# Distributed Probabilistic and Quantum Strategies

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**Abstract**

Building on a new definition and characterization of probabilistic event structures, a general definition of distributed probabilistic strategies is proposed. Probabilistic strategies are shown to compose, with probabilistic copy-cat strategies as identities. A higher-order probabilistic process language reminiscent of Milner's CCS is interpretable within probabilistic strategies. W.r.t. a new definition of quantum event structure, it is shown how consistent parts of a quantum event structure are automatically probabilistic event structures, and so possess a probability measure. This gives a non-traditional take on the consistent-histories approach to quantum theory. It leads to an extension to quantum strategies. Probabilistic games extend to games with payoff, symmetry and games of imperfect information.

## 1 Introduction

Concurrent strategies [15] are being investigated as a possible foundation for a generalized domain theory, in which concurrent games and strategies take over the roles of domains and continuous functions. One motivation is to broaden the range of applicability of denotational semantics. Hence it is important to see how concurrent strategies can be adapted to quantitative semantics, to probabilistic and quantum strategies.

Just as event structures can be thought of as models of distributed computation so are probabilistic event structures models of probabilistic distributed processes. Existing definitions of probabilistic event structures [1,10,17] are not general enough to ascribe probabilities to the results of the sometimes partial interaction between strategies. This paper first presents a new workable definition of probabilistic event structures, extending existing definitions. Probabilistic event structures are characterized as event structures with a continuous valuation on their domain of configurations. Probabilistic event structures possess a probabilistic measure on their

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configurations. Technically, probabilistic event structures are defined via 'drop functions' expressing the probability drops across general intervals of configurations of the event structure; drop functions provide a useful mathematical handle on probabilistic event structures and strategies.

This prepares the ground for a general definition of distributed probabilistic strategies, based on event structures. A probabilistic strategy for Player is a concurrent strategy whose behaviour is described by a probabilistic event structure when projected to just the Player moves. Probabilistic strategies are shown to compose—here 'drop functions' come into their own—with probabilistic copy-cat strategies as identities. The result of a play between Player and Opponent in a game will be a probabilistic event structure.

As an illustration of their expressive power, probabilistic strategies are shown to interpret a higher-order probabilistic process language reminiscent of Milner's CCS. Probabilistic strategies are easily extended to games with payoff and games of imperfect information. Their definition has been partly inspired by the work of Danos and Harmer on probabilistic HO games [3], and in an informal sense the definition here extends theirs from the sequential setting. (A formal connection must await the relation between concurrent games and HO games, being developed within concurrent games with symmetry [2].)

A novel application is to a new definition of quantum event structures and strategies. A quantum event structure is an event structure in which the events are interpreted as projection or unitary operators on a Hilbert space, so that concurrent events are associated with commuting operators; a configuration of the event structure is thought of as a partial-order history of the observations of a quantum experiment. Interestingly order-compatible families of configurations of a quantum event structure automatically determine a probabilistic event structures, and so possess probability distributions. This gives a non-traditional take on the consistent-histories approach to quantum theory, which provides consistency conditions on histories to pick out those subfamilies of histories over which it is meaningful to place a probability distribution. The approach via quantum event structures bypasses the consistency conditions usually invoked [7].

In a quantum game Player and Opponent interact to jointly create a probabilistic distributed experiment on a quantum system. Accordingly a quantum strategy is taken to be a distributed probabilistic strategy on a quantum event structure, according with work on quantum games [6]. There are similarities with the work of Delbecque [4], itself based on probabilistic HO games [3].

## 2 Event structures

### *2.1 Event structures and configurations*

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set $E$, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency*

*relation* Con consisting of finite subsets of $E$, which satisfy

$$\begin{array}{l}\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\ \{e\} \in \mathrm{Con} \text{ for all } e \in E,\\ Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \text{ and}\\ X \in \mathrm{Con} \;\&\; e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.\end{array}$$

The *configurations*, $\mathcal{C}^\infty(E)$, of an event structure $E$ consist of those subsets $x \subseteq E$ which are *(Consistent)* $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \mathrm{Con}$ and *(Down-closed)* $\forall e, e'.\ e' \leq e \in x \implies e' \in x$. Often we shall be concerned with just the finite configurations, $\mathcal{C}(E)$.

We say an event structure is *elementary* when the consistency relation consists of all finite subsets of events. Two events $e, e'$ which are both consistent and incomparable w.r.t. causal dependency in an event structure are regarded as *concurrent*, written $e \mathrel{co} e'$. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning $e$ and $e'$ are distinct with $e \leq e'$ and no event in between, will play an important role. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of $X$; note if $X \in \mathrm{Con}$, then $[X] \in \mathrm{Con}$ is a configuration.

**Notation 1** Let $E$ be an event structure. We use $x \mathrel{-\!\!\subset} y$ to mean $y$ covers $x$ in $\mathcal{C}^\infty(E)$, *i.e.* $x \subsetneq y$ in $\mathcal{C}^\infty(E)$ with nothing in between, and $x \overset{e}{\mathrel{-\!\!\!-\!\!\subset}} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{C}^\infty(E)$ and event $e \notin x$. We use $x \overset{e}{\mathrel{-\!\!\!-\!\!\subset}}$, expressing that event $e$ is enabled at configuration $x$, when $x \overset{e}{\mathrel{-\!\!\!-\!\!\subset}} y$ for some $y$. We write $\{x_i \mid i \in I\}\uparrow$ to indicate that a subset of configurations is compatible, *i.e.* bounded above by a configuration.

### *2.2 Maps and operations on event structures*

Let $E$ and $E'$ be event structures. A *map* of event structures $f : E \to E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}^\infty(E)$ its direct image $fx \in \mathcal{C}^\infty(E')$ and

$$e_1, e_2 \in x \;\&\; f(e_1) = f(e_2) \text{ (with both defined)} \implies e_1 = e_2.$$

Maps of event structures compose as partial functions, with identity maps given by identity functions. We will say the map is *total* if the function $f$ is total; then $f$ restricts to a bijection $x \cong fx$ for $x \in \mathcal{C}(E)$. A total map of event structures which preserves causal dependency is called *rigid*.

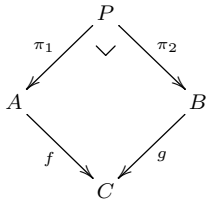
#### *2.2.1 Products*

The category of event structures with maps has *products* $A \times B$ with projections $\pi_1$ to $A$ and $\pi_2$ to $B$. It introduces arbitrary synchronizations between events of $A$ and events of $B$ in the manner of process algebra [13,21].

#### *2.2.2 Pullbacks*

Synchronized compositions of event structures $A$ and $B$ are obtained as restrictions $A \times B \restriction R$. The *restriction* of an event structure $E$ to a subset of events $R$, written

$E\restriction R$, is the event structure with events $E' = \{e \in E \mid [e] \subseteq R\}$ and causal dependency and consistency induced by $E$. We obtain *pullbacks* as a special case. Let $f : A \to C$ and $g : B \to C$ be maps of event structures. Defining $P =_{\text{def}} A \times B \restriction \{p \in A \times B \mid f\pi_1(p) = g\pi_2(p)\}$ we obtain a pullback square




in the category of event structures. When $f$ and $g$ are total the same construction gives the pullback in the category of event structures with *total* maps.

Some technology is needed to construct and analyse cleanly products, synchronized compositions and pullbacks (for instance, that of stable families [21]). Here it will suffice to have the following lemma giving a characterisation of finite configurations of pullbacks of total maps of event structures. (Its proof follows fairly directly from the construction of pullbacks in event structures from pullbacks in stable families [21].)

**Lemma 2.1** *Let $P, \pi_1, \pi_2$ form a pullback of total maps $f : A \to C$ and $g : B \to C$ in the category of event structures. Finite configurations of $P$ correspond to the composite bijections $\theta : x \cong fx = gy \cong y$ between configurations $x \in \mathcal{C}(A)$ and $y \in \mathcal{C}(B)$ s.t. $fx = gy$ for which the transitive relation generated on $\theta$ by taking $(a, b) \leq (a', b')$ if $a \leq_A a'$ or $b \leq_B b'$ is a partial order; the correspondence taking $z \in \mathcal{C}(P)$ to the composite bijection $\pi_1 z \cong f\pi_1 z = g\pi_2 z \cong \pi_2 z$ respects inclusion.*

*If $p' \rightarrowtriangle p$ in $P$ then $\pi_1(p') \rightarrowtriangle \pi_1(p)$ in $A$ or $\pi_2(p') \rightarrowtriangle \pi_2(p)$ in $B$.*

### *2.2.3 Projection*

Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of $E$ on $V$, to be $E{\downarrow}V =_{\text{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$. A partial map $f : E \to E'$ of event structures factors into a composition of a partial and total map $E \to E{\downarrow}V \to E'$ where: $V =_{\text{def}} \{e \in E \mid f(e) \text{ is defined}\}$ is the domain of definition of $f$; the partial map $E \to E{\downarrow}V$ acts as identity on $V$ and is undefined otherwise; and the total map $E{\downarrow}V \to E'$ acts as $f$.

### *2.2.4 Prefixes and sums*

The category of event structures has coproducts, a form of nondeterministic sum; a coproduct $\sum_{i \in I} E_i$ is obtained as the disjoint juxtaposition of an indexed collection of event structures, making events in distinct components inconsistent with each other. In practice, components of a sum are often prefixed by an event. The prefix of an event structure $A$, written $\bullet.A$, comprises the event structure in which all the events of $A$ are made to causally depend on an event $\bullet$.

## 3 Probabilistic event structures

A probabilistic event structure comprises an event structure $(E, \leq, \mathrm{Con})$ with a continuous valuation on its Scott open sets of configurations.Recall a *continuous valuation* is a function $w$ from the Scott-open subsets of $\mathcal{C}^\infty(E)$ to $[0,1]$ which is

*(normalized)* $w(\mathcal{C}^\infty(E)) = 1$; *(strict)* $w(\varnothing) = 0$;

*(monotone)* $U \subseteq V \implies w(U) \leq w(V)$;

*(modular)* $w(U \cup V) + w(U \cap V) = w(U) + w(V)$; and

*(continuous)* $w(\bigcup_{i\in I} U_i) = \sup_{i\in I} w(U_i)$ for *directed* unions $\bigcup_{i\in I} U_i$.

The idea: $w(U)$ is the probability of a result in open set $U$. Continuous valuations traditionally play the role of elements in probabilistic powerdomains [8]. Continuous valuations are determined by their restrictions to basic open sets $\widehat{x} =_{\mathrm{def}} \{y \in \mathcal{C}^\infty(E) \mid x \subseteq y\}$, for $x$ a finite configuration. This leads to an equivalent, more workable definition that we explain now. The description of a probabilistic event structure here extends the definitions mentioned in [17]. [2]

### *3.1 General intervals and drop functions*

Throughout this section assume $E$ is an event structure and $v : \mathcal{C}(E) \to \mathbb{R}$. Extend $\mathcal{C}(E)$ to a lattice $\mathcal{C}(E)^\top$ by adjoining an extra top element $\top$. Write its order as $x \sqsubseteq y$ and its finite join operations as $x \vee y$ and $\bigvee_{i\in I} x_i$. Extend $v$ to $v^\top : \mathcal{C}(E)^\top \to \mathbb{R}$ by taking $v^\top(\top) = 0$.

We are concerned with drops in value across general intervals $[y; x_1, \cdots, x_n]$, where $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)^\top$. The interval is thought of as specifying the set of configurations $\widehat{y} \setminus (\widehat{x}_1 \cup \cdots \cup \widehat{x}_n)$, *viz.* those configurations above or equal to $y$ and not above or equal to any $x_1, \cdots, x_n$. As such the intervals form a basis of the Lawson topology on $\mathcal{C}^\infty(E)^\top$.

Define the *drop functions* $d_v^{(n)}[y; x_1, \cdots, x_n] \in \mathbb{R}$ for $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)^\top$, by induction, taking

$$d_v^{(0)}[y;] =_{\mathrm{def}} v^\top(y) \text{ and}$$
$$d_v^{(n)}[y; x_1, \cdots, x_n] =_{\mathrm{def}} d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n],$$

for $n > 0$.

The following proposition shows how drop functions assign to general intervals $[y; x_1, \cdots, x_n]$ the value of being in $\widehat{y}$ minus the value of being in $\widehat{x}_1 \cup \cdots \cup \widehat{x}_n$, and that the latter is calculated using the inclusion-exclusion principle for sets; notice that an overlap $\bigcap_{i\in I} \widehat{x}_i$ equals $\widehat{\bigvee_{i\in I} x_i}$, where $\varnothing \neq I \subseteq \{1, \cdots, n\}$.

**Proposition 3.1** *Let $n \in \omega$. For $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$,*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = v(y) - \sum_{\varnothing \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i\in I} x_i)\,.$$

[2] Full proofs concerning probabilistic event structures, and the detailed properties of 'drop functions' on which they rely, in Sections 3.1 and 3.2, can be found in [21,23].

*For* $y, x_1, \cdots, x_n \in \mathcal{C}(E)$ *with* $y \subseteq x_1, \cdots, x_n$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] = v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i)\,,$$

*where the index* $I$ *ranges over sets satisfying* $\varnothing \neq I \subseteq \{1, \cdots, n\}$ *s.t.* $\{x_i \mid i \in I\}\uparrow$.

It will be important that drops across general intervals can be reduced to sums of drops across intervals based on coverings, as explained next.

**Lemma 3.2** *Let* $y \subseteq x_1, \cdots, x_n$ *in* $\mathcal{C}(E)$. *Then,* $d_v^{(n)}[y; x_1, \cdots, x_n]$ *is expressible as a sum of terms* $d_v^{(k)}[u; w_1, \cdots, w_k]$ *where* $y \subseteq u \multimap w_i$ *in* $\mathcal{C}(E)$ *and* $w_i \subseteq x_1 \cup \cdots \cup x_n$, *for all* $i$ *with* $1 \leq i \leq k$. *(*$x_1 \cup \cdots \cup x_n$ *need not be in* $\mathcal{C}(E)$*.)*

### *3.2 Probabilistic event structures*

A probabilistic event structure is an event structure associated with a $[0,1]$-valuation on configurations such that no general interval has a negative drop.

**Definition 3.3** Let $E$ be an event structure. A *configuration-valuation* on $E$ is function $v : \mathcal{C}(E) \to [0,1]$ such that $v(\varnothing) = 1$ and which satisfies the *drop condition*

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$$

for all $n \geq 1$ and $y, x_1, \cdots, x_n \in \mathcal{C}(E)$ with $y \subseteq x_1, \cdots, x_n$. A *probabilistic event structure* comprises an event structure $E$ together with a configuration-valuation $v : \mathcal{C}(E) \to [0,1]$. [3]

By Lemma 3.2, in showing we have a probabilistic event structure it suffices to verify the "drop condition" only for general intervals $[y; x_1, \cdots, x_n]$ where $y \multimap x_1, \cdots, x_n$.

**Theorem 3.4** *A configuration-valuation* $v$ *on an event structure* $E$ *extends uniquely to a continuous valuation* $w_v$ *on the open sets of* $\mathcal{C}^\infty(E)$ *(so* $v(x) = w_v(\widehat{x})$, *for all* $x \in \mathcal{C}(E)$*). Conversely, a continuous valuation on the open sets of* $\mathcal{C}^\infty(E)$ *restricts to a configuration-valuation on* $E$.

The above theorem also holds (with the same proof) for Scott domains. Now, by [11], Corollary 4.3:

**Theorem 3.5** *For a configuration-valuation* $v$ *on* $E$ *there is a unique probability measure* $\mu_v$ *on the Borel subsets of* $\mathcal{C}^\infty(E)$ *extending* $w_v$.

In particular, singleton sets of finite configurations are Borel sets for which there is a simple formula expressing their probability:

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[3] Samy Abbes has pointed out that the "drop condition" appears in early work of the Russian mathematician V.A.Rohlin [16](as relation (6) of Section 3, p.7), and Klaus Keimel that functions satisfying the "drop condition" are called "totally convex" or "completely monotone" in the literature [5]. The rediscovery of the "drop condition" and its reuse in the context of event structures was motivated by Lemma 3.2, tying it to occurrences of events.

**Proposition 3.6** *Let $E, v$ be a probabilistic event structure. For any finite configuration $y \in \mathcal{C}(E)$, the singleton set $\{y\}$ is a Borel subset with probability measure*

$$\mu_v(\{y\}) = \inf\{d_v^{(n)}[y; x_1, \cdots, x_n] \mid n \in \omega \ \&\ y \not\subseteq x_1, \cdots, x_n\}\,.$$

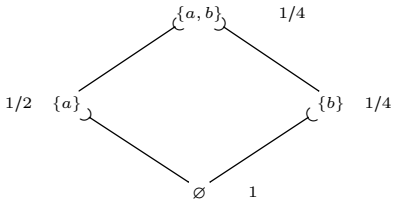
**Proof.** Let $y \in \mathcal{C}(E)$. Then $\{y\} = \widehat{y} \setminus U_y$ is clearly Borel as $U_y =_{\text{def}} \{x \in \mathcal{C}^\infty(E) \mid y \not\subseteq x\}$ is open. Hence $\mu_v(\{y\}) = v(y) - \mu_v(U_y)$ where

$$\begin{aligned}\mu_v(U_y) =& \sup\{\mu_v(\widehat{x}_1 \cup \cdots \widehat{x}_n) \mid y \not\subseteq x_1, \cdots, x_n\}\\ =& \sup\{\sum_{\varnothing \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) \mid y \not\subseteq x_1, \cdots, x_n\}\ ,\end{aligned}$$

because $U_y$ is the directed union of $\{\widehat{x}_1 \cup \cdots \widehat{x}_n \mid y \not\subseteq x_1, \cdots, x_n\}$, from which the result follows. □

In a probabilistic event structure $E, v$, when $y$ a finite configuration of $E$ has $v(y) > 0$ and $\mu_v(\{y\}) = 0$ we can understand $y$ as being a *transient* configuration on the way to a final result.

**Example 3.7** Consider the event structure comprising two concurrent events $a$ and $b$. It has configurations and configuration valuation $v$ as shown:




The probability $\mu_v(\{\{a, b\}\})$ of ending at the configuration $\{a, b\}$ is $1/4$; that of terminating at $\{a\}$ the drop $1/2 - 1/4 = 1/4$; that of terminating at $\{b\}$ the drop $1/4 - 1/4 = 0$ showing that $\{b\}$ is only a transient configuration; while the probability of terminating at $\varnothing$ is the drop $1 - 1/2 - 1/4 + 1/4 = 1/2$. □

**Remark.** In the definition of probabilistic event structures there are two different ways to say, for example, that events $e_1$ and $e_2$ do not occur together at a finite configuration $y$ where $y \xrightarrow{e_1} x_1$ and $y \xrightarrow{e_2} x_2$: either through $\{e_1, e_2\} \notin \text{Con}$; or via the configuration-valuation $v$ through $v(x_1 \cup x_2) = 0$. However, this seeming redundancy is exploited later in probabilistic strategies and quantum event structures, when we mix probability with nondeterminism and shall make use of both consistency and the valuation.

# 4 Probabilistic strategies

We show how concurrent strategies can be extended with probabilities, first reviewing the needed results from [15].

## *4.1 Strategies*

### *4.1.1 Event structures with polarity*

Both games and strategies in a game are represented in terms of event structures with polarity, which comprise $(E, pol)$ where $E$ is an event structure with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity + (Player) or − (Opponent) to its events. The events correspond to (occurrences of) moves. Maps of event structures with polarity are maps of event structures which preserve polarities.

The *dual*, $E^{\perp}$, of an event structure with polarity $E$ comprises the same underlying event structure $E$ but with a reversal of polarities. Let $A$ and $B$ be event structures with polarity. The operation $A \| B$, of *simple parallel composition*, juxtaposes disjoint copies of $A$ and $B$, maintaining their causal dependency and specifying a finite subset of events as consistent if it restricts to consistent subsets of $A$ and $B$. Polarities are unchanged. The empty game $\varnothing$ is the unit of $\|$.

### *4.1.2 Pre-strategies*

Let $A$ be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A *pre-strategy in* $A$ represents a nondeterministic play of the game and is defined to be a total map $\sigma : S \to A$ of event structures with polarity.

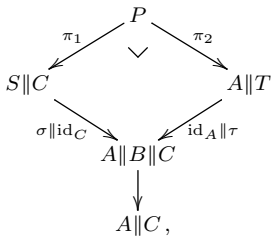
A map between pre-strategies, from $\sigma : S \to A$ and $\tau : T \to A$, is a map $f : S \to T$ such that $\sigma = \tau f$. Accordingly, $\sigma \cong \tau$ when there is an isomorphism $\theta : S \cong T$ such that $\sigma = \tau\theta$.

Let $A$ and $B$ be event structures with polarity. A *pre-strategy from $A$ to $B$* is a pre-strategy in $A^{\perp} \| B$. Write $\sigma : A \nrightarrow B$ to express that $\sigma$ is a pre-strategy from $A$ to $B$. Note that a pre-strategy $\sigma$ *in* a game $A$, a total map $\sigma : S \to A$, coincides with a pre-strategy *from* the empty game $\varnothing$ *to* the game $A$, *i.e.* $\sigma : \varnothing \nrightarrow A$.

Strategies are defined to be those pre-strategies for which copy-cat behaves as identity w.r.t composition, as we now explain.

### *4.1.3 Composing pre-strategies*

We can present the composition of pre-strategies via pullbacks. Given two pre-strategies $\sigma : S \to A^{\perp} \| B$ and $\tau : T \to B^{\perp} \| C$, ignoring polarities we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A \| B$ and $\tau : T \to B \| C$. Viewed this way we can form the pullback in the category of event structures as shown



where the map $A\|B\|C \to A\|C$ is undefined on $B$ and acts as identity on $A$ and $C$. The partial map from $P$ to $A\|C$ given by the diagram above (either way round the pullback square) factors as the composition of the partial map $P \to P \downarrow V$, where $V$ is the set of events of $P$ at which the map $P \to A\|C$ is defined, and a total map $P \downarrow V \to A\|C$. The resulting total map gives us the composition $\tau \odot \sigma : T \odot S =_{\text{def}} P \downarrow V \to A^{\perp}\|C$ once we reinstate polarities forced by those of $A$ and $C$.

In $T \odot S$ we have hidden the synchronization events over $B$ due to the instantiation of Opponent moves of $T$ in $B$ by Player moves of $S$, and *vice versa*. Later we shall also be concerned with the event structure $P$, composition before hiding, which we shall denote more descriptively by $T \circledast S$.

### *4.1.4 Concurrent copy-cat*

The copy-cat strategy from $A$ to $A$ is an instance of a pre-strategy, and a total map $\gamma_A : \mathbb{CC}_A \to A^{\perp}\|A$. It is based on the idea that Player moves, of +ve polarity, always copy previous corresponding moves of Opponent, of −ve polarity. For $c \in A^{\perp}\|A$ we use $\overline{c}$ to mean the corresponding copy of $c$, of opposite polarity, in the alternative component. Define $\mathbb{CC}_A$ to comprise the event structure with polarity $A^{\perp}\|A$ together with the extra causal dependencies generated by $\overline{c} \leq_{\mathbb{CC}_A} c$ for all events $c$ with $pol_{A^{\perp}\|A}(c) = +$. A finite subset of $\mathbb{CC}_A$ is consistent if its down-closure w.r.t. $\leq_{\mathbb{CC}_A}$ is consistent in $A^{\perp}\|A$. The *copy-cat* pre-strategy $\gamma_A : A \rightarrowtriangle A$ is defined to be the map $\gamma_A : \mathbb{CC}_A \to A^{\perp}\|A$ where $\gamma_A$ is the identity on the common set of events.

### *4.1.5 Strategies*

The main result of [15] is that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient for copy-cat to behave as identity w.r.t. the composition of pre-strategies. Receptivity ensures an openness to all possible moves of Opponent. Innocence restricts the behaviour of Player; Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$ beyond those already imposed by the game. A pre-strategy $\sigma$ is *receptive* iff $\sigma x \overset{a}{\text{—}\!\subset}$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{\text{—}\!\subset}$ & $\sigma(s) = a$. It is *innocent* iff $s \rightarrowtriangle s'$ & ($pol(s) = +$ or $pol(s') = -$) implies $\sigma(s) \rightarrowtriangle \sigma(s')$. The main result of [15] is that $\gamma_B \odot \sigma \odot \gamma_A \cong \sigma$ iff $\sigma$ is receptive and innocent. Copy-cats $\gamma_A : A \rightarrowtriangle A$ are receptive and innocent.

A *strategy* is a pre-strategy which is receptive and innocent. We obtain a bicategory in which the objects are event structures with polarity—the games, the arrows from $A$ to $B$ are strategies $\sigma : A \rightarrowtriangle B$ and 2-cells are total maps of pre-strategies with vertical composition the usual composition of such maps. Horizontal composition is given by the composition of strategies $\odot$. For future reference, recall from [15] that a strategy $\sigma : S \to A$ satisfies *strong receptivity*: whenever $\sigma x \subseteq^{-} y$ in $\mathcal{C}(A)$ there is a *unique* $x' \in \mathcal{C}(S)$ so that $x \subseteq^{-} x'$ & $\sigma x' = y$.

An event structure with polarity $S$ is *deterministic* iff

$$\forall X \subseteq_{\text{fin}} S.\ Neg[X] \in \text{Con}_S \implies X \in \text{Con}_S\,,$$

where $Neg[X] =_{\text{def}} \{s' \in S \mid pol(s') = - \;\&\; \exists s \in X.\ s' \leq s\}$. In other words, $S$ is deterministic iff any down-closed set of moves is consistent when its subset of Opponent moves is consistent. Say a strategy $\sigma : S \to A$ is deterministic if $S$ is deterministic. Copy-cat strategies $\gamma_A$ are deterministic iff the game $A$ is

**race-free:** for all $x \in \mathcal{C}(A)$ such that $x \xrightarrow{a} \subset$ and $x \xrightarrow{a'} \subset$ with $pol(a) = -$ and $pol(a') = +$, we have $x \cup \{a, a'\} \in \mathcal{C}(A)$.

We obtain a sub-bicategory of deterministic strategies between race-free games—in fact equivalent to an order-enriched category [15,20].

Strategies inherit a duality from pre-strategies. A pre-strategy $\sigma : A \nrightarrow B$ corresponds to a dual pre-strategy $\sigma^\perp : B^\perp \nrightarrow A^\perp$, arising from the correspondence between pre-strategies $\sigma : S \to A^\perp \| B$ and $\sigma^\perp : S \to (B^\perp)^\perp \| A^\perp$.

A more expansive treatment of strategies is to be found in [21].

### *4.2 Probabilistic strategies*

Without information about the stochastic rates of Player and Opponent we cannot hope to ascribe probabilities to outcomes of play in the presence of races, *i.e.* immediate conficts between moves of opposite polarities. Our results on probabilistic strategies depend on restricting to games which are race-free.

It will be convenient to define a probabilistic event structure in which some events are distinguished as Opponent events (where the other events may be Player events or "neutral" events due to synchronizations between Player and Opponent moves). Events which are not Opponent events we shall call $p$-events. For configurations $x, y$ we shall write $x \subseteq^p y$ if $x \subseteq y$ and $y \setminus x$ contains no Opponent events; we write $x \mathrel{-\!\!\subset^p} y$ when $x \mathrel{-\!\!\subset} y$ and $x \subseteq^p y$; we similarly write *e.g.* $x \subseteq^- y$, respectively $x \subseteq^+ y$, if $x \subseteq y$ and $y \setminus x$ comprises solely Opponent, respectively Player, events. We can now extend the notion of configuration-valuation to the situation where events carry polarities.

**Definition 4.1** Let $E$ be an event structure in which a specified subset of events are Opponent events. A *configuration-valuation* on $E$ is a function $v : \mathcal{C}(E) \to [0, 1]$ for which $v(\varnothing) = 1$,

$$x \subseteq^- y \implies v(x) = v(y) \tag{1}$$

for all $x, y \in \mathcal{C}(E)$, and satisfies the "drop condition"

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0 \tag{2}$$

for all $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{C}(E)$ with $y \subseteq^p x_1, \cdots, x_n$.

A *probabilistic event structure with polarity* comprises $E$ an event structure with polarity together with a configuration-valuation $v : \mathcal{C}(E) \to [0, 1]$.

As earlier, by Lemma 3.2, it suffices to verify the "drop condition" for general intervals $[y; x_1, \cdots, x_n]$ where $y \mathrel{-\!\!\subset^p} x_1, \cdots, x_n$.

**Definition 4.2** Let $A$ be a race-free event structure with polarity. A *probabilistic strategy* $v, \sigma$ *in* $A$ comprises $S, v$, a probabilistic event structure with polarity, and

a strategy $\sigma : S \to A$. [It follows that $S$ will also be race-free.]

Let $A$ and $B$ be a race-free event structures with polarity. A *probabilistic strategy from $A$ to $B$* is a probabilistic strategy in $A^{\perp} \| B$.

We extend the usual composition of strategies to probabilistic strategies. Assume probabilistic strategies $\sigma : S \to A^{\perp} \| B$, with configuration-valuation $v_S : \mathcal{C}(S) \to [0,1]$, and $\tau : T \to B^{\perp} \| C$ with configuration-valuation $v_T : \mathcal{C}(T) \to [0,1]$. We first define their composition before hiding, as the probabilistic event structure $T \circledast S, v$, tentatively taking $v : \mathcal{C}(T \circledast S) \to [0,1]$ to be $v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$ for $x \in \mathcal{C}(T \circledast S)$. We next present a key lemma in showing that $v$ is a configuration-valuation. The lemma reduces fulfilling the drop condition for $v$ to fulfilling the drop conditions for $v_S$ and $v_T$.

**Lemma 4.3** *Let $v : \mathcal{C}(T \circledast S) \to [0,1]$ be defined as above. Let $y, x_1, \cdots, x_n \in \mathcal{C}(T \circledast S)$ with $y \mathrel{-\!\!\subset^p} x_1, \cdots, x_n$. Assume that $\pi_1 y \mathrel{-\!\!\subset^+} \pi_1 x_i$ when $1 \le i \le m$ and $\pi_2 y \mathrel{-\!\!\subset^+} \pi_2 x_i$ when $m+1 \le i \le n$. Then in $\mathcal{C}(T \circledast S)$,*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = d_{v_S}^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] \times d_{v_T}^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n] .$$

**Proof.** Under the assumptions of the lemma, by Proposition 3.1,

$$d_v^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] = v_S(\pi_1 y) - \sum_{I_1} (-1)^{|I_1|+1} v_S(\bigcup_{i \in I_1} \pi_1 x_i) ,$$

where $I_1$ ranges over sets satisfying $\varnothing \neq I_1 \subseteq \{1, \cdots, m\}$ s.t. $\{\pi_1 x_i \mid i \in I_1\}\uparrow$. Similarly,

$$d_v^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n] = v_T(\pi_2 y) - \sum_{I_2} (-1)^{|I_2|+1} v_T(\bigcup_{i \in I_2} \pi_2 x_i) ,$$

where $I_2$ ranges over sets satisfying $\varnothing \neq I_2 \subseteq \{m+1, \cdots, n\}$ s.t. $\{\pi_2 x_i \mid i \in I_2\}\uparrow$.

Using Lemma 2.1, by the strong receptivity of $\tau$, when $\varnothing \neq I_1 \subseteq \{1, \cdots, m\}$,

$$\{\pi_1 x_i \mid i \in I_1\}\uparrow \text{ in } \mathcal{C}(S) \text{ iff } \{x_i \mid i \in I_1\}\uparrow \text{ in } \mathcal{C}(T \circledast S)$$

and, similarly by strong receptivity of $\sigma$, when $\varnothing \neq I_2 \subseteq \{m+1, \cdots, n\}$,

$$\{\pi_2 x_i \mid i \in I_2\}\uparrow \text{ in } \mathcal{C}(T) \text{ iff } \{x_i \mid i \in I_2\}\uparrow \text{ in } \mathcal{C}(T \circledast S) .$$

Hence

$$\bigcup_{i \in I_1} \pi_1 x_i = \pi_1 \bigcup_{i \in I_1} x_i \quad \text{and} \quad \bigcup_{i \in I_2} \pi_2 x_i = \pi_2 \bigcup_{i \in I_2} x_i .$$

Making these rewrites and taking the product

$$d_v^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] \times d_v^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n] ,$$

we obtain

$$\begin{aligned} v_S(\pi_1 y) \times v_T(\pi_2 y) \; & - \sum_{I_2} (-1)^{|I_2|+1} \; v_S(\pi_1 y) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i) \\ & - \sum_{I_1} (-1)^{|I_1|+1} \; v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 y) \\ & + \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|} \; v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i) \, . \end{aligned}$$

But at each index $I_2$,

$$v_S(\pi_1 y) = v_S(\pi_1 \bigcup_{i \in I_2} x_i)$$

as $\pi_1 y \subseteq^- \pi_1 \bigcup_{i \in I_2} x_i$. Similarly, at each index $I_1$,

$$v_T(\pi_2 y) = v_T(\pi_2 \bigcup_{i \in I_1} x_i) \, .$$

Hence the product becomes

$$\begin{aligned} v_S(\pi_1 y) \times v_T(\pi_2 y) \; & - \sum_{I_2} (-1)^{|I_2|+1} \; v_S(\pi_1 \bigcup_{i \in I_2} x_i) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i) \\ & - \sum_{I_1} (-1)^{|I_1|+1} \; v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 \bigcup_{i \in I_1} x_i) \\ & + \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|} \; v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i) \, . \end{aligned}$$

To simplify this further, we observe that

$$\{x_i \mid i \in I_1\}\uparrow \; \& \; \{x_i \mid i \in I_2\}\uparrow \iff \{x_i \mid i \in I_1 \cup I_2\}\uparrow \, .$$

The "$\Leftarrow$" direction is clear. We show "$\Rightarrow$." Assume $\{x_i \mid i \in I_1\}\uparrow$ and $\{x_i \mid i \in I_2\}\uparrow$. We obtain $\{\pi_1 x_i \mid i \in I_1\}\uparrow$ and $\{\pi_1 x_i \mid i \in I_2\}\uparrow$ as the projection map $\pi_1$ preserves consistency. Hence $\bigcup_{i \in I_1} \pi_1 x_i$ and $\bigcup_{i \in I_2} \pi_1 x_i$ are configurations of $S$. Furthermore, by assumption,

$$\pi_1 y \subseteq^+ \bigcup_{i \in I_1} \pi_1 x_i \quad \text{and} \quad \pi_1 y \subseteq^- \bigcup_{i \in I_2} \pi_1 x_i \, .$$

As $S$, a strategy over the race-free game $A^\perp \| B$, is automatically race-free, we obtain

$$\bigcup_{i \in I_1 \cup I_2} \pi_1 x_i \in \mathcal{C}(S) \, .$$

Similarly, because $T$ is race-free, we obtain

$$\bigcup_{i \in I_1 \cup I_2} \pi_2 x_i \in \mathcal{C}(T) \, .$$

Using Lemma 2.1, together these entail

$$\bigcup_{i \in I_1 \cup I_2} x_i \in \mathcal{C}(T \circledast S) \, ,$$

*i.e.* $\{x_i \mid i \in I_1 \cup I_2\}\uparrow$, as required. Notice too that

$$\pi_1 \bigcup_{i\in I_1} x_i \subseteq^- \pi_1 \bigcup_{i\in I_1\cup I_2} x_i \quad \text{and} \quad \pi_2 \bigcup_{i\in I_2} x_i \subseteq^- \pi_2 \bigcup_{i\in I_1\cup I_2} x_i\,,$$

which ensure

$$v_S(\pi_1 \bigcup_{i\in I_1} x_i) = v_S(\pi_1 \bigcup_{i\in I_1\cup I_2} x_i) \quad \text{and} \quad v_T(\pi_2 \bigcup_{i\in I_2} x_i) = v_T(\pi_2 \bigcup_{i\in I_1\cup I_2} x_i)\,,$$

so that

$$v(\bigcup_{i\in I_1\cup I_2} x_i) = v_S(\pi_1 \bigcup_{i\in I_1} x_i) \times v_T(\pi_2 \bigcup_{i\in I_2} x_i)\,.$$

We can now further simplify the product to

$$\begin{aligned} v(y) &- \sum_{I_2} (-1)^{|I_2|+1}\ v(\bigcup_{i\in I_2} x_i) \\ &- \sum_{I_1} (-1)^{|I_1|+1}\ v(\bigcup_{i\in I_1} x_i) \\ &+ \sum_{I_1,I_2} (-1)^{|I_1|+|I_2|}\ v(\bigcup_{i\in I_1\cup I_2} x_i)\,. \end{aligned}$$

Noting that any subset $I$ for which $\varnothing \neq I \subseteq \{1,\cdots,n\}$ either lies entirely within $\{1,\cdots,m\}$, entirely within $\{m+1,\cdots,n\}$, or properly intersects both, we have finally reduced the product to

$$v(y) - \sum_{I} (-1)^{|I|+1} v(\bigcup_{I} x_i)\,,$$

with indices those $I$ which satisfy $\varnothing \neq I \subseteq \{1,\cdots,n\}$ s.t. $\{x_i \mid i \in I\}\uparrow$, *i.e.* the product reduces to $d_v^{(n)}[y; x_1 \cdots, x_n]$ as required. □

**Corollary 4.4** *The assignment $v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$ to $x \in \mathcal{C}(T \circledast S)$ yields a configuration-valuation on $T \circledast S$, so a probabilistic event structure $T \circledast S, v$.*

**Proof.** Clearly,

$$v(\varnothing) = v_S(\pi_1\varnothing) \times v_T(\pi_2\varnothing) = 1 \times 1 = 1\,.$$

Assuming $x \mathrel{-\!\!\subset^-} y$ in $\mathcal{C}(T \circledast S)$, then either (i) $\pi_1 x \mathrel{-\!\!\subset^-} \pi_1 y$ and $\pi_2 x = \pi_2 y$ or (ii) $\pi_2 x \mathrel{-\!\!\subset^-} \pi_2 y$ and $\pi_1 x = \pi_1 y$. In either case, $v_S(\pi_1 x) = v_S(\pi_1 y)$ and $v_T(\pi_2 x) = v_T(\pi_2 y)$. Combined these two facts yield $v(x) = v(y)$. As $x \subseteq^- y$ is obtained as the reflexive transitive closure of $\mathrel{-\!\!\subset^-}$ it too entails $v(x) = v(y)$, requirement (1) of Definition 4.1. By Lemma 3.2 we need only verify requirement (2), the 'drop condition,' for $p$-covering intervals, which via Proposition 3.1 we can always permute into the form covered by Lemma 4.3—any $p$-event of $\mathcal{C}(T \circledast S)$ has a +ve component on one and only one side. □

We can now complete the definition of the composition of probabilistic strategies. Note that for $x \in \mathcal{C}(T \odot S)$ its down-closure within $T \circledast S$ forms the configuration $[x] \in \mathcal{C}(T \circledast S)$.

**Lemma 4.5** *Let $A$, $B$ and $C$ be race-free event structure with polarity. Let $\sigma : S \to A^{\perp} \| B$, with configuration-valuation $v_S : \mathcal{C}(S) \to [0,1]$, and $\tau : T \to B^{\perp} \| C$ with configuration-valuation $v_T : \mathcal{C}(T) \to [0,1]$ be probabilistic strategies. Assigning $v_S(\pi_1[x]) \times v_T(\pi_2[x])$ to $x \in \mathcal{C}(T \odot S)$ yields a configuration-valuation which with $\tau \odot \sigma : T \odot S \to A^{\perp} \| C$ forms a probabilistic strategy from $A$ to $C$.*

**Proof.** We use that $v(z) =_{\text{def}} v_S\pi_1(z) \times v_T\pi_2(z)$, for $z \in \mathcal{C}(T \circledast S)$, is a configuration-valuation on $T \circledast S$. For $x \in \mathcal{C}(T \odot S)$,

$$w(x) =_{\text{def}} v_S\pi_1[x] \times v_T\pi_2[x] = v([x])\,.$$

Consequently,

$$w(\varnothing) = v([\varnothing]) = v(\varnothing) = 1\,.$$

The function $w$ inherits requirement (1) to be a configuration-valuation from $v$. This is via the observation that for $p$ $-$ve,

$$x \overset{p}{\text{—}\!\subset} y \text{ in } T \odot S \implies [x] \overset{p}{\text{—}\!\subset} [y] \text{ in } T \circledast S\,.$$

From the observation , if $x \text{–}\!\subset^{-} y$ in $T \odot S$ then $[x] \text{–}\!\subset^{-} [y]$ in $T \circledast S$, so

$$w(x) = v([x]) = v([y]) = w(y)\,.$$

The proof of the observation relies on Lemma 2.1 specialised to the pullback $T \circledast S$, $\pi_1$, $\pi_2$ —refer to the pullback diagram of Section 4.1.3. Assume $x \overset{p}{\text{—}\!\subset} y$ in $T \odot S$. Supposing $q \rightarrowtriangle p$ in $T \circledast S$, by Lemma 2.1 we obtain

$$\text{(i) } \pi_1(q) \rightarrowtriangle \pi_1(p) \text{ in } S \| C \quad \text{or} \quad \text{(ii) } \pi_2(q) \rightarrowtriangle \pi_2(p) \text{ in } A \| T\,.$$

Suppose (i). There are two possibilities, either (a) $\pi_1(p)$ is in the component $S$ or (b) $\pi_1(p)$ is in the component $C$.

In case (a), under the composite map $(\sigma \| \mathrm{id}_C)\pi_1$ the event $p$ is sent to the component $A$ in $A \| B \| C$ —this is because $p$ being an event of $T \odot S$ cannot be sent to the component $B$. As $\sigma$ is $-$-innocent, the event $q$ must also be sent to the same component. This ensures that $q \in x$, so $q \in [x]$.

In case (b), $\pi_1(q) \rightarrowtriangle \pi_1(p)$ in the component $C$. From the properties of the pullback, it follows that $\tau\pi_2(q) \rightarrowtriangle \tau\pi_2(p)$ in the component $C$. This ensures that $q \in x$, so $q \in [x]$.

Thus, in either case, (a) or (b), any $\rightarrowtriangle$-predecessor of $p$ is in $[x]$, yielding $[x] \overset{p}{\text{—}\!\subset} [y]$ in $T \circledast S$. The case (ii) is proved similarly, using the $-$-innocence of $\tau$.

Similarly, $w$ inherits requirement (2) from $v$, as w.r.t. $w$,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &= w(y) - \sum_I (-1)^{|I|+1} w(\bigcup_{i \in I} x_i) \\ &= v(\bigcup y) - \sum_I (-1)^{|I|+1} v([\bigcup_{i \in I} x_i]) \\ &= v(\bigcup y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} [x_i]) \\ &\geq 0\,, \end{aligned}$$

whenever $y \subseteq^p x_1, \cdots, x_n$ in $\mathcal{C}(T \odot S)$. Above, the index $I$ ranges over sets satisfying $\varnothing \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\}\uparrow$. □

The assumption that games are race-free is needed for Corollary 4.4 and Lemmas 4.3, 4.5. Recall that race-freedom of a game $A$ ensures $!\mathbb{C}_A$ is deterministic [20,21] and hence its copy-cat strategy is easily turned into a probabilistic strategy, as is any deterministic strategy:

**Lemma 4.6** *Let $S$ be a deterministic event structure with polarity. Defining $v_S : \mathcal{C}(S) \to [0,1]$ to satisfy $v_S(x) = 1$ for all $x \in \mathcal{C}(S)$, we obtain a probabilistic event structure with polarity.*

**Proof.** Clearly, $v_S(\varnothing) = 1$ and

$$x \subseteq^- y \implies v_S(x) = v_S(y) = 1$$

for all $x, y \in \mathcal{C}(S)$. As $S$ is deterministic,

$$y \subseteq^+ x \;\&\; y \subseteq^+ x' \implies x \cup x' \in \mathcal{C}(S)\,,$$

for all $y, x, x' \in \mathcal{C}(S)$. For the remaining requirement, a simple induction shows that for all $n \geq 1$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] = 0$$

whenever $y \subseteq^+ x_1, \cdots, x_n$. The basis, when $n = 1$, is clear as

$$d_v^{(1)}[y; x] = v_S(y) - v_S(x) = 1 - 1 = 0$$

when $y \subseteq^+ x$. For the induction step, assuming $y \subseteq^+ x_1, \cdots, x_n$ with $n > 1$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n] = 0 - 0 = 0\,,$$

from the induction hypothesis. □

**Corollary 4.7** *f Let $A$ be a race-free game. The copy-cat strategy on $A$ comprising $\gamma_A : !\mathbb{C}_A \to A^\perp \| A$ with configuration-valuation $v_{!\mathbb{C}_A} : \mathcal{C}(!\mathbb{C}_A) \to [0,1]$ satisfying $v_{!\mathbb{C}_A}(x) = 1$, for all $x \in \mathcal{C}(!\mathbb{C}_A)$, forms a probabilistic strategy.*

Combining the results of this section:

**Theorem 4.8** *Race-free games with probabilistic strategies, with composition and copy-cat defined as in Lemma 4.5 and Corollary 4.7, inherit the structure of a bicategory from that of games with strategies. (2-cells $f : \sigma, v \Rightarrow \sigma', v'$ between probabilistic strategies are now 2-cells $f : \sigma \Rightarrow \sigma'$ of strategies for which $v$ is pointwise less or equal $v'f$.)*

# 5 A language of probabilistic strategies

As an indication of the expressivity of probabilistic strategies we sketch how they can straightforwardly interpret a simple language of probabilistic processes, reminiscent of a higher-order CCS. For this section only, write $\sigma : A$ to mean $\sigma$ is a probabilistic strategy in game $A$. Probabilistic strategies are closed under the following operations.

*Composition* $\sigma \odot \tau : A \| C$, if $\sigma : A \| B$ and $\tau : B^{\perp} \| C$. Hiding is automatic in a synchronized composition directly based on the composition of strategies.

*Simple parallel composition* $\sigma \| \tau : A \| B$, if $\sigma : A$ and $\tau : B$. Note that simple parallel composition can be regarded as a special case of synchronized composition: via the identification of $\sigma \| \tau$ with $\tau \odot \sigma$, taking $\sigma : A^{\perp} \nrightarrow \varnothing$ and $\tau : \varnothing \nrightarrow B$, the operation $\sigma \| \tau$ yields a probabilistic strategy. Supposing $\sigma : S \to A$ and $\tau : T \to B$ and $S$ and $T$ have configuration valuations $v_S$ and $v_T$, respectively, then the configuration valuation $v$ for $S \| T$ satisfies $v(x) = v_S(x_1) \times v_T(x_2)$, for $x \in \mathcal{C}(S \| T)$.

*Input prefixing* $\sum_{i \in I} \ominus . \sigma_i : \sum_{i \in I} \ominus . A_i$, if $\sigma_i : A_i$, for $i \in I$, where $I$ is countable. This prefixes both strategies and games with an initial Opponent move and then forms their sum (as an event structure).

*Output prefixing* $\sum_{i \in I} p_i \oplus . \sigma_i : \sum_{i \in I} \oplus . A_i$, if $\sigma_i : A_i$, for $i \in I$, where $I$ is countable, and $p_i \in [0, 1]$ for $i \in I$ with $\sum_{i \in I} p_i \leq 1$. This prefixes strategies with initial Player moves weighted by probabilities. If $\sum_{i \in I} p_i < 1$, there is non-zero probability of terminating without any move. By design $(\sum_{i \in I} \oplus . A_i)^{\perp} = \sum_{i \in I} \ominus . A_i^{\perp}$.

*General probabilistic sum* More generally we can define $\bigoplus_{i \in I} p_i \sigma_i : A$, for $\sigma_i : A$ and $I$ countable with sub-probability distribution $p_i, i \in I$. The operation makes the $+$-events of different components conflict and re-weights the configuration-valuation on the components according to the sub-probability distribution. In order for the sum to remain receptive, the initial $-$ve events of the components over a common event in the game $A$ must be identified.

*Relabelling*, the composition $f\sigma : B$, if $\sigma : A$ and $f : A \to B$ is itself a strategy, *i.e.* total, receptive and innocent. (This generalises to certain partial maps $f$.)

*Pullback* $f^{*}\sigma : A$, if $\sigma : B$ and $f : A \to B$ is a map of event structures with polarity,

possibly partial, which reflects +-consistency in the sense that

$$y \overset{+}{\longrightarrow\!\!\!\!\subset} x_1, \cdots, x_n \;\&\; \{fx_i \mid 1 \leq i \leq n\}\uparrow \Longrightarrow \{x_i \mid 1 \leq i \leq n\}\uparrow \; .$$

The strategy $f^*\sigma$ is got by the pullback

$$\begin{array}{ccc} S' & \xrightarrow{f'} & S \\ {\scriptstyle f^*\sigma}\downarrow & \lrcorner & \downarrow{\scriptstyle \sigma} \\ A & \xrightarrow[f]{} & B \, . \end{array}$$

Then, the map $f'$ also reflects +-consistency. This fact ensures we define a configuration-valuation $v_{S'}$ on $S'$ by taking $v_{S'}(x) = v_S(f'x)$, for $x \in \mathcal{C}(S')$. If $\sigma : S \to B$ is a strategy then so is $f^*\sigma : S' \to A$. Pullback along $f : A \to B$ may introduce causal links and events, present in $A$ but not in $B$. The pullback operation relies on both strategies and +-consistency-reflecting maps being stable under pullback, which we show at the end of this section. The pullback operation subsumes the operations of prefixing $\ominus.\sigma$ and $\oplus.\sigma$ and we can recover the previous prefix sums if we also have have sum types—see below.

*Sum types* If $A_i$, $i \in I$, is a countable family of games, we can form their sum, the game $\sum_{i\in I} A_i$ as the sum of event structures. If $\sigma : A_j$, for $j \in I$, we can create the probabilistic strategy $j\,\sigma : \sum_{i\in I} A_i$ in which we extend $\sigma$ with those initial $-$ve events needed to maintain receptivity. A probabilistic strategy of sum type $\sigma : \sum_{i\in I} A_i$ projects to a probabilistic strategy $(\sigma)_j : A_j$ where $j \in I$.

*Abstraction* $\lambda x : A.\sigma : A \multimap B$. Because probabilistic strategies form a monoidal-closed bicategory, with tensor $A \| B$ and function space $A \multimap B =_{\text{def}} A^\perp \| B$, they support an (linear) $\lambda$-calculus, which in this context permits process-passing as in [14].

*Recursive* types and probabilistic processes can be dealt with along standard lines [18].

The types as they stand are somewhat inflexible. These limitations can be remedied by introducing monads $T$ and new types of the form $T(A)$, though doing this in sufficient generality would involve the introduction of symmetry to games—see Section 7.

In the pullback operation we have relied on certain maps being stable under pullback. The following two propositions make good our debt, using techniques from open maps [9].

**Proposition 5.1** *If $\sigma : S \to B$ is a strategy then so is $f^*\sigma : S' \to A$.*

**Proof.** Define an *étale* map (w.r.t. to a path category $\mathcal{P}$) to be like an open map,

but where the lifting is unique. It is straightforward to show that the pullback of an étale map is étale. In fact, strategies can be regarded as étale maps, from which the proposition follows. Within the category of event structures with polarity and partial maps, take the path subcategory $\mathcal{P}$ to comprise all finite elementary event structures with polarity and take a typical map $f : p \to q$ in $\mathcal{P}$ to be a map such that:

(i) if $e \rightarrowtriangle_p e'$ with $e$ −ve and $e'$ +ve and both $f(e)$ and $f(e')$ defined, then $f(e) \rightarrowtriangle_q f(e')$; and

(ii) all events in $q$ not in the image $fp$ are −ve.

It can be checked that w.r.t. this choice of $\mathcal{P}$ the étale maps are precisely those maps which are strategies. □

**Proposition 5.2** *If $f : A \to B$ reflects +-consistency, then so does $f' : S' \to S$.*

**Proof.** As +-consistency-reflecting maps are special kinds of open maps, known to be stable under pullback. An appropriate path category comprises: all finite event structures with polarity for which there is a subset $M$ of $\leq$-maximal +-events s.t. a subset $X$ is consistent iff $X \cap M$ contains at most one event of $M$—all finite elementary event structures with polarity are included as $M$, the chosen subset of $\leq$-maximal +-events, may be empty; maps in the path category are rigid maps of event structures with polarity whose underlying functions are bijective on events.□

# 6 Quantum strategies

A more novel application is to a definition of quantum event structures and strategies. Throughout let $\mathcal{H}$ be a separable Hilbert space over the complex numbers. For operators $A, B$ on $\mathcal{H}$ we write $[A, B] =_{\text{def}} AB - BA$.

## *6.1 Quantum event structures*

**Definition 6.1** A *quantum event structure* (over $\mathcal{H}$) comprises an event structure $(E, \leq, \text{Con})$ together with an assignment $Q_e$ of projection or unitary operators on $\mathcal{H}$ to events $e \in E$ such that for all $e_1, e_2 \in E$,

$$e_1 \mathrel{co} e_2 \implies [Q_{e_1}, Q_{e_2}] = 0\,.$$

Given a finite configuration, $x \in \mathcal{C}(E)$, define the operator $A_x$ to be the composition $Q_{e_n} Q_{e_{n-1}} \cdots Q_{e_2} Q_{e_1}$ for some covering chain

$$\varnothing \xrightarrow{e_1} x_1 \xrightarrow{e_2} x_2 \cdots \xrightarrow{e_n} x_n = x$$

in $\mathcal{C}(E)$. This is well-defined as for any two covering chains up to $x$ the sequences of events are Mazurkiewicz trace equivalent, *i.e.* obtainable, one from the other,

by successively interchanging concurrent events. In particular $A_\varnothing$ is the identity operator on $\mathcal{H}$.

An *initial state* is given by a density operator $\rho$ on $\mathcal{H}$.

We regard $w \in \mathcal{C}^\infty(E)$ as a partial quantum experiment—it is 'partial' in the sense that it might extend to $w' \supseteq w$ in $\mathcal{C}^\infty(E)$. An experiment $w$ specifies which unitary operators (events of preparation) and projection operators (elementary positive tests) to apply and in which order. The order being partial permits commuting operators to be applied concurrently, independently of each other, perhaps in a distributed fashion.

Consider a quantum event structure with initial state $\rho$. While it does not make sense to attribute a probability distribution globally, over the whole space of configurations $\mathcal{C}^\infty(E)$, the next theorem says that with respect to any experiment $w$ there is a probability distribution $q_w$ over its possible outcomes. (Below, by an unnormalized density operator we mean a positive, self-adjoint operator with trace less than or equal to one.)

**Theorem 6.2** *Let $E, Q$ be a quantum event structure with initial state $\rho$. Each configuration $x \in \mathcal{C}(E)$ is associated with an unnormalized density operator $\rho_x =_{\text{def}} A_x \rho A_x^\dagger$ and a value in $[0,1]$ given by $v(x) =_{\text{def}} \mathrm{Tr}(\rho_x) = \mathrm{Tr}(A_x^\dagger A_x \rho)$. For any $w \in \mathcal{C}^\infty(E)$, the function $v$ restricts to a configuration-valuation $v_w$ on the elementary event structure $w$ (viz. the event structure with events $w$, and causal dependency and (trivial) consistency inherited from $E$); hence $v_w$ extends to a probability measure $q_w$ on $\mathcal{F}_w =_{\text{def}} \{x \in \mathcal{C}^\infty(E) \mid x \subseteq w\}$.*

**Proof.** We show $v$ restricts to a configuration-valuation on $\mathcal{F}_w$. As $A_\varnothing = \mathrm{id}_\mathcal{H}$, $v(\varnothing) = \mathrm{Tr}(\rho) = 1$. By Lemma 3.2, we need only to show $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$ when $y \xrightarrow{e_1} \subset x_1, \cdots, y \xrightarrow{e_n} \subset x_n$ in $\mathcal{F}_w$.

First, observe that if for some event $e_i$ the operator $Q_{e_i}$ is unitary, then $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$. W.l.o.g. suppose $e_n$ is assigned the unitary operator $U$. Then, $A_{x_n} = U A_y$ so

$$v(x_n) = \mathrm{Tr}(A_{x_n}^\dagger A_{x_n} \rho) = \mathrm{Tr}(A_y^\dagger U^\dagger U A_y \rho) = \mathrm{Tr}(A_y^\dagger A_y \rho) = v(y)\,.$$

Let $\varnothing \neq I \subseteq \{1, \cdots, n\}$. Then, either $\bigcup_{i \in I} x_i = \bigcup_{i \in I} x_i \cup x_n$ or $\bigcup_{i \in I} x_i \xrightarrow{e_n} \subset \bigcup_{i \in I} x_i \cup x_n$. In the either case—in the latter case by an argument similar to that above,

$$v(\bigcup_{i \in I} x_i) = v(\bigcup_{i \in I} x_i \cup x_n)\,.$$

Consequently,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n] \\ &= v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) - v(x_n) + \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i \cup x_n) \\ &= 0 \end{aligned}$$

—above index $I$ is understood to range over sets for which $\varnothing \neq I \subseteq \{1, \cdots, n\}$.

It remains to consider the case where all events $e_i$ are assigned projection operators $P_{e_i}$. As $x_1, \cdots, x_n \subseteq w$ we must have that all the projection operators $P_{e_1}, \cdots, P_{e_n}$ commute.

As $[P_{e_i}, P_{e_j}] = 0$, for $1 \leq i, j \leq n$, we can assume an orthonormal basis which extends the sub-basis of eigenvectors of all the projection operators $P_{e_i}$, for $1 \leq i \leq n$. Let $y \subseteq x \subseteq \bigcup_{1 \leq i \leq n} x_i$. Define $P_x$ to be the projection operator got as the composition of all the projection operators $P_e$ for $e \in x \setminus y$—this is a projection operator, well-defined irrespective of the order of composition as the relevant projection operators commute. Define $B_x$ to be the set of those basis vectors fixed by the projection operator $P_x$. In particular, $P_y$ is the identity operator and $B_y$ the set of all basis vectors. When $x, x' \in \mathcal{C}(E)$ with $y \subseteq x \subseteq \bigcup_{1 \leq i \leq n} x_i$ and $y \subseteq x' \subseteq \bigcup_{1 \leq i \leq n} x_i$,

$$B_{x \cup x'} = B_x \cap B_{x'} \,.$$

Also,

$$P_x |\psi\rangle = \sum_{i \in B_x} \langle i|\psi\rangle \, |i\rangle \,,$$

so

$$\langle \psi | P_x | \psi \rangle = \sum_{i \in B_x} \langle i|\psi\rangle\langle\psi|i\rangle = \sum_{i \in B_x} |\langle i|\psi\rangle|^2 \,,$$

for all $|\psi\rangle \in \mathcal{H}$.

Assume $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$, where the $\psi_k$ are normalised and all the $p_k$ are positive with sum $\sum_k p_k = 1$. For $x$ with $y \subseteq x \subseteq \bigcup_{1 \leq i \leq n} x_i$,

$$\begin{aligned}
v(x) &= \mathrm{Tr}(A_x^\dagger A_x \rho) \\
&= \mathrm{Tr}(A_y^\dagger P_x^\dagger P_x A_y \rho) \\
&= \mathrm{Tr}(A_y^\dagger P_x A_y \sum_k p_k |\psi_k\rangle\langle\psi_k|) \\
&= \sum_k p_k \, \mathrm{Tr}(A_y^\dagger P_x A_y |\psi_k\rangle\langle\psi_k|) \\
&= \sum_k p_k \langle A_y\psi_k | P_x | A_y\psi_k \rangle \\
&= \sum_{i \in B_x} \sum_k p_k |\langle i | A_y \psi_k \rangle|^2 \\
&= \sum_{i \in B_x} r_i \,,
\end{aligned}$$

where we define $r_i =_{\mathrm{def}} \sum_k p_k |\langle i|A_y\psi_k\rangle|^2$, necessarily a non-negative real for $i \in B_x$.

We now establish that

$$d_v^{(n)}[y; x_1, \cdots, x_n] = \sum_{i \in B_y \setminus B_{x_1} \cup \cdots \cup B_{x_n}} r_i \,,$$

for all $n \in \omega$, by mathematical induction—it then follows directly that its value is non-negative.

The base case of the induction, when $n = 0$, follows as

$$d_v^{(0)}[y;\,] = v(y) = \sum_{i \in B_y} r_i\,,$$

a special case of the result we have just established.

For the induction step, assume $n > 0$. Observe that

$$B_y \setminus B_{x_1} \cup \cdots \cup B_{x_{n-1}} = \left(B_y \setminus B_{x_1} \cup \cdots \cup B_{x_n}\right) \dot\cup \left(B_{x_n} \setminus B_{x_1 \cup x_n} \cup \cdots \cup B_{x_{n-1} \cup x_n}\right),$$

where as signified the outer union is disjoint. Hence,

$$\sum_{i \in B_y \setminus B_{x_1} \cup \cdots \cup B_{x_{n-1}}} r_i = \sum_{i \in B_y \setminus B_{x_1} \cup \cdots \cup B_{x_n}} r_i + \sum_{i \in B_{x_n} \setminus B_{x_1 \cup x_n} \cup \cdots \cup B_{x_{n-1} \cup x_n}} r_i\,,$$

By definition,

$$d_v^{(n)}[y; x_1, \cdots, x_n] =_{\text{def}} d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n]$$

—making use of the fact that we are only forming unions of compatible configurations. From the induction hypothesis,

$$d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] = \sum_{i \in B_y \setminus B_{x_1} \cup \cdots \cup B_{x_{n-1}}} r_i$$

$$\text{and } d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n] = \sum_{i \in B_{x_n} \setminus B_{x_1 \cup x_n} \cup \cdots \cup B_{x_{n-1} \cup x_n}} r_i\,.$$

Hence

$$d_v^{(n)}[y; x_1, \cdots, x_n] = \sum_{i \in B_y \setminus B_{x_1} \cup \cdots \cup B_{x_n}} r_i\,,$$

ensuring $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$, as required.

By Theorem 3.5, the configuration-valuation $v_w$ extends to a unique probability measure on $\mathcal{F}_w$. □

### *6.2 Quantum strategies*

A *quantum game* comprises $A, pol, Q, \rho$ where $A, pol$ is a race-free event structure with polarity, $A, Q$ is a quantum event structure with initial state $\rho$. A *strategy* in the quantum game comprises a probabilistic strategy in $A$, so a strategy $\sigma : S \to A$ together with configuration-valuation $v$ on $S$.

Given a strategy $v_S, \sigma : S \to A$ and counter-strategy $v_T, \tau : T \to A^\perp$ in a quantum game $A, Q, \rho$ we obtain as their composition before hiding the probabilistic event structure $T \circledast S$ with configuration-valuation $v(x) =_{\text{def}} v_S\pi_1(x) \times v_T\pi_2(x)$ on $x \in \mathcal{C}(T \circledast S)$—see Corollary 4.4. The event structure $T \circledast S$ is obtained as a pullback—Section 4.1.3—and is associated with a map $f =_{\text{def}} \sigma\pi_1 = \tau\pi_2 : T \circledast S \to A$. We can interpret $f : T \circledast S \to A$ as the probabilistic experiment which results from the interaction of the strategy $\sigma$ and the counter-strategy $\tau$. The event structure $T \circledast S$

carries a probability measure $\mu_v$. The probability that the play-off of $\sigma$ against $\tau$ produces a result in a Borel subset $U$ of of $\mathcal{C}^\infty(A)$, is given by the Lebesgue integral

$$\int q_w(U \cap \mathcal{F}_w)\, d\mu_v f^{-1}(w)\,.$$

Strategies in quantum games inherit the types and operations of probabilistic strategies, though additional constructs will be needed to introduce new entanglement across simple parallel compositions.

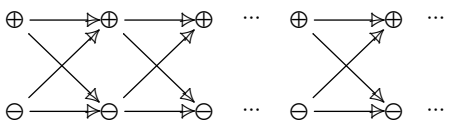
# 7 Extensions

As they stand the games here are games of *perfect information*. In games of *imperfect information* some moves are masked, or inaccessible, and strategies with dependencies on unseen moves are ruled out. It is straightforward to extend concurrent games to games with imperfect information in way that respects the operations of the bicategory of games [22] and does not disturb the addition of probability. A fixed preorder of *levels* $(\Lambda, \leq)$ is pre-supposed. The levels are to be thought of as levels of access, or permission. A $\Lambda$-game comprises a game $A$ with a *level function* $l : A \to \Lambda$ such that if $a \leq_A a'$ then $l(a) \leq l(a')$ for all $a, a' \in A$. A probabilistic $\Lambda$-strategy in the $\Lambda$-game is a probabilistic strategy $v_S, \sigma : S \to A$ for which if $s \leq_S s'$ then $l\sigma(s) \leq l\sigma(s')$ for all $s, s' \in S$. One interpretation of $\Lambda$, pertinent to the treatment of quantum strategies, is as space-time with $\lambda \leq \lambda'$ meaning there is a causal curve from $\lambda$ to $\lambda'$.

We can add *payoff* to a game $A$ as a Borel measurable function $X : \mathcal{C}^\infty(A) \to \mathbb{R}$. Given a probabilistic strategy $v_S, \sigma : S \to A$ and counter-strategy $v_T, \tau : T \to A^\perp$ we obtain their composition before hiding as their pullback $T \circledast S, \pi_1, \pi_2$, associated with the map $f =_{\text{def}} \sigma\pi_1 = \tau\pi_2 : T \circledast S \to A$. The event structure $T \circledast S$ comes equipped with a configuration-valuation $v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$, for $x \in \mathcal{C}(T \circledast S)$. The *expected payoff* is obtained as the Lebesgue integral

$$\mathbf{E}_{\sigma,\tau}(X) = \int X(fx)\; d\mu_v(x)\,.$$

In particular, *Blackwell games* [12] become a special case of probabilistic $\Lambda$-games with payoff. Blackwell games are games of imperfect information for which an appropriate choice of $\Lambda$ is the infinite elementary event structure:



A Blackwell game is given by $A$, a race-free concurrent game with payoff $X$, for which there is a (necessarily unique) polarity-preserving rigid map from $A$ to $\Lambda$—this map becomes the level function. Moves in $A$ occur in rounds comprising a choice of move for Opponent and a choice of move for Player made independently. Traditionally, in

Blackwell games a strategy (for Player) is a 'total' $\Lambda$-strategy in such a $\Lambda$-game—strategies are restricted to those assigning *total* probability distributions at each round. In fact, the existing literature is most often concerned with strategies which always progress, which we can express very generally by insisting non $\subseteq^+$-maximal finite configurations of the strategy are transient—*cf.* Proposition 3.6:

**Definition 7.1** Say a probabilistic strategy $\sigma : S \to A$ with configuration-valuation $v$ is *total* when $\inf\{d_v^{(n)}[y; x_1, \cdots, x_n] \mid n \in \omega \ \& \ y \not\subseteq^+ x_1, \cdots, x_n\} \neq 0$ implies $\sigma y$ is $\subseteq^+$-maximal, for all $y \in \mathcal{C}(S)$.

In the case of Blackwell games total strategies amount to those used traditionally.

There are several reasons to consider symmetry in games, situations where distinct plays are essentially similar to one another. Symmetry can help in the analysis of games, by for instance reducing the number of cases to consider. Symmetry can also help compensate for the overly-concrete nature of event structures in representing games; many useful operations on games which are not monads or comonads w.r.t. strategies become so *up to symmetry* [19,2] and this leads, for example, to richer type systems. Symmetry on an event structure can be captured through an *isomorphism family* which expresses when one finite configuration of the event structure is essentially the same as another [19]. It is a straightforward matter to ensure that configuration-valuations, attributing probability, respect the isomorphism family. The addition of symmetry to games meshes well with the introduction of probability. This should enable a formal connection with the probabilistic games of Danos and Harmer [3] which are based on HO games—allowing copying, so whose relation with concurrent games requires suitable (co)monads to exist, so symmetry.

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