

$$\frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)\,dx$$
$$\preccurlyeq [\mathcal{F}(u)\tilde{+}\mathcal{F}(\vartheta)]\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\,d\tau. \tag{21}$$

Combining (20) and (21), we have

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right) \preccurlyeq \frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)\,dx$$
$$\preccurlyeq [\mathcal{F}(u)\tilde{+}\mathcal{F}(\vartheta)]\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\,d\tau.$$

This completes the proof.

Note that, if $\mathcal{F}(x)$ is $(\hbar_1,\hbar_2)$-preconcave fuzzy-IVF then, integral inequality (19) is reversed

Remark 3.2. If $\hbar_2(\tau) \equiv 1$ then, Theorem 3.1 reduces to the result for $\hbar_1$-preinvex fuzzy-IVF:

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)}\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right) \preccurlyeq \frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)\,dx$$
$$\preccurlyeq [\mathcal{F}(u)\tilde{+}\mathcal{F}(\vartheta)]\int_0^1 \hbar_1(\tau)\,d\tau.$$

If $\hbar_1(\tau) = \tau^s$ and $\hbar_2(\tau) \equiv 1$ then, Theorem 3.1 reduces to the result for s-preinvex fuzzy-IVF:

$$2^{s-1}\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right) \preccurlyeq \frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)\,dx \preccurlyeq \frac{1}{s+1}[\mathcal{F}(u)\tilde{+}\mathcal{F}(\vartheta)].$$

If $\hbar_1(\tau) = \tau$ and $\hbar_2(\tau) \equiv 1$ then, Theorem 3.1 reduces to the result for preinvex fuzzy-IVF:

$$\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right) \preccurlyeq \frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)\,dx \preccurlyeq \frac{\mathcal{F}(u)\tilde{+}\mathcal{F}(\vartheta)}{2}.$$

If $\hbar_1(\tau) = \hbar_2(\tau) \equiv 1$ then, Theorem 3.1 reduces to the result for P-preinvex fuzzy-IVF:

$$\tfrac{1}{2}\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right) \preccurlyeq \frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)\,dx \preccurlyeq \mathcal{F}(u)\tilde{+}\mathcal{F}(\vartheta).$$

If $\mathcal{F}_*(u,\gamma) = \mathcal{F}^*(\vartheta,\gamma)$ with $\gamma = 1$ then, we obtain (4) from (19).

If $\mathcal{F}_*(u,\gamma) = \mathcal{F}^*(\vartheta,\gamma)$ with $\gamma = 1$ and $\hbar_2(\tau) \equiv 1$ then, we obtain (3) from (19).

Note that, if $\partial(\vartheta,u) = \vartheta - u$ then, above integral inequalities reduce to new ones.

Example 3.3. We consider $\hbar_1(\tau) = \tau$, $\hbar_2(\tau) \equiv 1$, for $\tau \in [0,1]$ and the fuzzy-IVF $\mathcal{F} : [u, u+\partial(\vartheta,u)] = [0, \partial(2,0)] \to \mathbb{F}_C(\mathbb{R})$ defined by,

$$\mathcal{F}(x)(\sigma) = \begin{cases} \frac{\sigma}{2x^2} & \sigma \in [0, 2x^2] \\ \frac{4x^2-\sigma}{2x^2} & \sigma \in (2x^2, 4x^2] \\ 0 & \text{otherwise,} \end{cases}$$

Then, for each $\gamma \in [0,1]$, we have $\mathcal{F}_\gamma(x) = \left[2\gamma x^2, (4-2\gamma)x^2\right]$. Since end point functions $\mathcal{F}_*(x,\gamma) = 2\gamma x^2$, $\mathcal{F}^*(x,\gamma) = (4-2\gamma)x^2$ are $(\hbar_1,\hbar_2)$-preinvex functions with respect to $\partial(\vartheta,u) = \vartheta - u$ for each $\gamma \in [0,1]$ then, $\mathcal{F}(x)$ is $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVF with respect to $\partial(\vartheta,u) = \vartheta - u$. Now we compute the following:

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right) \le \frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}_*(x,\gamma)\,dx$$
$$\le \left[\mathcal{F}_*(u,\gamma) + \mathcal{F}_*(\vartheta,\gamma)\right]\int_0^1 \hbar_1(\tau)\,\hbar_2(1-\tau)\,d\tau.$$

Now

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right) = \mathcal{F}_*(1,\gamma) = 2\gamma,$$

$$\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}_*(x,\gamma)\,dx = \frac{1}{2}\int_0^2 2\gamma x^2 dx = \frac{8\gamma}{3},$$

$$\left[\mathcal{F}_*(u,\gamma) + \mathcal{F}_*(\vartheta,\gamma)\right]\int_0^1 \hbar_1(\tau)\,\hbar_2(1-\tau)\,d\tau = 4\gamma,$$

for all $\gamma \in [0,1]$. That means

$$2\gamma \le \frac{8\gamma}{3} \le 4\gamma.$$

Similarly, it can be easily show that

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right) \le \frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}^*(x,\gamma)\,dx$$
$$\le \left[\mathcal{F}^*(u,\gamma) + \mathcal{F}^*(\vartheta,\gamma)\right]\int_0^1 \hbar_1(\tau)\,\hbar_2(1-\tau)\,d\tau.$$

for all $\gamma \in [0,1]$, such that

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right) = \mathcal{F}_*(1,\gamma) = 2(2-\gamma),$$
$$\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}^*(x,\gamma)\,dx = \frac{1}{2}\int_0^2 (4-2\gamma)x^2 dx = \frac{8(2-\gamma)}{3},$$

$$\left[\mathcal{F}^*(u,\gamma) + \mathcal{F}^*(\vartheta,\gamma)\right]\int_0^1 \hbar_1(\tau)\,\hbar_2(1-\tau)\,d\tau$$
$$= 4(2-\gamma),$$

From which, it follows that

$$(4-2\gamma) \le \frac{4(4-2\gamma)}{3} \le 2(4-2\gamma),$$

i.e.,

$$[2\gamma, 2(2-\gamma)] \le_I \left[\frac{8\gamma}{3}, \frac{8(2-\gamma)}{3}\right] \le_I [4\gamma, 4(2-1\gamma)], \text{ for all } \gamma \in [0,1].$$

Hence, the Theorem 3.1 is verified.

Theorem 3.4. *Let $\mathcal{F},\mathcal{J} : [u, u+\partial(\vartheta,u)] \to \mathbb{F}_C(\mathbb{R})$ be two $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVFs with nonnegative real-valued functions $\hbar_1,\hbar_2 : [0,1] \to \mathbb{R}$, whose γ-levels define the family of IVFs $\mathcal{F}_\gamma,\mathcal{J}_\gamma : [u,u+\partial(\vartheta,u)] \subset \mathbb{R} \to \mathcal{K}_C^+$ are given by $\mathcal{F}_\gamma(x) = \left[\mathcal{F}_*(x,\gamma), \mathcal{F}^*(x,\gamma)\right]$ and $\mathcal{J}_\gamma(x) = \left[\mathcal{J}_*(x,\gamma), \mathcal{J}^*(x,\gamma)\right]$ for all $x \in [u,u+\partial(\vartheta,u)]$ and for all $\gamma \in [0,1]$. If $\mathcal{F},\mathcal{J}$ and $\mathcal{F}\mathcal{J} \in \mathcal{IR}_{([u,u+\partial(\vartheta,u)],\gamma)}$ then,*

$$\frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)}\mathcal{F}(x)\tilde{\times}\mathcal{J}(x)\,dx$$
$$\preccurlyeq \mathcal{M}(u,\vartheta)\int_0^1\left[\hbar_1(\tau)\,\hbar_2(1-\tau)\right]^2 d\tau$$
$$\tilde{+}\mathcal{N}(u,\vartheta)\int_0^1 \hbar_1(\tau)\,\hbar_2(\tau)\,\hbar_1(1-\tau)\,\hbar_2(1-\tau)\,d\tau,$$

where $\mathcal{M}(u,\vartheta) = \mathcal{F}(u)\tilde{\times}\mathcal{J}(u)\tilde{+}\mathcal{F}(\vartheta)\tilde{\times}\mathcal{J}(\vartheta)$, $\mathcal{N}(u,\vartheta) = \mathcal{F}(u)\tilde{\times}\mathcal{J}(\vartheta)\tilde{+}\mathcal{F}(\vartheta)\tilde{\times}\mathcal{J}(u)$ with $\mathcal{M}_\gamma(u,\vartheta) = \left[\mathcal{M}_*((u,\vartheta),\gamma), \mathcal{M}^*((u,\vartheta),\gamma)\right]$ and $\mathcal{N}_\gamma(u,\vartheta) = \left[\mathcal{N}_*((u,\vartheta),\gamma), \mathcal{N}^*((u,\vartheta),\gamma)\right]$.

Proof. Since $\mathcal{F}$ and $\mathcal{J}$ both are $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVFs on $[u,u+\partial(\vartheta,u)]$ then, for each $\gamma \in [0,1]$ we have

$$\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)$$
$$\le \hbar_1(\tau)\,\hbar_2(1-\tau)\,\mathcal{F}_*(u,\gamma) + \hbar_1(1-\tau)\,\hbar_2(\tau)\,\mathcal{F}_*(\vartheta,\gamma),$$
$$\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)$$
$$\le \hbar_1(\tau)\,\hbar_2(1-\tau)\,\mathcal{F}^*(u,\gamma) + \hbar_1(1-\tau)\,\hbar_2(\tau)\,\mathcal{F}^*(\vartheta,\gamma).$$

And

$$\mathcal{J}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)$$
$$\le \hbar_1(\tau)\,\hbar_2(1-\tau)\,\mathcal{J}_*(u,\gamma) + \hbar_1(1-\tau)\,\hbar_2(\tau)\,\mathcal{J}_*(\vartheta,\gamma),$$
$$\mathcal{J}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)$$
$$\le \hbar_1(\tau)\,\hbar_2(1-\tau)\,\mathcal{J}^*(u,\gamma) + \hbar_1(1-\tau)\,\hbar_2(\tau)\,\mathcal{J}^*(\vartheta,\gamma).$$

From the definition of $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVFs it follows that $\mathcal{F}(x) \succcurlyeq \tilde{0}$ and $\mathcal{J}(x) \succcurlyeq \tilde{0}$, so

$$\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma) \times \mathcal{J}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)$$
$$\le \left(\hbar_1(\tau)\,\hbar_2(1-\tau)\,\mathcal{F}_*(u,\gamma) + \hbar_1(1-\tau)\,\hbar_2(\tau)\,\mathcal{F}_*(\vartheta,\gamma)\right)$$
$$\left(\hbar_1(\tau)\,\hbar_2(1-\tau)\,\mathcal{J}_*(u,\gamma) + \hbar_1(1-\tau)\,\hbar_2(\tau)\,\mathcal{J}_*(\vartheta,\gamma)\right)$$
$$= \mathcal{F}_*(u,\gamma)\times\mathcal{J}_*(u,\gamma)\left[\hbar_1(\tau)\,\hbar_2(1-\tau)\right]^2$$
$$+\mathcal{F}_*(\vartheta,\gamma)\times\mathcal{J}_*(\vartheta,\gamma)\left[\hbar_1(\tau)\,\hbar_2(1-\tau)\right]^2$$
$$+\mathcal{F}_*(u,\gamma)\times\mathcal{J}_*(\vartheta,\gamma)\,\hbar_1(\tau)\,\hbar_2(\tau)\,\hbar_1(1-\tau)\,\hbar_2(1-\tau)$$
$$+\mathcal{F}_*(\vartheta,\gamma)\times\mathcal{J}_*(u,\gamma)\,\hbar_1(\tau)\,\hbar_2(\tau)\,\hbar_1(1-\tau)\,\hbar_2(1-\tau),$$

$$\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\times\mathcal{J}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)$$
$$\leq\left(\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}^*(u,\gamma)+\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}^*(\vartheta,\gamma)\right)$$
$$\left(\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{J}^*(u,\gamma)+\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{J}^*(\vartheta,\gamma)\right)$$
$$=\mathcal{F}^*(u,\gamma)\times\mathcal{J}^*(u,\gamma)\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2$$
$$+\mathcal{F}^*(\vartheta,\gamma)\times\mathcal{J}^*(\vartheta,\gamma)\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2$$
$$+\mathcal{F}^*(u,\gamma)\times\mathcal{J}^*(\vartheta,\gamma)\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)$$
$$+\mathcal{F}^*(\vartheta,\gamma)\times\mathcal{J}^*(u,\gamma)\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau),$$

Integrating both sides of above inequality over [0, 1] we get

$$\int_0^1\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\times\mathcal{J}_*(u+(1-\tau)\partial(\vartheta,u)\gamma)$$
$$=\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}_*(x,\gamma)\times\mathcal{J}_*(x,\gamma)\,dx$$
$$\leq\begin{pmatrix}\mathcal{F}_*(u,\gamma)\times\mathcal{J}_*(u,\gamma)\\+\mathcal{F}_*(\vartheta,\gamma)\times\mathcal{J}_*(\vartheta,\gamma)\end{pmatrix}\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau$$
$$+\left(\mathcal{F}_*(u,\gamma)\times\mathcal{J}_*(\vartheta,\gamma)+\mathcal{F}_*(\vartheta,\gamma)\times\mathcal{J}_*(u,\gamma)\right)$$
$$.\int_0^1\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)\,d\tau,$$

$$\int_0^1\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\times\mathcal{J}^*(u+(1-\tau)\partial(\vartheta,u)\gamma)$$
$$=\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}^*(x,\gamma)\times\mathcal{J}^*(x,\gamma)\,dx$$
$$\leq\begin{pmatrix}\mathcal{F}^*(u,\gamma)\times\mathcal{J}^*(u,\gamma)\\+\mathcal{F}^*(\vartheta,\gamma)\times\mathcal{J}^*(\vartheta,\gamma)\end{pmatrix}\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau$$
$$+\left(\mathcal{F}^*(u,\gamma)\times\mathcal{J}^*(\vartheta,\gamma)+\mathcal{F}^*(\vartheta,\gamma)\times\mathcal{J}^*(u,\gamma)\right)$$
$$.\int_0^1\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)\,d\tau,$$

It follows that,

$$\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}_*(x,\gamma)\times\mathcal{J}_*(x,\gamma)\,dx$$
$$\leq\mathcal{M}_*((u,\vartheta),\gamma)\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau$$
$$+\mathcal{N}_*((u,\vartheta),\gamma)\int_0^1\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)\,d\tau,$$

$$\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}^*(x,\gamma)\times\mathcal{J}^*(x,\gamma)\,dx$$
$$\leq\mathcal{M}^*((u,\vartheta),\gamma)\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau$$
$$+\mathcal{N}^*((u,\vartheta),\gamma)\int_0^1\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)\,d\tau,$$

i.e.,

$$\frac{1}{\partial(\vartheta,u)}\left[\int_u^{u+\partial(\vartheta,u)}\begin{pmatrix}\mathcal{F}_*(x,\gamma)\times\\\mathcal{J}_*(x,\gamma)\,dx\end{pmatrix},\int_u^{u+\partial(\vartheta,u)}\begin{pmatrix}\mathcal{F}^*(x,\gamma)\\\times\mathcal{J}^*(x,\gamma)\end{pmatrix}dx\right]$$
$$\leq_I\left[\mathcal{M}_*((u,\vartheta),\gamma),\mathcal{M}^*((u,\vartheta),\gamma)\right]\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau$$
$$+\left[\mathcal{N}_*((u,\vartheta),\gamma),\mathcal{N}^*((u,\vartheta),\gamma)\right]$$
$$\int_0^1\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)\,d\tau.$$

Thus,

$$\frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)}\mathcal{F}(x)\tilde{\times}\mathcal{J}(x)\,dx$$
$$\preccurlyeq\mathcal{M}(u,\vartheta)\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau$$
$$\tilde{+}\mathcal{N}(u,\vartheta)\int_0^1\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)\,d\tau,$$

and the theorem has been established.

Following assumption is required to prove next result regarding the bi-function $\partial : K\times K\to\mathbb{R}$ which is known as

Condition C. Let K be an invex set with respect to ∂. For any $u,\vartheta\in K$ and $\tau\in[0,1]$,

$$\partial(\vartheta,u+\tau\partial(\vartheta,u))=(1-\tau)\,\partial(\vartheta,u),$$

$$\partial(u,u+\tau\xi(\vartheta,u))=-\tau\partial(\vartheta,u).$$

Clearly for $\tau=0$, we have $\partial(\vartheta,u)=0$ if and only if, $\vartheta=u$, for all $u,\vartheta\in K$. For the applications of Condition C, see [12,26,31,46,51].

Theorem 3.5. *Let $\mathcal{F},\mathcal{J} : [u,u+\partial(\vartheta,u)]\to\mathbb{F}_C(\mathbb{R})$ be two $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVFs with nonnegative real-valued functions $\hbar_1,\hbar_2 : [0,1]\to\mathbb{R}$ and $\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)\neq 0$, whose γ-levels define the family of IVFs $\mathcal{F}_\gamma,\mathcal{J}_\gamma : [u,u+\partial(\vartheta,u)]\subset\mathbb{R}\to\mathcal{K}_C^{\,+}$ are given by $\mathcal{F}_\gamma(x)=\left[\mathcal{F}_*(x,\gamma),\mathcal{F}^*(x,\gamma)\right]$ and $\mathcal{J}_\gamma(x)=\left[\mathcal{J}_*(x,\gamma),\mathcal{J}^*(x,\gamma)\right]$ for all $x\in[u,u+\partial(\vartheta,u)]$ and for all $\gamma\in[0,1]$. If $\mathcal{F},\mathcal{J}$ and $\mathcal{F}\mathcal{J}\in\mathcal{IR}_{([u,u+\partial(\vartheta,u)],\gamma)}$, and Condition C hold for ∂ then,*

$$\frac{1}{2\left[\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)\right]^2}\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right)\tilde{\times}\mathcal{J}\left(\frac{2u+\partial(\vartheta,u)}{2}\right)$$
$$\preccurlyeq\frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)}\mathcal{F}(x)\tilde{\times}\mathcal{J}(x)\,dx$$
$$\tilde{+}\mathcal{M}(u,\vartheta)\int_0^1\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)\,d\tau$$
$$\tilde{+}\mathcal{N}(u,\vartheta)\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau,$$

where $\mathcal{M}(u,\vartheta)=\mathcal{F}(u)\tilde{\times}\mathcal{J}(u)\tilde{+}\mathcal{F}(\vartheta)\tilde{\times}\mathcal{J}(\vartheta)$, $\mathcal{N}(u,\vartheta)=\mathcal{F}(u)\tilde{\times}\mathcal{J}(\vartheta)\tilde{+}\mathcal{F}(\vartheta)\tilde{\times}\mathcal{J}(u)$, and $\mathcal{M}_\gamma(u,\vartheta)=\left[\mathcal{M}_*((u,\vartheta),\gamma),\right.$ $\left.\mathcal{M}^*((u,\vartheta),\gamma)\right]$ and $\mathcal{N}_\gamma(u,\vartheta)=\left[\mathcal{N}_*((u,\vartheta),\gamma),\mathcal{N}^*((u,\vartheta),\gamma)\right]$.

Proof. Using Condition C, we can write

$$u+\tfrac{1}{2}\partial(\vartheta,u)=u+\tau\partial(\vartheta,u)+\tfrac{1}{2}\partial\left(u+(1-\tau)\partial(\vartheta,u),u+\tau\partial(\vartheta,u)\right).$$

By hypothesis, for each $\gamma\in[0,1]$, we have

$$\mathcal{F}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)\times\mathcal{J}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)$$

$$\mathcal{F}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)\times\mathcal{J}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)$$

$$=\mathcal{F}_*\left(u+\tau\partial(\vartheta,u)+\tfrac{1}{2}\partial\begin{pmatrix}u+(1-\tau)\partial(\vartheta,u),\\ u+\tau\partial(\vartheta,u)\end{pmatrix},\gamma\right)\times\mathcal{J}_*\left(u+\tau\partial(\vartheta,u)+\tfrac{1}{2}\partial\begin{pmatrix}u+(1-\tau)\partial(\vartheta,u),\\ u+\tau\partial(\vartheta,u)\end{pmatrix},\gamma\right),$$

$$=\mathcal{F}^*\left(u+\tau\partial(\vartheta,u)+\tfrac{1}{2}\partial\begin{pmatrix}u+(1-\tau)\partial(\vartheta,u),\\ u+\tau\partial(\vartheta,u)\end{pmatrix},\gamma\right)\times\mathcal{J}^*\left(u+\tau\partial(\vartheta,u)+\tfrac{1}{2}\partial\begin{pmatrix}u+(1-\tau)\partial(\vartheta,u),\\ u+\tau\partial(\vartheta,u)\end{pmatrix},\gamma\right),$$

$$\leq\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ +\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}_*(u+\tau\partial(\vartheta,u),\gamma)\end{bmatrix}+\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ +\mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}_*(u+\tau\partial(\vartheta,u),\gamma)\end{bmatrix},$$

$$\leq\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ +\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}^*(u+\tau\partial(\vartheta,u),\gamma)\end{bmatrix}+\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ +\mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}^*(u+\tau\partial(\vartheta,u),\gamma)\end{bmatrix},$$

$$\leq\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ +\mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}_*(u+\tau\partial(\vartheta,u),\gamma)\end{bmatrix}+\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\left(\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}_*(u,\gamma)+\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}_*(\vartheta,\gamma)\right)\\ \times\left(\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{J}_*(u,\gamma)+\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{J}_*(\vartheta,\gamma)\right)\\ +\left(\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}_*(u,\gamma)+\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}_*(\vartheta,\gamma)\right)\\ \times\left(\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{J}_*(u,\gamma)+\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{J}_*(\vartheta,\gamma)\right)\end{bmatrix},$$

$$\leq\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ +\mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}^*(u+\tau\partial(\vartheta,u),\gamma)\end{bmatrix}+\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\left(\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}^*(u,\gamma)+\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}^*(\vartheta,\gamma)\right)\\ \times\left(\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{J}^*(u,\gamma)+\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{J}^*(\vartheta,\gamma)\right)\\ +\left(\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}^*(u,\gamma)+\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}^*(\vartheta,\gamma)\right)\\ \times\left(\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{J}^*(u,\gamma)+\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{J}^*(\vartheta,\gamma)\right)\end{bmatrix},$$

$$=\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ +\mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}_*(u+\tau\partial(\vartheta,u),\gamma)\end{bmatrix}+2\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau).\mathcal{M}_*((u,\vartheta),\gamma)\\ +\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2\mathcal{N}_*((u,\vartheta),\gamma)\end{bmatrix},$$

$$=\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\\ +\mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma)\\ \times\mathcal{J}^*(u+\tau\partial(\vartheta,u),\gamma)\end{bmatrix}+2\left[\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\right]^2\begin{bmatrix}\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau).\mathcal{M}^*((u,\vartheta),\gamma)\\ +\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2\mathcal{N}^*((u,\vartheta),\gamma)\end{bmatrix},$$

integrating over $[0,1]$, we have

$$\frac{1}{2\left[\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)\right]^2}\mathcal{F}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)\times\mathcal{J}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)$$
$$\leq\frac{1}{\partial(\vartheta,u)}\int_0^{u+\partial(\vartheta,u)}\mathcal{F}_*(x,\gamma)\times\mathcal{J}_*(x,\gamma)\,dx$$
$$+\mathcal{M}_*((u,\vartheta),\gamma)\int_0^1\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)\,d\tau$$
$$+\mathcal{N}_*((u,\vartheta),\gamma)\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau,$$
$$\frac{1}{2\left[\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)\right]^2}\mathcal{F}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)\times\mathcal{J}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)$$
$$\leq\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}^*(x,\gamma)\times\mathcal{J}^*(x,\gamma)\,dx$$
$$+\mathcal{M}^*((u,\vartheta),\gamma)\int_0^1\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)\,d\tau$$
$$+\mathcal{N}^*((u,\vartheta),\gamma)\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau,$$

from which, we have

$$\frac{1}{2\left[\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)\right]^2}\left[\begin{matrix}\mathcal{F}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)\\ \times\mathcal{J}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)\end{matrix},\begin{matrix}\mathcal{F}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)\\ \times\mathcal{J}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)\end{matrix}\right]$$
$$\leq\frac{1}{l\,\partial(\vartheta,u)}\left[\int_u^{u+\partial(\vartheta,u)}\begin{matrix}\mathcal{F}_*(x,\gamma)\\ \times\mathcal{J}_*(x,\gamma)\,dx,\end{matrix}\int_u^{u+\partial(\vartheta,u)}\begin{matrix}\mathcal{F}^*(x,\gamma)\\ \times\mathcal{J}^*(x,\gamma)\,dx\end{matrix}\right]$$
$$+\int_0^1\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)\,d\tau$$
$$\left[\mathcal{M}_*((u,\vartheta),\gamma),\mathcal{M}^*((u,\vartheta),\gamma)\right]$$
$$+\left[\mathcal{N}_*((u,\vartheta),\gamma),\mathcal{N}^*((u,\vartheta),\gamma)\right]\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau,$$

i.e.,

$$\frac{1}{2\left[\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)\right]^2}\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right)\tilde{\times}\mathcal{J}\left(\frac{2u+\partial(\vartheta,u)}{2}\right)$$
$$\preccurlyeq\frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)}\mathcal{F}(x)\tilde{\times}\mathcal{J}(x)\,dx$$
$$\tilde{+}\mathcal{M}(u,\vartheta)\int_0^1\hbar_1(\tau)\hbar_2(\tau)\hbar_1(1-\tau)\hbar_2(1-\tau)\,d\tau$$
$$\tilde{+}\mathcal{N}(u,\vartheta)\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau,$$

hence, the required result.

Remark 3.6. If $\hbar_2(\tau)\equiv 1$, $\tau\in[0,1]$ then, above theorems reduces for $\hbar_1$-preinvex fuzzy-IVFs.

If $\hbar_1(\tau)=\tau$ and $\hbar_2(\tau)\equiv 1$, $\tau\in[0,1]$ then, above theorems reduces for preinvex fuzzy-IVFs.

If in the above theorem $\mathcal{F}_*(u,\gamma)=\mathcal{F}^*(u,\gamma)$ with $\gamma=1$ then, we obtain the appropriate theorems for $(\hbar_1,\hbar_2)$-preinvex functions, see [31].

If in the above theorem $\mathcal{F}_*(u,\gamma)=\mathcal{F}^*(u,\gamma)$ with $\gamma=1$ and $\hbar_2(\tau)\equiv 1$, $\tau\in[0,1]$ then, we obtain the appropriate theorems for $\hbar_1$-preinvex functions, see [30].

If in the above theorems $\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)=\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)$ with $\gamma=1$, $\partial(\vartheta,u)=\vartheta-u$, $\hbar_1(\tau)=\tau$ and $\hbar_2(\tau)\equiv 1$, $\tau\in[0,1]$ then, we obtain the appropriate theorems for convex functions, see [56].

If in the above theorems $\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)=\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)$ with $\gamma=1$, $\partial(\vartheta,u)=\vartheta-u$, $\hbar_1(\tau)=\tau^s$ and $\hbar_2(\tau)\equiv 1$, $\tau\in[0,1]$, $s\in[0,1]$ then, we obtain the appropriate theorems for s-convex functions in the second sense, see [57].

Example 3.7. We consider $\hbar_1(\tau)=\tau$, $\hbar_2(\tau)\equiv 1$, for $\tau\in[0,1]$, and the fuzzy-IVFs $\mathcal{F},\mathcal{J}:[u,u+\partial(\vartheta,u)]=[0,\partial(1,0)]\to\mathbb{F}_C(\mathbb{R})$ defined by,

$$\mathcal{F}(x)(\sigma)=\begin{cases}\frac{\sigma}{2x^2}, & \sigma\in\left[0,2x^2\right],\\ \frac{4x^2-\sigma}{2x^2}, & \sigma\in\left(2x^2,4x^2\right),\\ 0, & otherwise,\end{cases}$$

$$\mathcal{J}(x)(\sigma)=\begin{cases}\frac{\sigma}{x} & \sigma\in[0,x]\\ \frac{2x-\sigma}{x} & \sigma\in(x,2x]\\ 0 & otherwise,\end{cases}$$

Then, for each $\gamma\in[0,1]$, we have $\mathcal{F}_\gamma(x)=\left[2\gamma x^2,(4-2\gamma)x^2\right]$ and $\mathcal{J}_\gamma(x)=[\gamma x,(2-\gamma)x]$. Since end point functions $\mathcal{F}_*(x,\gamma)=2\gamma x^2$ and $\mathcal{F}^*(x,\gamma)=(4-2\gamma)x^2$ both are $(\hbar_1,\hbar_2)$-preinvex functions, and $\mathcal{J}_*(x,\gamma)=\gamma x$, and $\mathcal{J}^*(x,\gamma)=(2-\gamma)x$ both are also $(\hbar_1,\hbar_2)$ preinvex functions with respect to same $\partial(\vartheta,u)=\vartheta-u$, for each $\gamma\in[0,1]$ then, $\mathcal{F}$ and $\mathcal{J}$ both are $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVFs, respectively. Since $\mathcal{F}_*(x,\gamma)=2\gamma x^2$ and $\mathcal{F}^*(x,\gamma)=(4-2\gamma)x^2$, and $\mathcal{J}_*(x,\gamma)=\gamma x$, and $\mathcal{J}^*(x,\gamma)=(2-\gamma)x$ then,

$$\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}_*(x,\gamma)\times\mathcal{J}_*(x,\gamma)\,dx$$
$$=\int_0^1\left(2\gamma x^2\right)(\gamma x)\,dx=\frac{\gamma^2}{2},$$
$$\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}^*(x,\gamma)\times\mathcal{J}^*(x,\gamma)\,dx$$
$$=\int_0^1\left((4-2\gamma)u^2\right)((2-\gamma)u)\,dx=\frac{(2-\gamma)^2}{2},$$

$$\mathcal{M}_*((u,\vartheta),\gamma)\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau=\frac{2\gamma^2}{3},$$
$$\mathcal{M}^*((u,\vartheta),\gamma)\int_0^1\left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2d\tau=\frac{2(2-\gamma)^2}{3},$$

$$\mathcal{N}_*((u,\vartheta),\gamma)\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\hbar_1(1-\tau)\hbar_2(\tau)\,d\tau = 0$$
$$\mathcal{N}^*((u,\vartheta),\gamma)\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\hbar_1(1-\tau)\hbar_2(\tau)\,d\tau = 0,$$

for each $\gamma \in [0,1]$, that means

$$\frac{\gamma^2}{2} \le \frac{2\gamma^2}{3} + 0 = \frac{2\gamma^2}{3},$$
$$\frac{(2-\gamma)^2}{2} \le \frac{2(2-\gamma)^2}{3} + 0 = \frac{2(2-\gamma)^2}{3},$$

Hence, Theorem 3.4 is verified.

For Theorem 3.5, we have

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)\times\mathcal{J}_*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right) = \frac{\gamma^2}{2},$$
$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\mathcal{F}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right)\times\mathcal{J}^*\left(\frac{2u+\partial(\vartheta,u)}{2},\gamma\right) = \frac{(2-\gamma)^2}{2},$$

$$\mathcal{M}_*((u,\vartheta),\gamma)\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\hbar_1(1-\tau)\hbar_2(\tau)\,d\tau = \frac{\gamma^2}{3},$$
$$\mathcal{M}^*((u,\vartheta),\gamma)\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\hbar_1(1-\tau)\hbar_2(\tau)\,d\tau = \frac{(2-\gamma)^2}{3},$$

$$\mathcal{N}_*((u,\vartheta),\gamma)\int_0^1 \left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2 d\tau = 0,$$
$$\mathcal{N}^*((u,\vartheta),\gamma)\int_0^1 \left[\hbar_1(\tau)\hbar_2(1-\tau)\right]^2 d\tau = 0,$$

for each $\gamma \in [0,1]$, that means

$$\frac{\gamma^2}{2} \le \frac{\gamma^2}{2} + 0 + \frac{\gamma^2}{3} = \frac{5\gamma^2}{6},$$
$$\frac{(2-\gamma)^2}{2} \le \frac{(2-\gamma)^2}{2} + 0 + \frac{(2-\gamma)^2}{3} = \frac{5(2-\gamma)^2}{6},$$

hence, Theorem 3.5 is demonstrated.

Theorem 3.8. *(The second HH-Fejér inequality for $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVFs). Let $\mathcal{F} : [u, u+\partial(\vartheta,u)] \to \mathbb{F}_C(\mathbb{R})$ be a $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVF with $u < u+\partial(\vartheta,u)$ and nonnegative real-valued functions $\hbar_1,\hbar_2 : [0,1] \to \mathbb{R}$, whose γ-levels define the family of IVFs $\mathcal{F}_\gamma : [u, u+\partial(\vartheta,u)] \subset \mathbb{R} \to \mathcal{K}_C^+$ are given by $\mathcal{F}_\gamma(x) = \left[\mathcal{F}_*(x,\gamma), \mathcal{F}^*(x,\gamma)\right]$ for all $x \in [u, u+\partial(\vartheta,u)]$ and for all $\gamma \in [0,1]$. If $\mathcal{F} \in \mathcal{FR}_{([u,u+\partial(\vartheta,u)],\gamma)}$ and $\Omega : [u, u+\partial(\vartheta,u)] \to \mathbb{R}$, $\Omega(x) \ge 0$, symmetric with respect to $u + \frac{1}{2}\partial(\vartheta,u)$ then,*

$$\frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)\Omega(x)\,dx \preccurlyeq \left[\begin{array}{c} F(u) \\ \tilde{+}F(\vartheta)\end{array}\right]\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\Omega\left(u+\tau\partial(\vartheta,u)\right)d\tau. \tag{22}$$

Proof. Let $\mathcal{F}$ be a $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVF. Then, for each $\gamma \in [0,1]$, we have

$$\begin{aligned}
&\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\,\Omega(u+(1-\tau)\partial(\vartheta,u)) \\
&\le \left(\begin{array}{c}\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}_*(u,\gamma) \\ +\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}_*(\vartheta,\gamma)\end{array}\right)\Omega(u+(1-\tau)\partial(\vartheta,u)), \\
&\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\,\Omega(u+(1-\tau)\partial(\vartheta,u)) \\
&\le \left(\begin{array}{c}\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}^*(u,\gamma) \\ +\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}^*(\vartheta,\gamma)\end{array}\right)\Omega(u+(1-\tau)\partial(\vartheta,u)).
\end{aligned} \tag{23}$$

And

$$\begin{aligned}
&\mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma)\,\Omega(u+\tau\partial(\vartheta,u)) \\
&\le \left(\begin{array}{c}\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}_*(u,\gamma) \\ +\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}_*(\vartheta,\gamma)\end{array}\right)\Omega(u+\tau\partial(\vartheta,u)), \\
&\mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma)\,\Omega(u+\tau\partial(\vartheta,u)) \\
&\le \left(\begin{array}{c}\hbar_1(1-\tau)\hbar_2(\tau)\mathcal{F}^*(u,\gamma) \\ +\hbar_1(\tau)\hbar_2(1-\tau)\mathcal{F}^*(\vartheta,\gamma)\end{array}\right)\Omega(u+\tau\partial(\vartheta,u)).
\end{aligned} \tag{24}$$

After adding (23) and (24), and integrating over $[0,1]$, we get

$$\begin{aligned}
&\int_0^1 \begin{array}{c}\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma) \\ .\Omega\left(u+(1-\tau)\partial(\vartheta,u)\right)\end{array} d\tau + \int_0^1 \begin{array}{c}\mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma) \\ \Omega(u+\tau\partial(\vartheta,u))\end{array} d\tau \\
&\le \int_0^1 \left[\begin{array}{c}\mathcal{F}_*(u,\gamma)\left\{\begin{array}{c}\hbar_1(\tau)\hbar_2(1-\tau) \\ .\Omega\left(u+(1-\tau)\partial(\vartheta,u)\right) \\ +\hbar_1(1-\tau)\hbar_2(\tau) \\ .\Omega(u+\tau\partial(\vartheta,u))\end{array}\right\} \\ +\mathcal{F}_*(\vartheta,\gamma)\left\{\begin{array}{c}\hbar_1(1-\tau)\hbar_2(\tau) \\ .\Omega\left(u+(1-\tau)\partial(\vartheta,u)\right) \\ +\hbar_1(\tau)\hbar_2(1-\tau) \\ .\Omega(u+\tau\partial(\vartheta,u))\end{array}\right\}\end{array}\right] dt,
\end{aligned}$$

$$\int_0^1 \frac{\mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma)}{.\Omega(u+\tau\partial(\vartheta,u))}d\tau + \int_0^1 \frac{\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)}{.\Omega\left(u+(1-\tau)\partial(\vartheta,u)\right)}d\tau$$

$$\le \int_0^1 \left[\begin{array}{c} \mathcal{F}^*(u,\gamma)\left\{\begin{array}{c} \hbar_1(\tau)\hbar_2(1-\tau) \\ .\Omega\left(u+(1-\tau)\partial(\vartheta,u)\right) \\ +\hbar_1(1-\tau)\hbar_2(\tau) \\ .\Omega(u+\tau\partial(\vartheta,u)) \end{array}\right\} \\ +\mathcal{F}^*(\vartheta,\gamma)\left\{\begin{array}{c} \hbar_1(1-\tau)\hbar_2(\tau) \\ .\Omega\left(u+(1-\tau)\partial(\vartheta,u)\right) \\ +\hbar_1(\tau)\hbar_2(1-\tau) \\ .\Omega(u+\tau\partial(\vartheta,u)) \end{array}\right\} \end{array}\right] dt.$$

$$= 2\mathcal{F}_*(u,\gamma)\int_0^1 \frac{\hbar_1(\tau)\hbar_2(1-\tau)}{\Omega\left(u+(1-\tau)\partial(\vartheta,u)\right)}dt + 2\mathcal{F}_*(\vartheta,\gamma)\int_0^1 \frac{\hbar_1(\tau)\hbar_2(1-\tau)}{\Omega(u+\tau\partial(\vartheta,u))}dt,$$

$$= 2\mathcal{F}^*(u,\gamma)\int_0^1 \frac{\hbar_1(\tau)\hbar_2(1-\tau)}{\Omega(u+(1-\tau)\partial(\vartheta,u))}dt + 2\mathcal{F}^*(\vartheta,\gamma)\int_0^1 \frac{\hbar_1(\tau)\hbar_2(1-\tau)}{\Omega(u+\tau\partial(\vartheta,u))}dt.$$

Since Ω is symmetric then,

$$= 2\left[\begin{array}{c}\mathcal{F}_*(u,\gamma) \\ +\mathcal{F}_*(\vartheta,\gamma)\end{array}\right]\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\Omega\left(\begin{array}{c}u \\ +\tau\partial(\vartheta,u)\end{array}\right)dt,$$
$$= 2\left[\begin{array}{c}\mathcal{F}^*(u,\gamma) \\ +\mathcal{F}^*(\vartheta,\gamma)\end{array}\right]\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\Omega\left(\begin{array}{c}u \\ +\tau\partial(\vartheta,u)\end{array}\right)dt. \quad (25)$$

Since

$$\int_0^1 \mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\Omega\left(\begin{array}{c}u+ \\ (1-\tau)\partial(\vartheta,u)\end{array}\right)d\tau$$
$$= \int_0^1 \mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma)\Omega(u+\tau\partial(\vartheta,u))d\tau$$
$$= \frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)} \mathcal{F}_*(x,\gamma)\Omega(x)dx,$$
$$\int_0^1 \mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma)\Omega(u+\tau\partial(\vartheta,u))d\tau \quad (26)$$
$$= \int_0^1 \mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma)\Omega\left(\begin{array}{c}u+ \\ (1-\tau)\partial(\vartheta,u)\end{array}\right)d\tau$$
$$= \frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)} \mathcal{F}^*(x,\gamma)\Omega(x)dx.$$

From (25) and (26), we have

$$\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)} \mathcal{F}_*(x,\gamma)\Omega(x)dx \le \left[\begin{array}{c}\mathcal{F}_*(u,\gamma) \\ +\mathcal{F}_*(\vartheta,\gamma)\end{array}\right]\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\Omega\left(\begin{array}{c}u \\ +\tau\partial(\vartheta,u)\end{array}\right)dt,$$
$$\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)} \mathcal{F}^*(x,\gamma)\Omega(x)dx \le \left[\begin{array}{c}\mathcal{F}^*(u,\gamma) \\ +\mathcal{F}^*(\vartheta,\gamma)\end{array}\right]\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\Omega\left(\begin{array}{c}u \\ +\tau\partial(\vartheta,u)\end{array}\right)dt,$$

i.e.,

$$\left[\frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)} \frac{\mathcal{F}_*(x,\gamma)}{.\Omega(x)}dx, \frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)} \frac{\mathcal{F}^*(x,\gamma)}{.\Omega(x)}dx\right]$$
$$\le_I \left[\mathcal{F}_*(u,\gamma)+\mathcal{F}_*(\vartheta,\gamma), \mathcal{F}^*(u,\gamma)+\mathcal{F}^*(\vartheta,\gamma)\right]$$
$$.\int_0^1 \hbar_1(\tau)\hbar_2(1-\tau)\Omega(u+\tau\partial(\vartheta,u))dt,$$

hence

$$\frac{1}{\partial(\vartheta,u)}(FR)\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)\Omega(x)dx \preccurlyeq \left[\mathcal{F}(u)\tilde{+}\mathcal{F}(\vartheta)\right]\int_0^1 \frac{\hbar_1(\tau)\hbar_2(1-\tau)}{.\Omega(u+\tau\partial(\vartheta,u))d\tau},$$

then, we complete the proof.

Theorem 3.9. *(The first HH-Fejér inequality for $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVFs). Let $\mathcal{F}:[u,u+\partial(\vartheta,u)]\to\mathbb{F}_C(\mathbb{R})$ be a $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVF with $u<u+\partial(\vartheta,u)$ and nonnegative real-valued functions $\hbar_1,\hbar_2:[0,1]\to\mathbb{R}$, whose γ-levels define the family of IVFs $\mathcal{F}_\gamma:[u,u+\partial(\vartheta,u)]\subset\mathbb{R}\to\mathcal{K}_C^+$ are given by $\mathcal{F}_\gamma(x)=\left[\mathcal{F}_*(x,\gamma),\mathcal{F}^*(x,\gamma)\right]$ for all $x\in[u,u+\partial(\vartheta,u)]$ and for all $\gamma\in[0,1]$. If $\mathcal{F}\in\mathcal{FR}_{([u,u+\partial(\vartheta,u)],\gamma)}$ and $\Omega:[u,u+\partial(\vartheta,u)]\to\mathbb{R}, \Omega(x)\ge 0$, symmetric with respect to $u+\frac{1}{2}\partial(\vartheta,u)$, and $\int_u^{u+\partial(\vartheta,u)}\Omega(x)dx>0$, and Condition C holds for ∂ then,*

$$\mathcal{F}\left(u+\tfrac{1}{2}\partial(\vartheta,u)\right) \preccurlyeq \frac{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{\int_u^{u+\partial(\vartheta,u)}\Omega(x)dx}(FR)\int_u^{u+\partial(\vartheta,u)}\mathcal{F}(x)\Omega(x)dx. \quad (27)$$

Proof. Using Condition C, we can write

$$u+\tfrac{1}{2}\partial(\vartheta,u) = u+\tau\partial(\vartheta,u) + \tfrac{1}{2}\partial(u+(1-\tau)\partial(\vartheta,u), u+\tau\partial(\vartheta,u)).$$

Since $\mathcal{F}$ is a $(\hbar_1, \hbar_2)$-preinvex then, for $\gamma \in [0,1]$, we have

$$\begin{aligned}
&\mathcal{F}_*\left(u+\tfrac{1}{2}\partial(\vartheta,u),\gamma\right) \\
&= \mathcal{F}_*\left(u+\tau\partial(\vartheta,u)+\tfrac{1}{2}\partial\left(\begin{array}{c} u+(1-\tau)\partial(\vartheta,u), \\ u+\tau\partial(\vartheta,u)\end{array}\right),\gamma\right) \\
&\le \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\left(\begin{array}{c}\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma) \\ +\mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma)\end{array}\right), \\
&\mathcal{F}^*\left(u+\tfrac{1}{2}\partial(\vartheta,u),\gamma\right) \\
&= \mathcal{F}^*\left(u+\tau\partial(\vartheta,u)+\tfrac{1}{2}\partial\left(\begin{array}{c} u+(1-\tau)\partial(\vartheta,u), \\ u+\tau\partial(\vartheta,u)\end{array}\right),\gamma\right) \\
&\le \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\left(\begin{array}{c}\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma) \\ +\mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma)\end{array}\right),
\end{aligned} \tag{28}$$

By multiplying (28) by $\Omega(u+(1-\tau)\partial(\vartheta,u)) = \Omega(u+\tau\partial(\vartheta,u))$ and integrate it by τ over $[0,1]$, we obtain

$$\begin{aligned}
&\mathcal{F}_*\left(u+\tfrac{1}{2}\partial(\vartheta,u),\gamma\right)\int_0^1 \Omega(u+\tau\partial(\vartheta,u))\,d\tau \\
&\le \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\left(\begin{array}{c}\displaystyle\int_0^1 \begin{array}{c}\mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma) \\ \Omega(u+(1-\tau)\partial(\vartheta,u))\end{array} d\tau \\ \displaystyle +\int_0^1 \begin{array}{c}\mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma) \\ \Omega(u+\tau\partial(\vartheta,u))\end{array} d\tau\end{array}\right), \\
&\mathcal{F}^*\left(u+\tfrac{1}{2}\partial(\vartheta,u),\gamma\right)\int_0^1 \Omega(u+\tau\partial(\vartheta,u))\,d\tau \\
&\le \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\left(\begin{array}{c}\displaystyle\int_0^1 \begin{array}{c}\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma) \\ \Omega(u+(1-\tau)\partial(\vartheta,u))\end{array} d\tau \\ \displaystyle +\int_0^1 \begin{array}{c}\mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma) \\ .\Omega(u+\tau\partial(\vartheta,u))\end{array} d\tau\end{array}\right),
\end{aligned} \tag{29}$$

Since

$$\begin{aligned}
&\int_0^1 \mathcal{F}_*(u+(1-\tau)\partial(\vartheta,u),\gamma)\,\Omega(u+(1-\tau)\partial(\vartheta,u))\,d\tau \\
&= \int_0^1 \mathcal{F}_*(u+\tau\partial(\vartheta,u),\gamma)\,\Omega(u+\tau\partial(\vartheta,u))\,d\tau \\
&= \frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)} \mathcal{F}_*(x,\gamma)\,\Omega(x)\,dx, \\
&\int_0^1 \mathcal{F}^*(u+\tau\partial(\vartheta,u),\gamma)\,\Omega(u+\tau\partial(\vartheta,u))\,d\tau \\
&= \int_0^1 \begin{array}{c}\mathcal{F}^*(u+(1-\tau)\partial(\vartheta,u),\gamma) \\ .\Omega(u+(1-\tau)\partial(\vartheta,u))\end{array} d\tau \\
&= \frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)} \mathcal{F}^*(x,\gamma)\,\Omega(x)\,dx
\end{aligned} \tag{30}$$

From (29) and (30), we have

$$\begin{aligned}
&\mathcal{F}_*\left(u+\tfrac{1}{2}\partial(\vartheta,u),\gamma\right) \\
&\le \frac{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{\int_u^{u+\partial(\vartheta,u)}\Omega(x)\,dx}\int_u^{u+\partial(\vartheta,u)} \mathcal{F}_*(x,\gamma)\,\Omega(x)\,dx, \\
&\mathcal{F}^*\left(u+\tfrac{1}{2}\partial(\vartheta,u),\gamma\right) \\
&\le \frac{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{\int_u^{u+\partial(\vartheta,u)}\Omega(x)\,dx}\int_u^{u+\partial(\vartheta,u)} \mathcal{F}^*(x,\gamma)\,\Omega(x)\,dx,
\end{aligned}$$

From which, we have

$$\begin{aligned}
&\left[\mathcal{F}_*\left(u+\tfrac{1}{2}\partial(\vartheta,u),\gamma\right),\mathcal{F}^*\left(u+\tfrac{1}{2}\partial(\vartheta,u),\gamma\right)\right] \\
&\le_I \frac{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{\int_u^{u+\partial(\vartheta,u)}\Omega(x)\,dx}\left[\begin{array}{c}\displaystyle\int_u^{u+\partial(\vartheta,u)} \mathcal{F}_*(x,\gamma).\Omega(x)\,dx, \\ \displaystyle\int_u^{u+\partial(\vartheta,u)} \mathcal{F}^*(x,\gamma).\Omega(x)\,dx\end{array}\right],
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\mathcal{F}\left(u+\tfrac{1}{2}\partial(\vartheta,u)\right) \\
&\preccurlyeq \frac{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{\int_u^{u+\partial(\vartheta,u)}\Omega(x)\,dx}(FR)\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)\,\Omega(x)\,dx,
\end{aligned}$$

this completes the proof.

Remark 3.10. If $\hbar_2(\tau) \equiv 1$, $\tau \in [0,1]$ then, inequalities in Theorems 3.8 and 3.9 reduce for $\hbar_1$-preinvex fuzzy-IVFs which are also new one.

If $\hbar_1(\tau) = \tau$ and $\hbar_2(\tau) \equiv 1$, $\tau \in [0,1]$ then, inequalities in Theorems 3.8 and 3.9 reduce for preinvex fuzzy-IVFs which are also new one.

If in the Theorems 3.8 and 3.9 $\hbar_2(\tau) \equiv 1$ and $\partial(\vartheta,u) = \vartheta - u$ then, we obtain the appropriate theorems for $\hbar_1$-convex fuzzy-IVFs which are also new one.

If in the Theorems 3.8 and 3.9 $\hbar_1(\tau) = \tau$, $\hbar_2(\tau) \equiv 1$ and $\partial(\vartheta,u) = \vartheta - u$ then, we obtain the appropriate theorems for convex fuzzy-IVFs which are also new one.

If $\mathcal{F}_*(u,\gamma) = \mathcal{F}^*(u,\gamma)$ with $\gamma = 1$ and $\hbar_2(\tau) \equiv 1$ then, Theorems 3.8 and 3.9 reduces to classical first and second *HH*-Fejér inequality for $\hbar$-preinvex function, see [30].

If in the Theorems 3.8 and 3.9 $\mathcal{F}_*(u,\gamma) = \mathcal{F}^*(u,\gamma)$ with $\gamma = 1$, $\hbar_2(\tau) \equiv 1$ and $\partial(\vartheta,u) = \vartheta - u$ then, we obtain the appropriate theorems for $\hbar$-convex function, see [39].

If $\Omega(x) = 1$ then, combining Theorems 3.8 and 3.9, we get Theorem 3.1.

Example 3.11. We consider $\hbar_1(\tau)=\tau$, $\hbar_2(\tau)=1$ for $\tau\in[0,1]$ and the fuzzy-IVF $\mathcal{F}:[1,1+\partial(5,1)]\to\mathbb{F}_C(\mathbb{R})$ defined by,

$$\mathcal{F}(x)(\sigma)=\begin{cases}\frac{\sigma-e^x}{e^x}, & \sigma\in[e^x,2e^x],\\ \frac{4e^x-\sigma}{2e^x}, & \sigma\in(2e^x,4e^x),\\ 0, & otherwise,\end{cases}$$

Then, for each $\gamma\in[0,1]$, we have $\mathcal{F}_\gamma(x)=[(1+\gamma)e^x,2(2-\gamma)e^x]$. Since end point functions $\mathcal{F}_*(x,\gamma),\mathcal{F}^*(x,\gamma)$ are $(\hbar_1,\hbar_2)$-preinvex functions $\partial(y,x)=y-x$ for each $\gamma\in[0,1]$ then, $\mathcal{F}(x)$ is $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVF. If

$$\Omega(x)=\begin{cases}x-1, & \sigma\in\left[1,\frac{5}{2}\right],\\ 4-x, & \sigma\in\left(\frac{5}{2},4\right],\end{cases}$$

then, we have

$$\frac{1}{\partial(4,1)}\int_1^{1+\partial(5,1)}\mathcal{F}_*(x,\gamma)\Omega(x)\,dx$$
$$=\frac{1}{3}\int_1^4\mathcal{F}_*(x,\gamma)\Omega(x)\,dx$$
$$=\frac{1}{3}\int_1^{\frac{5}{2}}\mathcal{F}_*(x,\gamma)\Omega(x)\,dx+\frac{1}{3}\int_{\frac{5}{2}}^4\mathcal{F}_*(x,\gamma)\Omega(x)\,dx,$$
$$\frac{1}{\partial(4,1)}\int_1^{1+\partial(5,1)}\mathcal{F}^*(x,\gamma)\Omega(x)\,dx$$
$$=\frac{1}{3}\int_1^4\mathcal{F}^*(x,\gamma)\Omega(x)\,dx$$
$$=\frac{1}{3}\int_1^{\frac{5}{2}}\mathcal{F}^*(x,\gamma)\Omega(x)\,dx+\frac{1}{3}\int_{\frac{5}{2}}^4\mathcal{F}^*(x,\gamma)\Omega(x)\,dx,$$

$$\begin{aligned}&=\tfrac{1}{3}(1+\gamma)\int_1^{\frac{5}{2}}e^x(x-1)\,dx+\frac{1}{3}(1+\gamma)\int_{\frac{5}{2}}^4e^x(4-x)\,dx\\&\approx 11(1+\gamma),\\&=\tfrac{2}{3}(2-\gamma)\int_1^{\frac{5}{2}}e^x(x-1)\,dx+\frac{2}{3}(2-\gamma)\int_{\frac{5}{2}}^4e^x(4-x)\,dx\\&\approx 21(2-\gamma),\end{aligned}\tag{31}$$

and

$$\left[\mathcal{F}_*(u,\gamma)+\mathcal{F}_*(\vartheta,\gamma)\right]\int_0^1\begin{matrix}\hbar_1(\tau)\hbar_2(1-\tau)\\ .\Omega(u+\tau\partial(\vartheta,u))\end{matrix}\,d\tau$$
$$\left[\mathcal{F}^*(u,\gamma)+\mathcal{F}^*(\vartheta,\gamma)\right]\int_0^1\begin{matrix}\hbar_1(\tau)\hbar_2(1-\tau)\\ .\Omega(u+\tau\partial(\vartheta,u))\end{matrix}\,d\tau$$

$$\begin{aligned}&=(1+\gamma)\left[e+e^4\right]\left[\int_0^{\frac{1}{2}}3\tau^2dx+\int_{\frac{1}{2}}^1\tau(3-3\tau)\,d\tau\right]\\&\approx\tfrac{43}{2}(1+\gamma),\\&=2(2-\gamma)\left[e+e^4\right]\left[\int_0^{\frac{1}{2}}3\tau^2dx+\int_{\frac{1}{2}}^1\tau(3-3\tau)\,d\tau\right]\\&\approx 43(2-\gamma),\end{aligned}\tag{32}$$

From (31) and (32), we have

$$[11(1+\gamma),21(2-\gamma)]\ \le_l\left[\frac{43}{2}(1+\gamma),43(2-\gamma)\right],$$

for each $\gamma\in[0,1]$. Hence, Theorem 3.8 is verified.

For Theorem 3.9, we have

$$\begin{aligned}\mathcal{F}_*\left(u+\tfrac{1}{2}\partial(\vartheta,u),\gamma\right)&\approx\tfrac{64}{5}(1+\gamma),\\ \mathcal{F}^*\left(u+\tfrac{1}{2}\partial(\vartheta,u),\gamma\right)&\approx\tfrac{122}{5}(2-\gamma),\end{aligned}\tag{33}$$

$$\int_u^{u+\partial(\vartheta,u)}\Omega(x)\,dx=\int_1^{\frac{5}{2}}(x-1)\,dx+\int_u^{u+\partial(\vartheta,u)}(4-x)\,dx=\frac{9}{4},$$

$$\begin{aligned}&\frac{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{\int_u^{u+\partial(\vartheta,u)}\Omega(x)\,dx}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}_*(x,\gamma)\Omega(x)\,dx\\&\approx\frac{73}{5}(1+\gamma)\\&\frac{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}{\int_u^{u+\partial(\vartheta,u)}\Omega(x)\,dx}\int_u^{u+\partial(\vartheta,u)}\mathcal{F}^*(x,\gamma)\Omega(x)\,dx\\&\approx\frac{293}{10}(2-\gamma)\end{aligned}\tag{34}$$

From (33) and (34), we have

$$\left[\frac{64}{5}(1+\gamma),\frac{122}{5}(2-\gamma)\right]\ \le_l\left[\frac{73}{5}(1+\gamma),\frac{293}{10}(2-\gamma)\right].$$

Hence, Theorem 3.9 is verified.

4. CONCLUSION AND FUTURE STUDY

In this article, we introduced new class of nonconvex functions is known as $(\hbar_1,\hbar_2)$-preinvex fuzzy-IVFs. With the help of this class, we derived some new *HH*-inequalities by means of fuzzy order relation. To strengthen our result, we provided some examples to illustrate the validation of our results. Moreover, several new and previously known results also obtained. In future, we will try to explore these concepts for fuzzy fractional integral operators.

CONFLICTS OF INTEREST

The authors declare that they have no competing interests.

AUTHORS' CONTRIBUTIONS

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Funding Statement

The research is supported by the National Natural Science Foundation of China (Grant No. 61673169).

ACKNOWLEDGMENTS

The authors would like to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research and academic environments.

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