Concurrent Strategies

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***Abstract*—A bicategory of very general nondeterministic concurrent games and strategies is presented. The intention is to formalize distributed games in which both Player (or a team of players) and Opponent (or a team of opponents) can interact in highly distributed fashion, without, for instance, enforcing that their moves alternate.**

I. Introduction

This paper characterizes *nondeterministic concurrent strategies* in *concurrent games* within a very general model of concurrent/distributed computation.

More precisely, games and strategies are represented as event structures with polarities to distinguish the moves of Player and Opponent (more accurately thought of as teams of players and opponents)—*cf.* [1]. A total map $\sigma : S \to A$ of event structures with polarity can be understood as a pre-strategy in a game A—the map ensures that Player and Opponent respect the constraints of the game. Following Joyal's exposition of Conway games [2], a pre-strategy from a game A to a game B is understood as a pre-strategy in a composite game $A^\perp \| B$, got by setting the dual game of A, reversing the roles of Player and Opponent, in parallel with B. Within this general scheme, concurrent strategies—pre-strategies for which copy-cat strategies behave as identities w.r.t. composition of pre-strategies—are characterized as those pre-strategies which satisfy the two conditions of *receptivity* and *innocence*.

It is sketched how (bi)categories of *stable spans*, certain *profunctors*, Berry's *stable functions*, and *simple games* arise as sub(bi)categories of concurrent games. The important special case of *deterministic* concurrent strategies coincides with the *receptive ingenuous* strategies of Melliès and Mimram [3]. Deterministic strategies find direct expression as *closure operators*, an elegant formulation of deterministic concurrent strategies in early work of Abramsky and Melliès [4]. The relation with other work is ongoing and unfinished. There are clear expressions of innocence as *"saturation"* conditions in early concurrent games of Laird [5], Ghica and Murawski [6]. We have been inspired by the paper of Faggian and Piccolo [7], which in part communicates an idea of Hyland on extending the copy-cat strategy to partial orders of moves, a precursor to the distributed copy-cat here and in [7].[1]

Not surprisingly, the proofs here are reminiscent of certain proofs in distributed algorithms (we are most familiar with similar dependency-chasing proofs in security protocols). More intriguing is the prospect that proofs of distributed algorithms could embed into the general conceptual framework of concurrent games proposed here.

[1] We adopt the term *innocence* from [7]—it is not directly related to innocence in Hyland-Ong games [8].

II. Event structures and stable families

We quickly review event structures and the broader model of stable families, their properties and constructions.

An *event structure* comprises $(E, \mathrm{Con}, \leq)$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$
$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$
$$Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}$$
$$X \in \mathrm{Con} \ \&\ e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.$$

The (finite) *configurations*, $\mathcal{C}(E)$, of an event structure E consist of those finite subsets $x \subseteq E$ which are

Consistent: $x \in \mathrm{Con}$, and
Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

Two events which are both consistent and incomparable w.r.t. causal dependency in an event structure are regarded as *concurrent*. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, will play a very important role.

Operations such as synchronized parallel composition are awkward to define directly on the simple event structures above. It is useful to broaden event structures to stable families, where operations are often carried out more easily, and then turned into event structures by the operation Pr below.

A *stable family* comprises $\mathcal{F}$, a nonempty family of finite subsets, called *configurations*, satisfying:
Completeness: $\forall Z \subseteq \mathcal{F}.\ Z\uparrow \implies \bigcup Z \in \mathcal{F}$;
Coincidence-freeness: For all $x \in \mathcal{F}$, $e, e' \in x$ with $e \neq e'$,

$$\exists y \in \mathcal{F}.\ y \subseteq x \ \&\ (e \in y \iff e' \notin y);$$

Stability: $\forall Z \subseteq \mathcal{F}.\ Z \neq \varnothing \ \&\ Z\uparrow \implies \bigcap Z \in \mathcal{F}$.
($Z\uparrow$ means $\exists x \in \mathcal{F} \forall z \in Z.\ z \subseteq x$, and expresses the compatibility of Z.) We call elements of $\bigcup \mathcal{F}$ *events* of $\mathcal{F}$.

Proposition 1. *Let x be a configuration of a stable family $\mathcal{F}$. For $e, e' \in x$ define*

$$e' \leq_x e \text{ iff } \forall y \in \mathcal{F}.\ y \subseteq x \ \&\ e \in y \implies e' \in y.$$

When $e \in x$ define the prime configuration

$$[e]_x = \bigcap \{y \in \mathcal{F} \mid y \subseteq x \ \&\ e \in y\} \ .$$

Then $\leq_x$ is a partial order and $[e]_x$ is a configuration such that

$$[e]_x = \{e' \in x \mid e' \leq_x e\}.$$

Moreover the configurations $y \subseteq x$ are exactly the down-closed subsets of $\leq_x$.

Proposition 2. *Let $\mathcal{F}$ be a stable family. Then,* $\mathrm{Pr}(\mathcal{F}) =_{\mathrm{def}} (P, \mathrm{Con}, \leq)$ *is an event structure where:*

$$P = \{[e]_x \mid e \in x \ \&\ x \in \mathcal{F}\}\ ,$$
$$Z \in \mathrm{Con} \textit{ iff } Z \subseteq P \ \&\ \bigcup Z \in \mathcal{F} \ \textit{ and},$$
$$p \leq p' \textit{ iff } p, p' \in P \ \&\ p \subseteq p'\ .$$

A (partial) map of stable families $f : \mathcal{F} \to \mathcal{G}$ is a partial function f from the events of $\mathcal{F}$ to the events of $\mathcal{G}$ such that for all configurations $x \in \mathcal{F}$,

$$fx \in \mathcal{G} \ \&\ (\forall e_1, e_2 \in x.\ f(e_1) = f(e_2) \implies e_1 = e_2)\ .$$

Maps of event structures are maps of their stable families of configurations. Maps compose as functions. We say a map is *total* when it is total as a function. Say a total map of event structures is *rigid* when it preserves causal dependency.

Pr is the right adjoint of the "inclusion" functor, taking an event structure E to the stable family $\mathcal{C}(E)$. The unit of the adjunction $E \to \mathrm{Pr}(\mathcal{C}(E))$ takes and event e to the prime configuration $[e] =_{\mathrm{def}} \{e' \in E \mid e' \leq e\}$. The counit $max : \mathcal{C}(\mathrm{Pr}(\mathcal{F})) \to \mathcal{F}$ takes prime configuration $[e]_x$ to e.

Proposition 3. *Let E and E' be event structures. Suppose*

$$\theta_x : x \cong \theta_x x, \ \textit{indexed by } x \in \mathcal{C}(E),$$

is a family of bijections such that whenever $\theta_y : y \cong \theta_y y$ is in the family then its restriction $\theta_z : z \cong \theta_z z$ is also in the family, whenever $z \in \mathcal{C}(E)$ and $z \subseteq y$. Then, $\theta =_{\mathrm{def}} \bigcup_{x \in \mathcal{C}(E)} \theta_x$ is the unique total map of event structures from E to E' such that $\theta\, x = \theta_x x$ for all $x \in \mathcal{C}(E)$.

Proposition 4. *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of stable families. Let $e, e' \in x$, a configuration of $\mathcal{F}$. If $f(e)$ and $f(e')$ are defined and $f(e) \leq_{fx} f(e')$ then $e \leq_x e'$.*

Definition 5. Let $\mathcal{F}$ be a stable family. We use $x \multimap\!\!\subset y$ to mean y covers x in $\mathcal{F}$, *i.e.* $x \subset y$ in $\mathcal{F}$ with nothing in between, and $x \overset{e}{\longrightarrow\!\!\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{F}$ and event $e \notin x$. We sometimes use $x \overset{e}{\longrightarrow\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \overset{e}{\longrightarrow\!\!\subset} y$ for some y. W.r.t. $x \in \mathcal{F}$, write $[e)_x =_{\mathrm{def}} \{e' \in E \mid e' \leq_x e \ \&\ e' \neq e\}$, so, for example, $[e)_x \overset{e}{\longrightarrow\!\!\subset} [e]_x$. The relation of *immediate* dependence of event structures generalizes: with respect to $x \in \mathcal{F}$, the relation $e \rightarrowtriangle_x e'$ means $e \leq_x e'$ with $e \neq e'$ and no event in between.

III. Process operations

A. Products

Let $\mathcal{A}$ and $\mathcal{B}$ be stable families with events A and B, respectively. Their product, the stable family $\mathcal{A} \times \mathcal{B}$, has events comprising pairs in $A \times_* B =_{\mathrm{def}} \{(a, *) \mid a \in A\} \cup \{(a, b) \mid a \in A \ \&\ b \in B\} \cup \{(*, b) \mid b \in B\}$, the product of sets with partial functions, with (partial) projections π_1 and π_2—treating $*$ as 'undefined'—with configurations

$x \in \mathcal{A} \times \mathcal{B}$ iff
x is a finite subset of $A \times_* B$ s.t. $\pi_1 x \in \mathcal{A} \ \&\ \pi_2 x \in \mathcal{B}$,
$\forall e, e' \in x.\ \pi_1(e) = \pi_1(e')$ or $\pi_2(e) = \pi_2(e') \Rightarrow e = e'$, &
$\forall e, e' \in x.\ e \neq e' \Rightarrow \exists y \subseteq x.\ \pi_1 y \in \mathcal{A} \ \&\ \pi_2 y \in \mathcal{B}$ &
$(e \in y \iff e' \notin y)\ .$

Right adjoints preserve products. Consequently we obtain a product of event structures A and B by first regarding them as stable families $\mathcal{C}(A)$ and $\mathcal{C}(B)$, forming their product $\mathcal{C}(A) \times \mathcal{C}(B), \pi_1, \pi_2$, and then constructing the event structure

$$A \times B =_{\mathrm{def}} \mathrm{Pr}(\mathcal{C}(A) \times \mathcal{C}(B))$$

and its projections as $\Pi_1 =_{\mathrm{def}} \pi_1 max$ and $\Pi_2 =_{\mathrm{def}} \pi_2 max$.

Lemma 6. *Suppose $e \rightarrowtriangle_x e'$ in a product of stable families $\mathcal{A} \times \mathcal{B}, \pi_1, \pi_2$.*
(i) If $e = (a, *)$ then $e' = (a', b)$ or $e' = (a', *)$ with $a \rightarrowtriangle_{\pi_1 x} a'$ in $\mathcal{A}$.
(ii) If $e' = (a', *)$ then $e = (a, b)$ or $e = (a, *)$ with $a \rightarrowtriangle_{\pi_1 x} a'$ in $\mathcal{A}$.
(iii) If $e = (a, b)$ and $e' = (a', b')$ then $a \rightarrowtriangle_{\pi_1 x} a'$ in $\mathcal{A}$ or $b \rightarrowtriangle_{\pi_2 x} b'$ in $\mathcal{B}$.

B. Restriction

The *restriction* of $\mathcal{F}$ to a subset of events R is the stable family $\mathcal{F} \upharpoonright R =_{\mathrm{def}} \{x \in \mathcal{F} \mid x \subseteq R\}$. Defining $E \upharpoonright R$, the restriction of an event structure E to a subset of events R, to have events $E' = \{e \in E \mid [e] \subseteq R\}$ with causal dependency and consistency induced by E, we obtain $\mathcal{C}(E \upharpoonright R) = \mathcal{C}(E) \upharpoonright R$.

Proposition 7. *Let $\mathcal{F}$ be a stable family and R a subset of its events. Then,* $\mathrm{Pr}(\mathcal{F} \upharpoonright R) = \mathrm{Pr}(\mathcal{F}) \upharpoonright max^{-1} R$.

C. Synchronized compositions

Synchronized parallel compositions are obtained as restrictions of products to those events which are allowed to synchronize or occur asynchronously. For example, the synchronized composition of Milner's CCS on stable families $\mathcal{A}$ and $\mathcal{B}$ (with labelled events) is defined as $\mathcal{A} \times \mathcal{B} \upharpoonright R$ where R comprises events which are pairs $(a, *)$, $(*, b)$ and (a, b), where in the latter case the events a of $\mathcal{A}$ and b of $\mathcal{B}$ carry complementary labels. Similarly, synchronized compositions of event structures A and B are obtained as restrictions $A \times B \upharpoonright R$. By Proposition 7, we can equivalently form a synchronized composition of event structures by forming the synchronized composition of their stable families of configurations, and then obtaining the resulting event structure—this has the advantage of eliminating superfluous events earlier.

D. Projection

Event structures support a simple form of hiding. Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ iff $v \leq v' \ \&\ v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con} \ \&\ X \subseteq V$.

IV. Event structures with polarities

We shall represent both a game and a strategy in a game as an event structure with polarity, comprising an event structure together with a polarity function $pol : E \to \{+,-\}$ ascribing a polarity $+$ or $-$ to its events E. The events correspond to (occurrences of) moves. The two polarities $+/-$ express the dichotomy: Player/Opponent; Process/Environment; or Ally/Enemy. Maps of event structures with polarity are maps of event structures which preserve polarity.

A. Operations

1) Dual: The *dual*, $E^\perp$, of an event structure with polarity E comprises a copy of the event structure E but with a reversal of polarities. It obviously extends to a functor. Write $\overline{e} \in E^\perp$ for the event complementary to $e \in E$ and *vice versa*.

2) Simple parallel composition: This operation simply juxtaposes two event structures with polarity. Let $(A, \leq_A, \mathrm{Con}_A, pol_A)$ and $(B, \leq_B, \mathrm{Con}_B, pol_B)$ be event structures with polarity. The events of $A \| B$ are $(\{1\} \times A) \cup (\{2\} \times B)$, their polarities unchanged, with: the only relations of causal dependency given by $(1, a) \leq (1, a')$ iff $a \leq_A a'$ and $(2, b) \leq (2, b')$ iff $b \leq_B b'$; a subset of events C is consistent in $A \| B$ iff $\{a \mid (1, a) \in C\} \in \mathrm{Con}_A$ and $\{b \mid (2, b) \in C\} \in \mathrm{Con}_B$. The operation extends to a functor—put the two maps in parallel.

B. Categories for games

We remark that event structures with polarity appear to provide a rich environment in which to explore structural properties of games and strategies. There are adjunctions

$$\begin{tikzcd}
\mathcal{PA}_r \arrow[r, hook, shift right] \arrow[d, shift right] \arrow[r, phantom, "\top"] \arrow[d, phantom, "\vdash"] & \mathcal{PF}_r \arrow[l, shift right] \arrow[r, hook, shift right] \arrow[d, shift right] \arrow[r, phantom, "\top"] \arrow[d, phantom, "\vdash"] & \mathcal{PE}_r \arrow[l, shift right] \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] & \mathcal{PE}_t \arrow[l, shift right] \\
\mathcal{PA}_r^{\#} \arrow[u, hook, shift right] \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] & \mathcal{PF}_r^{\#} \arrow[u, hook, shift right] \arrow[l, shift right] & &
\end{tikzcd}$$

relating $\mathcal{PE}_t$, the category of event structures with polarity with total maps, to subcategories $\mathcal{PE}_r$, with rigid maps, $\mathcal{PF}_r$ of forest-like (or filiform) event structures with rigid maps, and $\mathcal{PA}_r$, its full subcategory where polarities alternate along a branch; in $\mathcal{PF}_r^{\#}$ and $\mathcal{PA}_r^{\#}$ distinct branches are inconsistent. We shall mainly be considering games in $\mathcal{PE}_t$. Lamarche games and those of sequential algorithms belong to $\mathcal{PA}_r$ [9]. Conway games inhabit $\mathcal{PF}_r^{\#}$, in fact a coreflective subcategory of $\mathcal{PE}_t$ as the inclusion is now full; Conway's 'sum' is obtained by applying the right adjoint to the $\|$-composition of Conway games in $\mathcal{PE}_t$. Further refinements are possible. The 'simple games' of [10], [11] belong to $\mathcal{PA}_r^{-\#}$, the coreflective subcategory of $\mathcal{PA}_r^{\#}$ comprising "polarized" games, starting with moves of Opponent. The 'tensor' of simple games is recovered by applying the right adjoint of $\mathcal{PA}_r^{-\#} \hookrightarrow \mathcal{PE}_t$ to their $\|$-composition in $\mathcal{PE}_t$. Generally, the right adjoints, got by composition, from $\mathcal{PE}_t$ to the other categories fail to conserve immediate causal dependency. Such facts led Melliès *et al.* to the insight that uses of pointers in game semantics can be an artifact of working with models of games which do not take account of the independence of moves [1], [3].

V. Pre-strategies

Let A be an event structure with polarity, thought of as a game (sometimes called an "arena" in game semantics); its events stand for the possible occurrences of moves of Player ($+$) and Opponent ($-$) and its causal dependency and consistency relations the constraints imposed by the game. A *pre-strategy* in A is a total map $\sigma : S \to A$ from an event structure with polarity S. The $+$-events of S stand for the moves of Player, generally in answer to the moves of Opponent, the $-$-events of S.

We shall later refine the definition of pre-strategy to that of *strategy*. For example, in a *strategy* we expect that it should not be possible for Player to affect the moves of Opponent beyond the dictates of the game. This and other concerns are not reflected adequately in the definition of pre-strategy as it stands. What is captured by taking a pre-strategy to be a total map $\sigma : S \to A$ is that the behaviour of Player and Opponent as narrated by S respects the constraints of game A; every move of Player and Opponent is a move allowed by the game. Note that pre-strategies (and strategies) are nondeterministic in that Player moves may be inconsistent, and are not necessarily determined by the preceding Player and Opponent moves. A pre-strategy represents a nondeterministic play of the game.

Let A and B be event structures with polarity. Following Joyal [2], a pre-strategy from A to B is a pre-strategy in $A^\perp \| B$, so a total map $\sigma : S \to A^\perp \| B$. It thus determines a span

$$\begin{tikzcd}
& S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\
A^\perp & & B\,,
\end{tikzcd}$$

of event structures with polarity where σ_1, σ_2 are *partial* maps. In fact, a pre-strategy from A to B corresponds to such spans where for all $s \in S$ either, but not both, $\sigma_1(s)$ or $\sigma_2(s)$ is defined. Two pre-strategies will be essentially the same when they are isomorphic as spans. We write $\sigma \cong \tau$, for pre-strategies σ and τ from A to B when their spans are isomorphic. We write $\sigma : A \mathrel{+\!\!\!\rightarrow} B$ to express that σ is a pre-strategy from A to B. The notation raises the question of how pre-strategies compose and the nature of identities.

A. Concurrent copy-cat

Identities on games are given by copy-cat strategies—strategies for Player based on copying the latest moves made by Opponent.

Let A be an event structure with polarity. The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}_A \to A^\perp \| A$. It describes a concurrent, or distributed, strategy based on the idea that Player moves, of $+$ve polarity, always copy previous corresponding moves of Opponent, of $-$ve polarity.

For $c \in A^\perp \| A$ we use $\overline{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component, *i.e.*

$$\overline{(1, a)} = (2, \overline{a}) \text{ and } \overline{(2, a)} = (1, \overline{a})\,.$$

Proposition 8. *Let A be an event structure with polarity. There is an event structure with polarity $\mathbb{C}_A$ having the same events, consistency and polarity as $A^\perp \| A$ but with causal dependency $\le_{\mathbb{C}_A}$ given as the transitive closure of the relation*

$$\le_{A^\perp\|A} \cup \{(\overline{c}, c) \mid c \in A^\perp \| A \ \&\ pol_{A^\perp\|A}(c) = +\}\,.$$

Moreover,
(i) $c \rightarrowtriangle c'$ in $\mathbb{C}_A$ iff

$$c \rightarrowtriangle c' \text{ in } A^\perp\|A \text{ or } pol_{A^\perp\|A}(c') = + \ \&\ \overline{c} = c'\,;$$

(ii) $x \in \mathcal{C}(\mathbb{C}_A)$ iff

$$x \in \mathcal{C}(A^\perp\|A) \ \&\ \forall c \in x.\ pol_{A^\perp\|A}(c) = + \implies \overline{c} \in x\,.$$

Proof. It can first be checked that defining

$$\begin{aligned} c \le_{\mathbb{C}_A} c' \text{ iff } &(i)\ c \le_{A^\perp\|A} c' \text{ or}\\ &(ii)\ \exists c_0 \in A^\perp\|A.\ pol_{A^\perp\|A}(c_0) = + \ \&\\ &\qquad c \le_{A^\perp\|A} \overline{c_0} \ \&\ c_0 \le_{A^\perp\|A} c'\,, \end{aligned}$$

yields a partial order. Note that

$$c \le_{A^\perp\|A} d \ \text{ iff } \ \overline{c} \le_{A^\perp\|A} \overline{d}\,,$$

used in verifying transitivity and antisymmetry. The relation $\le_{\mathbb{C}_A}$ is clearly the transitive closure of $\le_{A^\perp\|A}$ together with all extra causal dependencies $(\overline{c}, c)$ where $pol_{A^\perp\|A}(c) = +$. The remaining properties required for $\mathbb{C}_A$ to be an event structure follow routinely.
(i) From the above characterization of $\le_{\mathbb{C}_A}$.
(ii) From $\mathbb{C}_A$ and $A^\perp\|A$ sharing the same consistency relation and the extra causal dependency adjoined to $\mathbb{C}_A$. □

Based on Proposition 8, define the *copy-cat* pre-strategy from A to A to be the pre-strategy $\gamma_A : \mathbb{C}_A \to A^\perp\|A$ where $\mathbb{C}_A$ comprises the event structure with polarity $A^\perp\|A$ together with extra causal dependencies $\overline{c} \le_{\mathbb{C}_A} c$ for all events c with $pol_{A^\perp\|A}(c) = +$, and γ_A is the identity on the set of events common to both $\mathbb{C}_A$ and $A^\perp\|A$.

B. Composing pre-strategies

Consider two pre-strategies $\sigma : A \rightarrow\!\!\!\!\!\!\!+\ B$ and $\tau : B \rightarrow\!\!\!\!\!\!\!+\ C$ as spans:

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & & & T \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ A^\perp & & B & B^\perp & & C\,. \end{tikzcd}$$

We show how to define their composition $\tau\odot\sigma : A \rightarrow\!\!\!\!\!\!\!+\ C$. If we ignore polarities the partial maps of event structures σ_2 and τ_1 have a common codomain, the underlying event structure of B and $B^\perp$. The composition $\tau\odot\sigma$ will be constructed as a synchronized composition of S and T, in which output events of S synchronize with input events of T, followed by an operation of hiding 'internal' synchronization events. Only those events s from S and t from T for which $\sigma_2(s) = \overline{\tau_1(t)}$ synchronize; note that then s and t must have opposite polarities as this is so for their images $\sigma_2(s)$ in B and $\tau_1(t)$ in $B^\perp$. The event resulting from the synchronization of s and t has indeterminate polarity and will be hidden in the composition $\tau\odot\sigma$.

Formally, we use the construction of synchronized composition and projection of Section III-C. Via projection we hide all those events with undefined polarity.

We first define the composition of the families of configurations of S and T as a synchronized composition of stable families. We form the product of stable families $\mathcal{C}(S)\times\mathcal{C}(T)$ with projections π_1 and π_2, and then form a restriction:

$$\mathcal{C}(T)\odot\mathcal{C}(S) =_{\text{def}} \mathcal{C}(S)\times\mathcal{C}(T) \upharpoonright R$$

where

$$\begin{aligned} R \ = \ &\{(s,*) \mid s \in S \ \&\ \sigma_1(s) \text{ is defined}\} \cup\\ &\{(s,t) \mid s \in S \ \&\ t \in T \ \&\ \sigma_2(s) = \overline{\tau_1(t)} \text{ with both defined}\} \cup\\ &\{(*,t) \mid t \in T \ \&\ \tau_2(t) \text{ is defined}\}\,. \end{aligned}$$

The stable family $\mathcal{C}(T)\odot\mathcal{C}(S)$ is the synchronized composition of the stable families $\mathcal{C}(S)$ and $\mathcal{C}(T)$ in which synchronizations are between events of S and T which project, under σ_2 and τ_1 respectively, to complementary events in B and $B^\perp$. The stable family $\mathcal{C}(T)\odot\mathcal{C}(S)$ represents all the configurations of the composition of pre-strategies, including internal events arising from synchronizations. We obtain the synchronized composition as an event structure by forming $\Pr(\mathcal{C}(T)\odot\mathcal{C}(S))$, in which events are the primes of $\mathcal{C}(T)\odot\mathcal{C}(S)$. This synchronized composition still has internal events.

To obtain the composition of pre-strategies we hide the internal events due to synchronizations. The event structure of the composition of pre-strategies is defined to be

$$T\odot S =_{\text{def}} \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \downarrow V\,,$$

the projection onto "visible" events,

$$\begin{aligned} V = \ &\{p \in \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \mid \exists s \in S.\ max(p) = (s,*)\} \cup\\ &\{p \in \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \mid \exists t \in T.\ max(p) = (*,t)\}\,. \end{aligned}$$

Finally, the composition $\tau\odot\sigma$ is defined by the span

$$\begin{tikzcd} & T\odot S \arrow[dl, "\upsilon_1"'] \arrow[dr, "\upsilon_2"] & \\ A^\perp & & C \end{tikzcd}$$

where υ_1 and υ_2 are maps of event structures, which on events p of $T\odot S$ act so $\upsilon_1(p) = \sigma_1(s)$ when $max(p) = (s,*)$ and $\upsilon_2(p) = \tau_2(t)$ when $max(p) = (*,t)$, and are undefined elsewhere.

Proposition 9. *Above, υ_1 and υ_2 are partial maps of event structures with polarity, which together define a pre-strategy $\upsilon : A \rightarrow\!\!\!\!\!\!\!+\ C$. For $x \in \mathcal{C}(T\odot S)$,*

$$\upsilon_1 x = \sigma_1\pi_1 \bigcup x \ \text{ and } \ \upsilon_2 x = \tau_2\pi_2 \bigcup x\,.$$

Proof. Consider the two maps of event structures

$$\begin{aligned} u_1 &: \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \xrightarrow{\Pi_1} S \xrightarrow{\sigma_1} A^\perp\,,\\ u_2 &: \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \xrightarrow{\Pi_2} T \xrightarrow{\tau_2} C\,, \end{aligned}$$

where Π_1, Π_2 are (restrictions of) projections of the product of event structures. *E.g.* for $p \in \Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$, $\Pi_1(p) = s$ precisely when $max(p) = (s, *)$, so $\sigma_1(s)$ is defined, or when $max(p) = (s,t)$, so $\sigma_1(s)$ is undefined. The partial functions υ_1 and υ_2 are restrictions of the two maps u_1 and u_2 to the projection set V. But V consists exactly of those events in $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$ where u_1 or u_2 is defined. It follows that υ_1 and υ_2 are maps of event structures.

Clearly one and only one of υ_1, υ_2 are defined on any event in $T \odot S$ so they form a pre-strategy. Their effect on $x \in \mathcal{C}(T \odot S)$ follows directly from their definition. □

Proposition 10. *Let* $\sigma : A \mathrel{\longrightarrow\!\!\!\!\!\!\!+} B$, $\tau : B \mathrel{\longrightarrow\!\!\!\!\!\!\!+} C$ *and* $\upsilon : C \mathrel{\longrightarrow\!\!\!\!\!\!\!+} D$ *be pre-strategies. The two compositions* $\upsilon \odot (\tau \odot \sigma)$ *and* $(\upsilon \odot \tau) \odot \sigma$ *are isomorphic.*

Proof. The natural isomorphism $S \times (T \times U) \cong (S \times T) \times U$, associated with the product of event structures S, T, U, restricts to the required isomorphism of spans as the synchronizations involved in successive compositions are disjoint. □

Remark. We have chosen to project away from internal events, rather than treat them as events of neutral polarity, to obviate the extra bicategorical complications internal events involve.

C. Duality

A pre-strategy $\sigma : A \mathrel{\longrightarrow\!\!\!\!\!\!\!+} B$ corresponds to a dual pre-strategy $\sigma^\perp : B^\perp \mathrel{\longrightarrow\!\!\!\!\!\!\!+} A^\perp$. This duality arises from the correspondence

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ A^\perp & & B \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} & S \arrow[dl, "\sigma_2"'] \arrow[dr, "\sigma_1"] & \\ (B^\perp)^\perp & & A^\perp \,. \end{tikzcd}$$

It is easy to check that the dual of copy-cat, $\gamma_A^\perp$, is isomorphic, as a span, to the copy-cat of the dual, $\gamma_{A^\perp}$, for A an event structure with polarity. It is also straightforward, though more involved, to show that the dual of a composition of pre-strategies $(\tau \odot \sigma)^\perp$ is isomorphic as a span to the composition $\sigma^\perp \odot \tau^\perp$. Duality, as usual, will save us work.

VI. Strategies

This section is devoted to the main result of this paper: that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient in order for copy-cat to behave as identity w.r.t. the composition of pre-strategies. It becomes compelling to define a *(nondeterministic) concurrent strategy*, in general, as a pre-strategy which is receptive and innocent.

A. Necessity of receptivity and innocence

The properties of *receptivity* and *innocence* of a pre-strategy, described below, will play a central role.

Receptivity. Say a pre-strategy $\sigma : S \to A$ is *receptive* when $\sigma x \overset{a}{\longrightarrow\!\!\!\!\!\subset}$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{\longrightarrow\!\!\!\!\!\subset}$ & $\sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$. Receptivity ensures that no Opponent move which is possible is disallowed.

Innocence. Say a pre-strategy σ is *innocent* when it is both $+$-innocent and $-$-innocent:

$+$-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.
$-$-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

The definition of a pre-strategy $\sigma : S \to A$ ensures that the moves of Player and Opponent respect the causal constraints of the game A. Innocence restricts Player further. Locally, within a configuration, Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$. Thus innocence gives Player the freedom to await Opponent moves before making their move, but prevents Player having any influence on the moves of Opponent beyond those stipulated in the game A; more surprisingly, innocence also disallows any immediate causality of the form $\oplus \rightarrowtriangle \oplus$, purely between Player moves, not already stipulated in the game A.

Two important consequences of $-$-innocence:

Lemma 11. *Let* $\sigma : S \to A$ *be a pre-strategy. Suppose, for* $s, s' \in S$, *that*

$$[s] \uparrow [s'] \ \&\ pol_S(s) = pol_S(s') = - \ \&\ \sigma(s) = \sigma(s')\,.$$

(i) If σ *is* $-$*-innocent, then* $[s) = [s')$.
(ii) If σ *is receptive and* $-$*-innocent, then* $s = s'$.
[$x \uparrow y$ *expresses the compatibility of* $x, y \in \mathcal{C}(S)$.*]*

Proof. (i) Assume the property above holds of $s, s' \in S$. Assume σ is $-$-innocent. Suppose $s_1 \rightarrowtriangle s$. Then by $-$-innocence, $\sigma(s_1) \rightarrowtriangle \sigma(s)$. As $\sigma(s') = \sigma(s)$ and σ is a map of event structures there is $s_2 < s'$ s.t. $\sigma(s_2) = \sigma(s_1)$. But s_1, s_2 both belong to the configuration $[s] \cup [s']$ so $s_1 = s_2$, as σ is a map, and $s_1 < s'$. Symmetrically, if $s_1 \rightarrowtriangle s'$ then $s_1 < s$. It follows that $[s) = [s')$. (ii) Now both $[s) \overset{s}{\longrightarrow\!\!\!\!\!\subset}$ and $[s) \overset{s'}{\longrightarrow\!\!\!\!\!\subset}$ with $\sigma(s) = \sigma(s')$ where both s, s' have $-$ve polarity. If, further, σ is receptive, $s = s'$. □

Let x and x' be configurations of an event structure with polarity. Write $x \subseteq^- x'$ to mean $x \subseteq x'$ and $pol(x' \setminus x) \subseteq \{-\}$, *i.e.* the configuration x' extends the configuration x solely by events of $-$ve polarity. In the presence of $-$-innocence, receptivity strengthens to the following useful property:

Lemma 12. *Let* $\sigma : S \to A$ *be a* $-$*-innocent pre-strategy. The pre-strategy* σ *is* receptive *iff whenever* $\sigma x \subseteq^- y$ *there is a* unique $x' \in \mathcal{C}(S)$ *so that* $x \subseteq x'$ & $\sigma x' = y$. *Diagrammatically,*

$$\begin{tikzcd} x \arrow[r, dotted, no head, "\subseteq"] \arrow[d, mapsto, "\sigma"'] & x' \arrow[d, dotted, mapsto, "\sigma"] \\ \sigma x \arrow[r, phantom, "\subseteq^-"] & y\,. \end{tikzcd}$$

[It will necessarily be the case that $x \subseteq^- x'$.*]*

Proof. *"if":* Clear. *"Only if":* Assuming $\sigma x \subseteq^- y$ we can form a covering chain

$$\sigma x \overset{a_1}{\longrightarrow\!\!\!\!\!\subset} y_1 \cdots \overset{a_n}{\longrightarrow\!\!\!\!\!\subset} y_n = y\,.$$

By repeated use of receptivity we obtain the existence of x' where $x \subseteq x'$ and $\sigma x' = y$. To show the uniqueness of x' suppose $x \subseteq z, z'$ and $\sigma z = \sigma z' = y$. Suppose that $z \neq z'$. Then, without loss of generality, suppose there is a $\leq_S$-minimal

$s' \in z'$ with $s' \notin z$. Then $[s') \subseteq z$. Now $\sigma(s') \in y$ so there is $s \in z$ for which $\sigma(s) = \sigma(s')$. We have $[s), [s') \subseteq z$ so $[s) \uparrow [s')$. By Lemma 11(ii) we deduce $s = s'$ so $s' \in z$, a contradiction. Hence, $z = z'$. □

It is useful to define innocence and receptivity on partial maps of event structures with polarity.

Definition 13. Let $f : S \to A$ be a partial map of event structures with polarity. Say f is *receptive* when

$$f(x) \xrightarrow{a}\subset \ \&\ pol_A(a) = - \implies \exists! s \in S.\ x \xrightarrow{s}\subset \ \&\ f(s) = a$$

for all $x \in \mathcal{C}(S)$, $a \in A$.

Say f is *innocent* when it is both +-innocent and −-innocent, *i.e.*

$$s \rightarrowtriangle s' \ \&\ pol(s) = + \ \&\ f(s) \text{ is defined} \implies f(s') \text{ is defined} \ \&\ f(s) \rightarrowtriangle f(s'),$$

$$s \rightarrowtriangle s' \ \&\ pol(s') = - \ \&\ f(s') \text{ is defined} \implies f(s) \text{ is defined} \ \&\ f(s) \rightarrowtriangle f(s').$$

Proposition 14. *A pre-strategy $\sigma : A \twoheadrightarrow B$ is receptive, respectively +/−-innocent, iff both the partial maps σ_1 and σ_2 of its span are receptive, respectively +/−-innocent.*

Proposition 15. *For $\sigma : A \twoheadrightarrow B$ a pre-strategy, σ_1 is receptive, respectively +/−-innocent, iff $(\sigma^\perp)_2$ is receptive, respectively +/−-innocent; σ is receptive and innocent iff $\sigma^\perp$ is receptive and innocent.*

The next lemma will play a major role in importing receptivity and innocence to compositions of pre-strategies.

Lemma 16. *For pre-strategies $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$, if σ_1 is receptive, respectively +/−-innocent, then $(\tau \odot \sigma)_1$ is receptive, respectively +/−-innocent.*

Proof. Abbreviate $\tau \odot \sigma$ to υ.
Receptivity: We show the receptivity of υ_1 assuming that σ_1 is receptive. Let $x \in \mathcal{C}(T \odot S)$ such that $\upsilon_1 x \xrightarrow{a}\subset$ in $\mathcal{C}(A^\perp)$ with $pol_{A^\perp}(a) = -$. By Proposition 9, $\sigma_1 \pi_1 \bigcup x \xrightarrow{a}\subset$ with $\pi_1 \bigcup x \in \mathcal{C}(S)$. As σ_1 is receptive there is a unique $s \in S$ such that $\pi_1 \bigcup x \xrightarrow{s}\subset$ in S and $\sigma_1(s) = a$. It follows that $\bigcup x \xrightarrow{(s,*)}\subset z$, for some z, in $\mathcal{C}(T) \odot \mathcal{C}(S)$. Defining $p =_{\text{def}} [(s,*)]_z$ we obtain $x \xrightarrow{p}\subset$ and $\upsilon_1(p) = a$, with p the unique such event.
Innocence: Assume that σ_1 is innocent. To show the +-innocence of υ_1 we first establish a property of the $\rightarrowtriangle$-relation in the event structure $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$, the synchronized composition of event structures S and T, before projection to V:

> *If $e \rightarrowtriangle e'$ in $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$ with $e \in V$, $pol(e) = +$ and $\upsilon_1(e)$ defined, then $e' \in V$ and $\upsilon_1(e')$ is defined.*

Assume $e \rightarrowtriangle e'$ in $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$, $e \in V$, $pol(e) = +$ and $\upsilon_1(e)$ is defined. From the definition of $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$, the event e is a prime configuration of $\mathcal{C}(T) \odot \mathcal{C}(S)$ where $max(e)$ must have the form $(s,*)$, for some event s of S where $\sigma_1(s)$ is defined. By Lemma 6, $max(e')$ has the form $(s',*)$ or (s',t) with $s \rightarrowtriangle s'$ in S. Now, as $s \rightarrowtriangle s'$ and $pol(s) = +$, from the +-innocence of σ_1, we obtain $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ in $A^\perp \| A$. Whence $\sigma_1(s')$ is defined ensuring $max(e') = (s',*)$. It follows that $e' \in V$ and $\upsilon_1(e')$ is defined.

Now suppose $e \rightarrowtriangle e'$ in $T \odot S$. Then either
(i) $e \rightarrowtriangle e'$ in $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$, or
(ii) $e \rightarrowtriangle e_1 < e'$ in $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$ for some 'invisible' event $e_1 \notin V$.

But the above argument shows that case (ii) cannot occur when $pol(e) = +$ and $\upsilon_1(e)$ is defined. It follows that whenever $e \rightarrowtriangle e'$ in $T \odot S$ with $pol(e) = +$ and $\upsilon_1(e)$ defined, then $\upsilon_1(e')$ is defined and $\upsilon_1(e) \rightarrowtriangle \upsilon_1(e')$, as required.

The argument showing −-innocence of υ_1 assuming that of σ_1 is similar. □

Corollary 17. *For pre-strategies $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$, if τ_2 is receptive, respectively +/−-innocent, then $(\tau \odot \sigma)_2$ is receptive, respectively +/−-innocent.*

Proof. By duality using Lemma 16: if τ_2 is receptive, respectively +/−-innocent, then $(\tau^\perp)_1$ is receptive, respectively +/−-innocent, and hence $(\sigma^\perp \odot \tau^\perp)_1 = ((\tau \odot \sigma)^\perp)_1 = (\tau \odot \sigma)_2$ is receptive, respectively +/−-innocent. □

Lemma 18. *For an event structure with polarity A, the pre-strategy copy-cat $\gamma_A : A \twoheadrightarrow A$ is receptive and innocent.*

Proof. *Receptive:* Suppose $x \in \mathcal{C}(\mathbb{C}_A)$ such that $\gamma_A x \xrightarrow{c}\subset$ in $\mathcal{C}(A^\perp \| A)$ where $pol_{A^\perp \| A}(c) = -$. Now $\gamma_A x = x$ and $x' =_{\text{def}} x \cup \{c\} \in \mathcal{C}(A^\perp \| A)$. Proposition 8(ii) characterizes those configurations of $A^\perp \| A$ which are also configurations of $\mathbb{C}_A$: the characterization applies to x and to its extension $x' = x \cup \{c\}$ because of the −ve polarity of c. Hence $x' \in \mathcal{C}(\mathbb{C}_A)$ and $x \xrightarrow{c}\subset x'$ in $\mathcal{C}(\mathbb{C}_A)$, and clearly c is unique so $\gamma_A(c) = c$.
−-Innocent: Suppose $c \rightarrowtriangle c'$ in $\mathbb{C}_A$ and $pol(c') = -$. By Proposition 8(i), $c \rightarrowtriangle c'$ in $A^\perp \| A$. The argument for +-innocence is similar. □

Theorem 19. *Let $\sigma : A \twoheadrightarrow B$ be a pre-strategy from A to B. If $\sigma \odot \gamma_A \cong \sigma$ and $\gamma_B \odot \sigma \cong \sigma$, then σ is receptive and innocent.*

Let $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$ be pre-strategies which are both receptive and innocent. Then their composition $\tau \odot \sigma : A \twoheadrightarrow C$ is receptive and innocent.

Proof. We know the copy-cat pre-strategies γ_A and γ_B are receptive and innocent—Lemma 18. Assume $\sigma \odot \gamma_A \cong \sigma$ and $\gamma_B \odot \sigma \cong \sigma$. By Lemma 16, $(\sigma \odot \gamma_A)_1$ is receptive and innocent so σ_1 is receptive and innocent. From its dual, Corollary 17, $(\gamma_B \odot \sigma)_2$ so σ_2 is receptive and innocent. Hence σ is receptive and innocent.

Assume that $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$ are receptive and innocent. The fact that σ is receptive and innocent ensures that $(\tau \odot \sigma)_1$ is receptive and innocent, that τ is receptive and innocent that $(\tau \odot \sigma)_2$ is too. Combining, we obtain that $\tau \odot \sigma$ is receptive and innocent. □

In other words, if a pre-strategy is to compose well with copy-cat, in the sense that copy-cat behaves as an identity w.r.t. composition, the pre-strategy must be receptive and innocent. Copy-cat behaving as identity is a hallmark of game-based semantics, so any sensible definition of concurrent strategy will have to ensure receptivity and innocence.

B. *Sufficiency of receptivity and innocence*

In fact, as we will now see, not only are the conditions of receptivity and innocence on pre-strategies necessary to ensure that copy-cat acts as identity. They are also sufficient.

Technically, this section establishes that for a pre-strategy $\sigma : A \mathrel{-\!\!\!+\!\!\!\rightarrowtriangle} B$ which is receptive and innocent both the compositions $\sigma \odot \gamma_A$ and $\gamma_B \odot \sigma$ are isomorphic to σ. We shall concentrate on the isomorphism from $\sigma \odot \gamma_A$ to σ. The isomorphism from $\gamma_B \odot \sigma$ to σ follows by duality.

Recall, from Section V-B, the construction of the pre-strategy $\sigma \odot \gamma_A$ as a total map $S \odot \mathbb{C}\mathbb{C}_A \to A^\perp \| B$. The event structure $S \odot \mathbb{C}\mathbb{C}_A$ is built from the synchronized composition of stable families $\mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}\mathbb{C}_A)$, a restriction of the product of stable families to events

$$\{(c,*) \mid c \in \mathbb{C}\mathbb{C}_A \ \&\ \gamma_{A1}(c) \text{ is defined}\} \cup$$
$$\{(c,s) \mid c \in \mathbb{C}\mathbb{C}_A \ \&\ s \in S \ \&\ \gamma_{A2}(c) = \overline{\sigma_1(s)}\} \cup$$
$$\{(*,s) \mid s \in S \ \&\ \sigma_2(t) \text{ is defined}\} :$$

$$\begin{tikzcd}
 & & \mathcal{C}(S)\odot\mathcal{C}(\mathbb{C}\mathbb{C}_A) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
 & \mathcal{C}(\mathbb{C}\mathbb{C}_A) \arrow[dl, "\gamma_{A1}"'] \arrow[dr, "\gamma_{A2}"] & & \mathcal{C}(S) \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\
\mathcal{C}(A^\perp) & & \mathcal{C}(A)\ \mathcal{C}(A^\perp) & & \mathcal{C}(B)
\end{tikzcd}$$

Finally $S \odot \mathbb{C}\mathbb{C}_A$ is obtained from the prime configurations of $\mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ whose maximum events are defined under $\gamma_{A1}\pi_1$ or $\sigma_2\pi_2$.

We will first present the putative isomorphism from $\sigma \odot \gamma_A$ to σ as a total map of event structures $\theta : S \odot \mathbb{C}\mathbb{C}_A \to S$. The definition of θ depends crucially on the lemmas below. They involve special configurations of $\mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}\mathbb{C}_A)$, *viz.* those of the form $\bigcup x$, where x is a configuration of $S \odot \mathbb{C}\mathbb{C}_A$.

Lemma 20. *For* $x \in \mathcal{C}(S \odot \mathbb{C}\mathbb{C}_A)$*,*

$$(c,s) \in \bigcup x \implies (\overline{c},*) \in \bigcup x\,.$$

Proof. The case when $pol(c) = +$ follows directly because then $\overline{c} \rightarrowtriangle c$ in $\mathbb{C}\mathbb{C}_A$ so $(\overline{c},*) \rightarrowtriangle_{\bigcup x} (c,s)$.
Suppose the lemma fails in the case when $pol(c) = -$, so there is a $\le_{\bigcup x}$-maximal $(c,s) \in \bigcup x$ such that

$$pol(c) = - \ \&\ (\overline{c},*) \notin \bigcup x\,. \qquad (\dagger)$$

The event (c,s) cannot be maximal in $\bigcup x$ as its maximal events take the form $(c',*)$ or $(*,s')$. There must be $e \in \bigcup x$ for which

$$(c,s) \rightarrowtriangle_{\bigcup x} e\,.$$

Consider the possible forms of e:
Case $e = (c',s')$*:* Then, by Lemma 6, either $c \rightarrowtriangle c'$ in $\mathbb{C}\mathbb{C}_A$ or $s \rightarrowtriangle s'$ in S. However if $s \rightarrowtriangle s'$ then, as $pol(s) = +$ by innocence, $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ in $A^\perp$, so $\gamma_{A2}(c) \rightarrowtriangle \gamma_{A2}(c')$ in A; but then $c \rightarrowtriangle c'$ in $\mathbb{C}\mathbb{C}_A$. Either way, $c \rightarrowtriangle c'$ in $\mathbb{C}\mathbb{C}_A$.

Suppose $pol(c') = +$. Then,

$$(c,s) \rightarrowtriangle_{\bigcup x} (\overline{c},*) \rightarrowtriangle_{\bigcup x} (\overline{c'},*) \rightarrowtriangle_{\bigcup x} (c',s')\,.$$

But this contradicts $(c,s) \rightarrowtriangle_{\bigcup x} (c',s')$.

Suppose $pol(c') = -$. Because (c,s) is maximal such that $(\dagger)$, $(\overline{c'},*) \in \bigcup x$. But $(\overline{c},*) \rightarrowtriangle_{\bigcup x} (\overline{c'},*)$ whence $(\overline{c},*) \in \bigcup x$, contradicting $(\dagger)$.
Case $e = (*,s')$*:* Now $(c,s) \rightarrowtriangle_{\bigcup x} (*,s')$. By Lemma 6, $s \rightarrowtriangle s'$ in S with $pol(s) = +$. By innocence, $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ and in particular $\sigma_1(s')$ is defined, which forbids $(*,s')$ as an event of $\mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}\mathbb{C}_A)$.
Case $e = (c',*)$*:* Now $(c,s) \rightarrowtriangle_{\bigcup x} (c',*)$. By Lemma 6, $c \rightarrowtriangle c'$ in $\mathbb{C}\mathbb{C}_A$. Because (c,s) and $(c',*)$ are events of $\mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ we must have $\gamma_2(c)$ and $\gamma_1(c')$ are defined—they are in different components of $\mathbb{C}\mathbb{C}_A$. By Proposition 8, $c' = \overline{c}$, contradicting $(\dagger)$.

In all cases we obtain a contradiction—hence the lemma. □

Lemma 21. *For* $x \in \mathcal{C}(S \odot \mathbb{C}\mathbb{C}_A)$*,*

$$\sigma_1\pi_2 \bigcup x \subseteq^- \gamma_{A1}\pi_1 \bigcup x\,.$$

Proof. As a direct corollary of Lemma 20, we obtain:

$$\sigma_1\pi_2 \bigcup x \subseteq \gamma_{A1}\pi_1 \bigcup x\,.$$

The current lemma will follow provided all events of +ve polarity in $\gamma_{A1}\pi_1 \bigcup x$ are in $\sigma_1\pi_2 \bigcup x$. However, $(\overline{c},s) \rightarrowtriangle_{\bigcup x} (c,*)$, for some $s \in S$, when $pol(c) = +$. □

Lemma 22. *For* $x \in \mathcal{C}(S \odot \mathbb{C}\mathbb{C}_A)$*,*

$$\sigma\pi_2 \bigcup x \subseteq^- \sigma \odot \gamma_A\, x\,.$$

Proof.

$$\begin{aligned}
\sigma\pi_2 \bigcup x &= \sigma_1\pi_2 \bigcup x \ \cup\ \sigma_2\pi_2 \bigcup x \\
&\subseteq^- \gamma_{A1}\pi_1 \bigcup x \ \cup\ \sigma_2\pi_2 \bigcup x\,, \quad \text{by Lemma 21} \\
&= \sigma \odot \gamma_A\, x\,, \quad \text{by Proposition 9.}
\end{aligned}$$

□

Lemma 22 is the key to defining a map $\theta : S \odot \mathbb{C}\mathbb{C}_A \to S$ via the following map-lifting property of receptive maps:

Lemma 23. *Let* $\sigma : S \to C$ *be a total map of event structures with polarity which is receptive and* $-$*-innocent. Let* $p : \mathcal{C}(V) \to \mathcal{C}(S)$ *be a monotonic function, i.e. such that* $p(x) \subseteq p(y)$ *whenever* $x \subseteq y$ *in* $\mathcal{C}(V)$*. Let* $\upsilon : V \to C$ *be a total map of event structures with polarity such that*

$$\forall x \in \mathcal{C}(V).\ \sigma p(x) \subseteq^- \upsilon\, x\,.$$

Then, there is a unique total map of event structures with polarity $\theta : V \to S$ *such that* $\forall x \in \mathcal{C}(V).\ p(x) \subseteq^- \theta\, x$ *and*

$\upsilon = \sigma\theta$:

$$\begin{tikzcd} V \arrow[r, dashed, "p"] \arrow[r, bend left=60, dotted, "\theta"] \arrow[dr, "\upsilon"'] & S \arrow[d, "\sigma"] \\ & C\,. \end{tikzcd}$$

[We use a broken arrow to signify that p is not a map of event structures.]

Proof. Let $x \in \mathcal{C}(V)$. Then $\sigma p(x) \subseteq^- \upsilon\, x$. Define $\Theta(x)$ to be the unique configuration of $\mathcal{C}(S)$, determined by the receptivity of σ, such that

$$\begin{tikzcd} p(x) \arrow[r, dotted, no head, "\subseteq^-"] \arrow[d, mapsto, "\sigma"'] & \Theta(x) \arrow[d, dotted, mapsto, "\sigma"] \\ \sigma p(x) \arrow[r, phantom, "\subseteq^-"] & \upsilon\, x\,. \end{tikzcd}$$

Define θ_x to be the composite bijection

$$\theta_x : x \;\cong\; \upsilon x \;\cong\; \Theta(x)$$

where the bijection $x \cong \upsilon x$ is that determined locally by the total map of event structures υ, and the bijection $\upsilon x \cong \Theta(x)$ is the inverse of the bijection $\sigma \upharpoonright \Theta(x) : \Theta(x) \cong \upsilon\, x$ determined locally by the total map σ.

Now, let $y \in \mathcal{C}(V)$ with $x \subseteq y$. We claim that θ_x is the restriction of θ_y. This will follow once we have shown that $\Theta(x) \subseteq \Theta(y)$. Then, treating the inclusions as inclusion maps, both squares in the diagram below will commute:

$$\begin{array}{ccccc} \theta_y : y & \cong & \upsilon\, y & \cong & \Theta(y) \\ \rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} \\ \theta_x : x & \cong & \upsilon\, x & \cong & \Theta(x) \end{array}$$

This will make the composite rectangle commute, *i.e.* make θ_x the restriction of θ_y.

To show $\Theta(x) \subseteq \Theta(y)$ we suppose otherwise. Then there is an event $s \in \Theta(x)$ of minimum depth w.r.t. $\leq_S$ such that $s \notin \Theta(y)$. Note that $pol(s) = -$, as otherwise $s \in p(x) \subseteq p(y) \subseteq \Theta(y)$. As $\sigma(s) \in \upsilon\, x \subseteq \upsilon\, y$ there is $s' \in \Theta(y)$ such that $\sigma(s') = \sigma(s)$. From the minimality of s, both $[s), [s') \subseteq \Theta(y)$ ensuring the compatibility of $[s)$ and $[s')$. By Lemma 11(ii), $s = s'$ and $s \in \Theta(y)$—a contradiction.

By Proposition 3, the family θ_x, $x \in \mathcal{C}(V)$, determines the unique total map $\theta : V \to S$ such that $\theta\, x = \Theta(x)$. By construction, $p(x) \subseteq^- \theta\, x$, for all $x \in \mathcal{C}(V)$, and $\upsilon = \sigma\theta$. This property in itself ensures that $\theta\, x = \Theta(x)$ so determines θ uniquely. $\square$

In Lemma 23, instantiate $p : \mathcal{C}(S{\odot}\mathbb{C}_A) \to \mathcal{C}(S)$ to the function $p(x) = \pi_2 \bigcup x$ for $x \in \mathcal{C}(S{\odot}\mathbb{C}_A)$, the map σ to the pre-strategy $\sigma : S \to A^\perp \| B$ and υ to the pre-strategy $\sigma{\odot}\gamma_A$. By Lemma 22, $\sigma\pi_2 \bigcup x \subseteq^- \sigma{\odot}\gamma_A\, x$, so the conditions of Lemma 23 are met and we obtain a total map $\theta : S{\odot}\mathbb{C}_A \to S$ such that $\pi_2 \bigcup x \subseteq^- \theta\, x$, for all $x \in \mathcal{C}(S{\odot}\mathbb{C}_A)$, and $\sigma\theta = \sigma{\odot}\gamma_A$:

$$\begin{tikzcd} S{\odot}\mathbb{C}_A \arrow[r, dashed, "p"] \arrow[r, bend left=60, dotted, "\theta"] \arrow[dr, "\sigma\odot\gamma_A"'] & S \arrow[d, "\sigma"] \\ & A^\perp \| B\,. \end{tikzcd}$$

The next lemma is used in showing θ is an isomorphism.

Lemma 24. *(i) Let $z \in \mathcal{C}(S){\odot}\mathcal{C}(\mathbb{C}_A)$. If $e \leq_z e'$ and $\pi_2(e)$ and $\pi_2(e')$ are defined, then $\pi_2(e) \leq_S \pi_2(e')$. (ii) The map π_2 is surjective on configurations.*

Proof. (i) It suffices to show when

$$e \rightarrowtriangle_z e_1 \rightarrowtriangle_z \cdots \rightarrowtriangle_z e_{n-1} \rightarrowtriangle_z e'$$

with $\pi_2(e)$ and $\pi_2(e')$ defined and all $\pi_2(e_i)$, $1 \leq i \leq n-1$, undefined, that $\pi_2(e) \leq_S \pi_2(e')$.
Case $n = 1$, so $e \rightarrowtriangle_z e'$: Use Lemma 6. If either e or e' has the form $(*, s)$ then the other event must have the form $(*, s')$ or (c', s') with $s \rightarrowtriangle s'$ in S. In the remaining case $e = (c, s)$ and $e' = (c', s')$ with either (1) $c \rightarrowtriangle c'$ in $\mathbb{C}_A$, and $\gamma_{A2}(c) \rightarrowtriangle \gamma_{A2}(c')$ in A, or (2) $s \rightarrowtriangle s'$ in S. If (1), $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ in $A^\perp$ where $s, s' \in \pi_2 z$. By Proposition 4, $s \leq_S s'$. In either case (1) or (2), $\pi_2(e) \leq_S \pi_2(e')$.
Case $n > 1$: Each e_i has the form $(c_i, *)$, for $1 \leq i \leq n-1$. By Lemma 6, events e and e' must have the form (c, s) and (c', s') with $c \rightarrowtriangle c_1$ and $c_{n-1} \rightarrowtriangle c'$ in $\mathbb{C}_A$. As $\gamma_{A1}(c)$ and $\gamma_{A2}(c_1)$ are defined, $c_1 = \overline{c}$ and similarly $c_{n-1} = \overline{c'}$. Again by Lemma 6, $c_i \rightarrowtriangle c_{i+1}$ in $\mathbb{C}_A$ for $1 \leq i \leq i-2$. Consequently $\gamma_{A2}(c) \leq_A \gamma_{A2}(c')$. Now, $s, s' \in \pi_2 z$ with $\sigma_1(s) \leq_{A^\perp} \sigma_1(s')$. By Proposition 4, $s \leq_S s'$, as required.
(ii) Let $y \in \mathcal{C}(S)$. Then $\sigma_1 y \in \mathcal{C}(A^\perp)$ and by the clear surjectivity of γ_{A2} on configurations there exists $w \in \mathcal{C}(\mathbb{C}_A)$ such that $\gamma_{A2} w = \sigma_1 y$. Now let

$$\begin{aligned} z = &\{(c, *) \mid c \in w \ \&\ \gamma_{A1}(c) \text{ is defined}\} \\ &\cup \{(c, s) \mid c \in w \ \&\ s \in y \ \&\ \gamma_{A2}(c) = \sigma_1(s)\} \\ &\cup \{(*, s) \mid s \in y \ \&\ \sigma_2(s) \text{ is defined}\}\,. \end{aligned}$$

Then, from the definition of the product of stable families—III-A, it can be checked that $z \in \mathcal{C}(S){\odot}\mathcal{C}(\mathbb{C}_A)$. By construction, $\pi_2 z = y$. Hence π_2 is surjective on configurations. $\square$

Theorem 25. *$\theta : \sigma{\odot}\gamma_A \cong \sigma$, an isomorphism of pre-strategies.*

Proof. We show θ is an isomorphism of event structures by showing θ is rigid and both surjective and injective on configurations (Lemma 3.3 of [12]). The rest is routine.
Rigid: It suffices to show $p \rightarrowtriangle p'$ in $S{\odot}\mathbb{C}_A$ implies $\theta(p) \leq_S \theta(p')$. Suppose $p \rightarrowtriangle p'$ in $S{\odot}\mathbb{C}_A$ with $max(p) = e$ and $max(p') = e'$. Take $x \in \mathcal{C}(S{\odot}\mathbb{C}_A)$ containing p' so p too. Then

$$e \rightarrowtriangle_{\bigcup x} e_1 \rightarrowtriangle_{\bigcup x} \cdots \rightarrowtriangle_{\bigcup x} e_{n-1} \rightarrowtriangle_{\bigcup x} e'$$

where $e, e' \in V$ and $e_i \notin V$ for $1 \leq i \leq n-1$. (V consists of 'visible' events of the form $(c, *)$ with $\gamma_{A1}(c)$ defined, or $(*, s)$, with $\sigma_2(s)$ defined.)

Case $n = 1$, so $e \rightarrowtriangle_{\bigcup x} e'$: By Lemma 6, either (i) $e = (*, s)$ and $e' = (*, s')$ with $s \rightarrowtriangle s'$ in S, or (ii) $e = (c, *)$ and $e' = (c', *)$ with $c \rightarrowtriangle c'$ in $\mathbb{CC}_A$.
If (i), we observe, via $\sigma\theta = \sigma \odot \gamma_A$, that $s \in \pi_2 \bigcup x \subseteq \theta x$ and $\theta(p) \in \theta x$ with $\sigma(\theta(p)) = \sigma(s)$, so $\theta(p) = s$ by the local injectivity of σ. Similarly, $\theta(p') = s'$, so $\theta(p) \le_S \theta(p')$.
If (ii), we obtain $\theta(p), \theta(p') \in \theta x$ with $\sigma_1\theta(p) = \gamma_{A1}(c)$, $\sigma_1\theta(p') = \gamma_{A1}(c')$ and $\gamma_{A1}(c) \rightarrowtriangle \gamma_{A1}(c')$ in $A^\perp$. By Proposition 4, $\theta(p) \le_S \theta(p')$.
Case $n > 1$: Note $e_i = (c_i, s_i)$ for $1 \le i \le n-1$, and that $s_1 \le_S s_{n-1}$ by Lemma 24(i). Consider the case in which $e = (c, *)$ and $e' = (c', *)$—the other cases are similar. By Lemma 6, $c \rightarrowtriangle c_1$ and $c_{n-1} \rightarrowtriangle c'$ in $\mathbb{CC}_A$. But $\gamma_{A1}(c)$ and $\gamma_{A2}(c_1)$ are defined, so $c_1 = \overline{c}$, and similarly $c_{n-1} = \overline{c'}$. We remark that $\theta(p) = s_1$, by the local injectivity of σ, as both $s_1 \in \pi_2 \bigcup x \subseteq \theta x$ and $\theta(p) \in \theta x$ with $\sigma(\theta(p)) = \sigma(s_1)$. Similarly $\theta(p') = s_{n-1}$, whence $\theta(p) \le_S \theta(p')$.
Surjective: Let $y \in \mathcal{C}(S)$. By Lemma 24(ii), there is $z \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{CC}_A)$ such that $\pi_2 z = y$. Let

$$z' = z \cup \{(c, *) \mid pol(c) = + \;\&\; \exists s \in S.\ (\overline{c}, s) \in z\}\,.$$

It is straightforward to check $z' \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{CC}_A)$. Now let

$$z'' = z' \smallsetminus \{(c, *) \mid pol(c) = - \;\&\; \forall s \in S.\ (\overline{c}, s) \notin z'\}\,.$$

Then $z'' \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{CC}_A)$ by the following argument. The set z'' is certainly consistent, so it suffices to show

$$pol(c) = - \;\&\; (c, *) \le_{z'} e \in z'' \implies \exists s \in S.\ (\overline{c}, s) \in z'\,,$$

for all $c \in \mathbb{CC}_A$ and $e \in z''$. This we do by induction on the number of events between $(c, *)$ and e. Suppose

$$pol(c) = - \;\&\; (c, *) \rightarrowtriangle_{z'} e_1 \le_{z'} e \in z'\,.$$

In the case where $e_1 = (c_1, s_1)$, we deduce $c \rightarrowtriangle c_1$ in $\mathbb{CC}_A$ and as $\gamma_{A1}(c)$ is defined while $\gamma_{A2}(c_1)$ is defined, we must have $c_1 = \overline{c}$, as required. In the case where $e_1 = (c_1, *)$ and $pol(c_1) = -$, by induction, we obtain $(\overline{c_1}, s_1) \in z'$ for some $s_1 \in S$. Also $c \rightarrowtriangle c_1$, so $\overline{c} \rightarrowtriangle \overline{c_1}$ in $\mathbb{CC}_A$. As z' is a configuration we must have $(\overline{c}, s) \le_{z'} (\overline{c_1}, s_1)$, for some $s \in S$, so $(\overline{c}, s) \in z'$. In the case where $e_1 = (c_1, *)$ and $pol(c_1) = +$, we have $c \rightarrowtriangle c_1$ in $\mathbb{CC}_A$. Moreover, $(\overline{c}_1, s) \in z'$, for some $s \in S$, as z' is a configuration and $\overline{c_1} \rightarrowtriangle c_1$ in $\mathbb{CC}_A$. Again, from the fact that z' is a configuration, there must be $(\overline{c}, s) \in z'$ for some $s \in S$. We have exhausted all cases and conclude $z'' \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{CC}_A)$ with $\theta z'' = \pi_2 z = y$, as required to show θ is surjective on configurations.
Injective: Abbreviate $\sigma \odot \gamma_A$ to υ. Assume $\theta x = \theta y$, where $x, y \in \mathcal{C}(S \odot \mathbb{CC}_A)$. Via the commutativity $\upsilon = \sigma\theta$, we observe

$$\upsilon x = \sigma\theta\, x = \sigma\theta\, y = \upsilon y\,.$$

Recall by Proposition 9, that $\upsilon_1 x = \gamma_{A1}\pi_1 \bigcup x = \pi_1 \bigcup x$. It follows that

$$(c, *) \in \bigcup x \iff c \in \upsilon_1 x \iff c \in \upsilon_1 y \iff (c, *) \in \bigcup y\,.$$

Observe

$$(*, s) \in \bigcup x \iff \sigma_2(s) \text{ is defined } \&\ s \in \theta x\ :$$

"$\Rightarrow$" by the local injectivity of σ_2, as $p =_{\text{def}} [(*, s)]_{\bigcup x}$ yields $\theta(p) \in \theta x$ and $s \in \pi_2 \bigcup x \subseteq \theta x$ with $\sigma_2(\theta(p)) = \sigma_2(s)$, so $\theta(p) = s$; "$\Leftarrow$" as $\sigma_2(s)$ defined and $s \in \theta x$ entails $s = \theta(p)$ for some $p \in x$, necessarily with $max(p) = (*, s)$. Hence

$$\begin{aligned}(*, s) \in \bigcup x &\iff \sigma_2(s) \text{ is defined } \&\ s \in \theta x \\ &\iff \sigma_2(s) \text{ is defined } \&\ s \in \theta y \\ &\iff (*, s) \in \bigcup y\,.\end{aligned}$$

Assuming $(c, s) \in \bigcup x$ we now show $(c, s) \in \bigcup y$. (The converse holds by symmetry.) There is $p \in x$, such that $(c, s) \in p$. If $max(p) = (*, s')$ (also in $\bigcup y$ as it is visible) then as π_2 is rigid, $s \le s'$ and we must have $(c', s) \in \bigcup y$. Otherwise, $max(p) = (d, *)$ and we can suppose (by taking p minimal) that $(c, s) \le_{\bigcup x} (d', s') \rightarrowtriangle_{\bigcup x} (d, *)$. But then $\theta(p) = s' \in \theta x = \theta y$. Also $s \le_S s'$, by the rigidity of π_2, and, as we have seen before, $d' = \overline{d}$ with d' $-$ve. Hence s' is +ve and as θy is a $-$ve extension of $\pi_2 \bigcup y$ we must have $s' \in \pi_2 \bigcup y$. Hence there is $(*, s')$ or (c'', s') in $\bigcup y$, and as $s \le_S s'$ there is some $(c', s) \in \bigcup y$. In both cases, $\gamma_{A2}(c') = \overline{\sigma_1(s)} = \gamma_{A2}(c)$, so $c' = c$, and thus $(c, s) \in \bigcup y$.

We conclude $\bigcup x = \bigcup y$, so $x = y$, as required for injectivity.
□

C. The bicategory of concurrent games and strategies

Define a *strategy* to be a pre-strategy which is receptive and innocent. We obtain a bicategory, **Games**, in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \twoheadrightarrow B$ and the 2-cells are maps of spans. The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the functoriality of synchronized composition). The isomorphisms expressing associativity and the identity of copy-cat are those of Proposition 10 and Theorem 25 with its dual.

An alternative description of concurrent strategies exhibits the correspondence between innocence and earlier "saturation conditions," *reflecting* specific independence, in [5], [6], [3]:

Proposition 26. *A strategy S in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that*
(i) $\sigma x \overset{a}{\longrightarrow\!\!\subset} \;\&\; pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{\longrightarrow\!\!\subset} \;\&\; \sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$.
(ii)(+) If $x \overset{e}{\longrightarrow\!\!\subset} x_1 \overset{e'}{\longrightarrow\!\!\subset} \;\&\; pol_S(e) = +$ in $\mathcal{C}(S)$ and $\sigma x \overset{\sigma(e')}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(A)$, then $x \overset{e'}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(S)$.
(ii)(−) If $x \overset{e}{\longrightarrow\!\!\subset} x_1 \overset{e'}{\longrightarrow\!\!\subset} \;\&\; pol_S(e') = -$ in $\mathcal{C}(S)$ and $\sigma x \overset{\sigma(e')}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(A)$, then $x \overset{e'}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(S)$.

D. Deterministic strategies

Say an event structure with polarity S is *deterministic* iff

$$\forall X \subseteq_{\text{fin}} S.\ Neg[X] \in \mathrm{Con}_S \implies X \in \mathrm{Con}_S\,,$$

where $Neg[X] =_{\text{def}} \{s' \in S \mid \exists s \in X.\ pol_S(s') = - \;\&\; s' \le s\}$. Say a strategy $\sigma : S \to A$ is deterministic if S is deterministic.

Proposition 27. *An event structure with polarity S is deterministic iff $x \overset{s}{\longrightarrow\!\!\!\!\subset}$ & $x \overset{s'}{\longrightarrow\!\!\!\!\subset}$ & $pol_S(s) = +$ implies $x \cup \{s, s'\} \in \mathcal{C}(S)$, for all $x \in \mathcal{C}(S)$.*

A copy-cat strategy can fail to be deterministic in the two ways allowed by Proposition 27.

Example 28. (i) Take A to consist of two +ve events and one −ve event, with any two but not all three events consistent. The construction of $\mathbb{C}\mathbb{C}_A$ is pictured:

$$\begin{array}{cccc} & \ominus \rightarrow \oplus & \\ A^\perp & \ominus \rightarrow \oplus & A \\ & \oplus \leftarrow \ominus & \end{array}$$

$\mathbb{C}\mathbb{C}_A$ is not deterministic: take x to be the set of all three −ve events in $\mathbb{C}\mathbb{C}_A$ and s, s' to be the two +ve events in the A component.
(ii) Take A to consist of two events, one +ve and one −ve event, inconsistent with each other. The construction $\mathbb{C}\mathbb{C}_A$:

$$\begin{array}{cccc} A^\perp & \ominus \rightarrow \oplus & A \\ & \oplus \leftarrow \ominus & \end{array}$$

To see $\mathbb{C}\mathbb{C}_A$ is not deterministic, take x to be the singleton set consisting *e.g.* of the −ve event on the left and s, s' to be the +ve and −ve events on the right.

Lemma 29. *Let A be an event structure with polarity. The copy-cat strategy γ_A is deterministic iff A satisfies*

$$\forall x \in \mathcal{C}(A).\ x \overset{a}{\longrightarrow\!\!\!\!\subset} \ \&\ x \overset{a'}{\longrightarrow\!\!\!\!\subset} \ \&\ pol_A(a) = + \ \&\ pol_A(a') = - \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad (\ddagger)$$

Lemma 30. *The composition $\tau \odot \sigma$ of two deterministic strategies σ and τ is deterministic.*

We obtain a sub-bicategory **DGames** of **Games**; its objects satisfy (‡) of Lemma 29 and its maps are deterministic strategies. Moreover **DGames** can be made equivalent to a order-enriched category via the following proposition, which ensures deterministic strategies in A correspond to certain subfamilies of the configurations of A.

Proposition 31. *A deterministic strategy is injective on configurations (and therefore mono as a map of event structures).*

A deterministic strategy $\sigma : S \to A$ determines, as the image of the configurations $\mathcal{C}(S)$, a subfamily $F =_{\mathrm{def}} \sigma\mathcal{C}(S)$ of configurations of $\mathcal{C}(A)$, which satisfies:
reachability: $\varnothing \in F$ and if $x \in F$ there is a covering chain $\varnothing \overset{a_1}{\longrightarrow\!\!\!\!\subset} x_1 \overset{a_2}{\longrightarrow\!\!\!\!\subset} \cdots \overset{a_k}{\longrightarrow\!\!\!\!\subset} x_k = x$ within F;
determinacy: If $x \overset{a}{\longrightarrow\!\!\!\!\subset}$ and $x \overset{a'}{\longrightarrow\!\!\!\!\subset}$ in F with $pol_A(a) = +$, then $x \cup \{a, a'\} \in F$;
receptivity: If $x \in F$ and $x \overset{a}{\longrightarrow\!\!\!\!\subset}$ in $\mathcal{C}(A)$ and $pol_A(a) = -$, then $x \cup \{a\} \in F$;
+-innocence: If $x \overset{a}{\longrightarrow\!\!\!\!\subset} x_1 \overset{a'}{\longrightarrow\!\!\!\!\subset}$ & $pol_A(a) = +$ in F and $x \overset{a'}{\longrightarrow\!\!\!\!\subset}$ in $\mathcal{C}(A)$, then $x \overset{a'}{\longrightarrow\!\!\!\!\subset}$ in F;
cube: In F,

$$\begin{tikzcd} & x_1 \arrow[r, "e"] \arrow[dr, "b"] & y_1 \arrow[dr, "b"] & \\ x \arrow[ur, "a"] \arrow[dr, "b"'] & & y \arrow[r, "e"] & z \\ & x_2 \arrow[r, "e"'] \arrow[ur, "a"'] & y_2 \arrow[ur, "a"'] & \end{tikzcd} \quad \text{implies} \quad x \overset{e}{\longrightarrow\!\!\!\!\subset}\,.$$

(Here receptivity implies −-innocence.)

Theorem 32. *A subfamily $F \subseteq \mathcal{C}(A)$ satisfies the conditions above iff there is a deterministic strategy $\sigma : S \to A$ such that $F = \sigma\mathcal{C}(S)$, the image of $\mathcal{C}(S)$ under σ.*

VII. Related work—early results

1) Stable spans, profunctors and stable functions: The sub-bicategory of **Games** where the events of games are purely +ve is equivalent to the bicategory of stable spans [12]. In this case, strategies correspond to *stable spans*:

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ A^\perp & & B \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} & S^+ \arrow[dl, "\sigma_1^-"'] \arrow[dr, "\sigma_2^+"] & \\ A & & B \end{tikzcd}\,,$$

where S^+ is the projection of S to its +ve events; σ_2^+ is the restriction of σ_2 to S^+, necessarily a rigid map by innocence; σ_2^- is a *demand map* taking $x \in \mathcal{C}(S^+)$ to $\sigma_1^-(x) = \sigma_1[x]$; here $[x]$ is the down-closure of x in S. Composition of stable spans coincides with composition of their associated profunctors. If we further restrict strategies to be deterministic (and, strictly, event structures to be countable) we obtain a bicategory equivalent to Berry's *dI-domains and stable functions*.

2) Ingenuous strategies: Via Theorem 32, deterministic concurrent strategies coincide with the *receptive ingenuous* strategies of Melliès and Mimram [3].

3) Closure operators: In [4], deterministic strategies are presented as closure operators. A deterministic strategy $\sigma : S \to A$ determines a closure operator φ on possibly infinite configurations $\mathcal{C}^\infty(S)$: for $x \in \mathcal{C}^\infty(S)$,

$$\varphi(x) = x \cup \{s \in S \mid pol(s) = + \ \&\ Neg[\{s\}] \subseteq x\}\,.$$

Clearly φ preserves intersections of configurations and is continuous. The closure operator φ on $\mathcal{C}^\infty(S)$ induces a *partial* closure operator φ_p on $\mathcal{C}^\infty(A)$. This in turn determines a closure operator $\varphi_p^\top$ on $\mathcal{C}^\infty(A)^\top$, where configurations are extended with a top $\top$, *cf.* [4]: take $y \in \mathcal{C}^\infty(A)^\top$ to the least, fixed point of φ_p above y, if such exists, and $\top$ otherwise.

4) Simple games: *"Simple games"* [10], [11] arise when we restrict **Games** to objects and deterministic strategies in $\mathcal{PA}_r^{-\#}$, described in Section IV-B.

5) Extensions: Games, such as those of [8], [13], allowing copying are being systematized through the use of monads and comonads [11], work now feasible on event structures with symmetry [12]. Nondeterministic strategies can potentially support probability as probabilistic or stochastic event structures [14] to become probabilistic or stochastic strategies.

6) Other models: Event structures occupy a central position amongst models for concurrency. The techniques here transfer to other models such as Mazurkiewicz trace languages, asynchronous transition systems and Petri nets, some of which would appear more suitable for algorithmic and logical considerations in that they support looping behaviour.

ACKNOWLEDGMENT

The authors thank the anonymous referees. Thanks to Pierre-Louis Curien, Marcelo Fiore, Jonathan Hayman, Martin Hyland, Paul-André Melliès, Samuel Mimram and Gordon Plotkin for helpful remarks. GW acknowledges with gratitude the support of a Royal Society Leverhulme Trust Senior Fellowship and Advanced Grant ECSYM of the ERC.

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