Ch 2.

Relations & functions

- The extension of a property $P(x)$, $x \in \mathcal{U}$, is a set $\{x \in \mathcal{U} \mid P(x)\}$.

  What is the extension of a property

  $$P(x,y), \quad x \in \mathcal{U},\ y \in V,$$

  or $$P(x,y,z), \quad x \in \mathcal{U},\ y \in V,\ z \in W\ ?$$

- In the 19c. it was recognised that functions were often, but not always, associated with expressions eg. x^2+3 for $x \in \mathbb{R}$.

Pairs & Products

$\{a, b\}$ unordered pair of a, b.

(a, b) ordered pair of a, b.

$(a, b) = (a', b') \iff a = a'$ & $b = b'$.

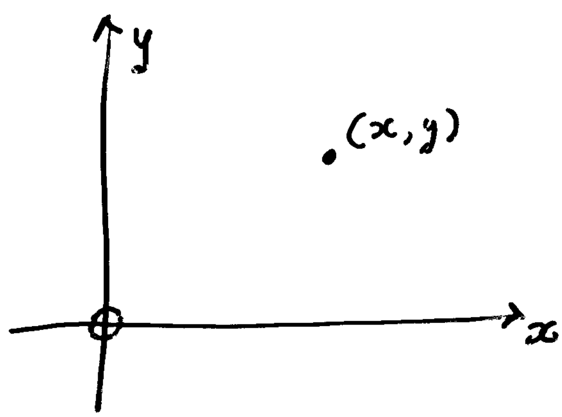
We could define $(a, b) =_{def} \{ \{a\}, \{a, b\} \}$.

The product of sets X and Y

$$X \times Y =_{def} \{ (a, b) \mid a \in X \ \& \ b \in Y \}$$

Eg.

$\mathbb{R} \times \mathbb{R}$



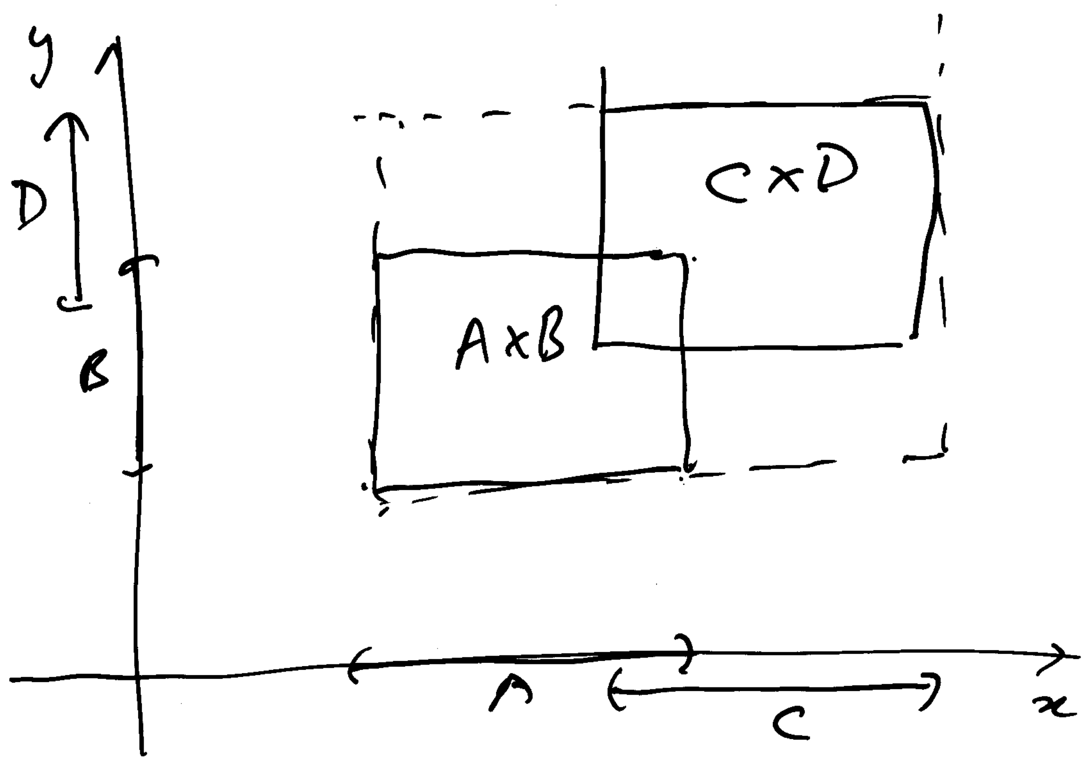
Some laws:

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

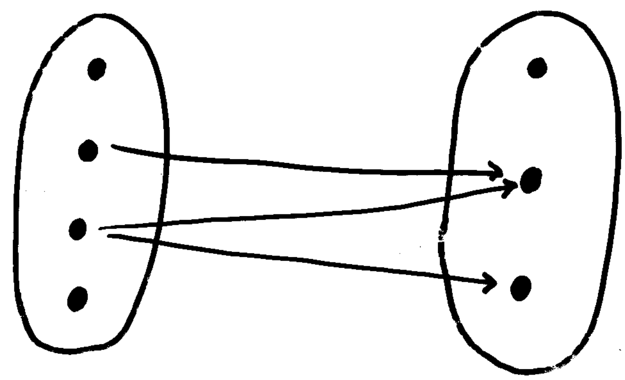
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$$



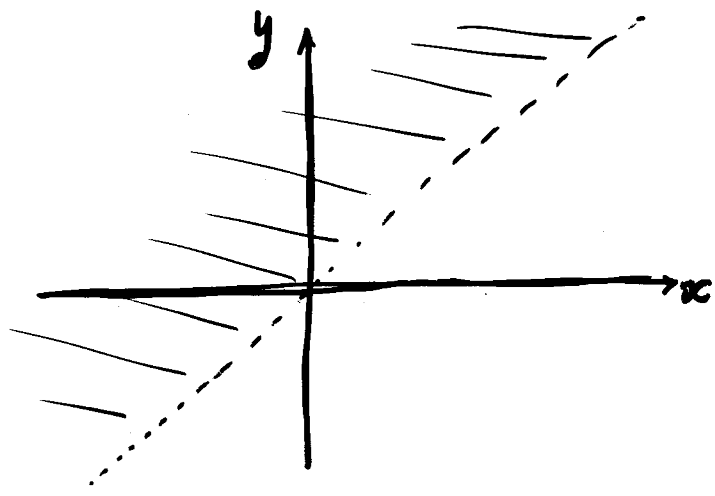
A binary relation between sets X, Y is a subset $R \subseteq X \times Y$



$(x,y) \in R$ often written $x \, R \, y$.

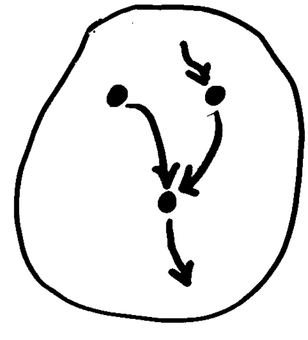
E.g.

- $\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x < y\}$



- $\{(x,y) \mid x \text{ is parent of } y\} \subseteq P \times P$

  P is set of people



$$f : X \to Y$$

A function from a set X to set Y is a relation $f \subseteq X \times Y$ such that:

(1) $(x, y) \in f$ & $(x, y') \in f \Rightarrow y = y'$ for all $x \in X$, $y, y' \in Y$;

(2) $\forall x \in X \ \exists y \in Y. \ (x, y) \in f$

Write $f(x)$ for the unique y s.t. $(x, y) \in f$.

$$f : X \rightharpoonup Y$$

A partial function from X to Y is a relation $f \subseteq X \times Y$ s.t. (1).

Special functions

Let $f : X \to Y$.

f is injective (1-1) iff
$\forall x, x' \in X.\ f(x) = f(x') \Rightarrow x = x'$

f is surjective (onto) iff
$\forall y \in Y\ \exists x \in X.\ y = f(x)$.

f is bijective (1-1 correspondence) iff
f is injective and surjective.

Proposition 2.9 [P. 32]
$f : X \to Y$ is bijective iff it has an inverse
i.e. $g : Y \to X$ s.t. $g(f(x)) = x$ forall $x \in X$
and $f(g(y)) = y$ forall $y \in Y$.

Composing relations and functions.

$$R \subseteq X \times Y \qquad S \subseteq Y \times Z$$

Their composition:

$$S \circ R =_{def} \{ (x,z) \in X \times Z \mid \exists y \in Y.\ (x,y) \in R \ \&\ (y,z) \in S \}$$

Identity:

$$id_X \subseteq X \times X$$

$$id_X =_{def} \{ (x,x) \mid x \in X \}$$

Associativity:

$R \subseteq X \times Y$, $S \subseteq Y \times Z$, $T \subseteq Z \times W$

$$T \circ (S \circ R) = (T \circ S) \circ R$$

Composition of functions / partial fns is a function / partial function.

Equivalence relations.

An equivalence relation on a set X is a relation

$$R \subseteq X \times X$$

which is

reflexive: $\forall x \in X. \quad x R x$

symmetric: $\forall x, y \in X. \ x R y \Rightarrow y R x$

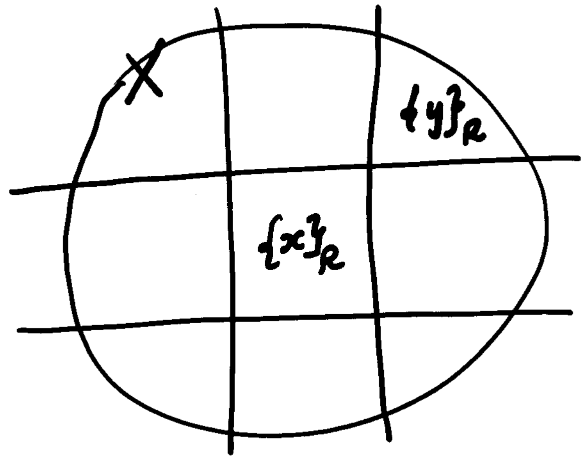
transitive: $\forall x, y, z \in X. \ x R y \ \& \ y R z \Rightarrow x R z$

Let $x \in X$. Its equivalence class

$$\{x\}_R =_{def} \{y \in X \mid y R x\}$$

Theorem 2.12 [P.34]

$\{\{x\}_R \mid x \in X\}$ is a partition of the set X.



Partition:

- $x \in \{x\}_R$
- $\{x\}_R \cap \{y\}_R \neq \emptyset \Rightarrow \{x\}_R = \{y\}_R$

(1) $\{x\}_R \cap \{y\}_R \neq \emptyset \Rightarrow xRy$

(2) $xRy \Rightarrow \{x\}_R \underset{\subseteq \checkmark}{\overset{\supseteq ?}{=}} \{y\}_R$

(1) zRx & zRy

xRz & zRy $\therefore$ xRy.

(2) Assume xRy.

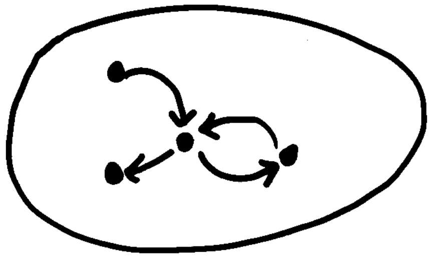
Let $z \in \{x\}_R$ i.e. zRx. $\therefore$ zRy $\therefore z \in \{y\}_R$.

Let $w \in \{y\}_R$, i.e. wRy. Have yRx.

$\therefore$ wRx i.e. $w \in \{x\}_R$.

Relations as structure – other examples

Directed graphs $\quad (X, R)$ where $R \subseteq X \times X$.



Partial orders $\quad (P, \leq)$ where $\leq \subseteq P \times P$

s.t.

refl. $\quad p \leq p$

tran. $\quad p \leq q \;\&\; q \leq r \Rightarrow p \leq r$

antisym. $\quad p \leq q \;\&\; q \leq p \Rightarrow p = q$

Cf. $\subseteq$ on sets

least upper bounds $\bigvee$ (cf. $\cup$)

greatest lower bounds $\bigwedge$ (cf. $\cap$)

Direct and inverse image

$$R \subseteq X \times Y$$

Let $A \subseteq X$. Its direct image under R

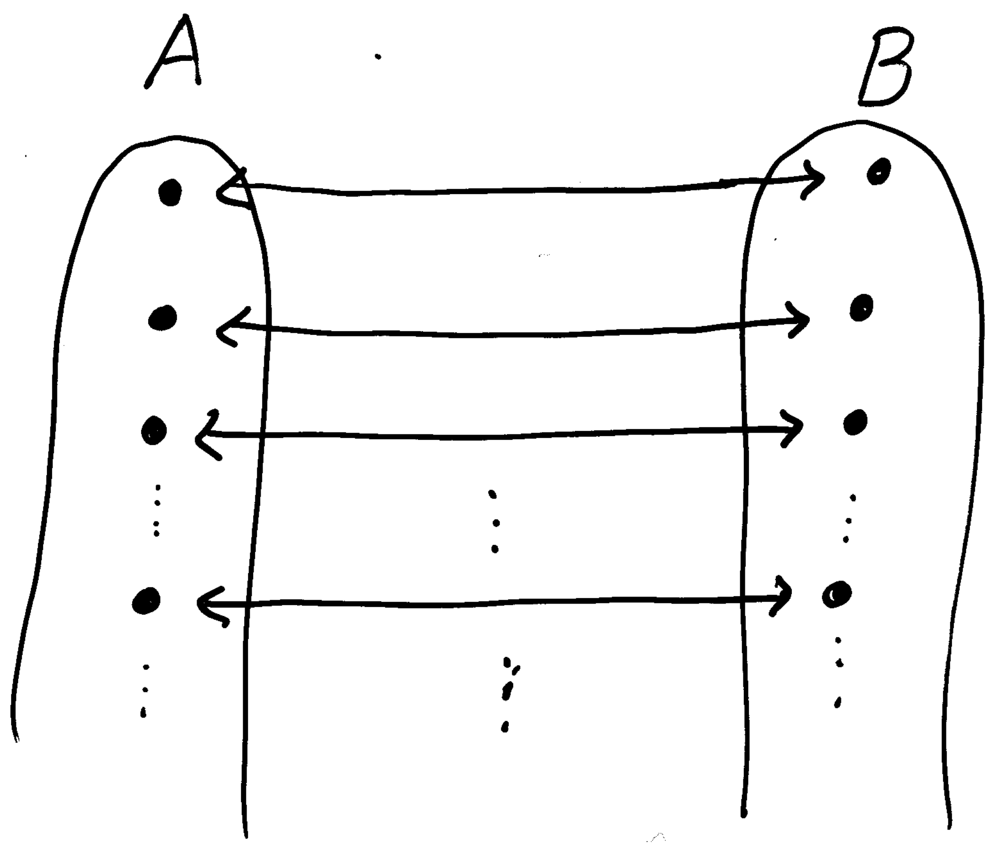
$$RA = \{ y \in Y \mid \exists x \in A.\ (x,y) \in R \}$$

Let $B \subseteq Y$. Its inverse image under R

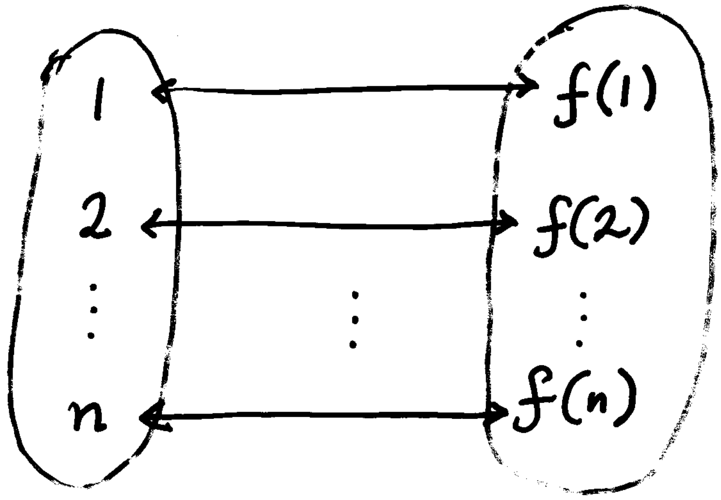
$$R^{-1}B = \{ x \in X \mid \exists y \in B.\ (x,y) \in R \}$$

Size of sets
— countability.

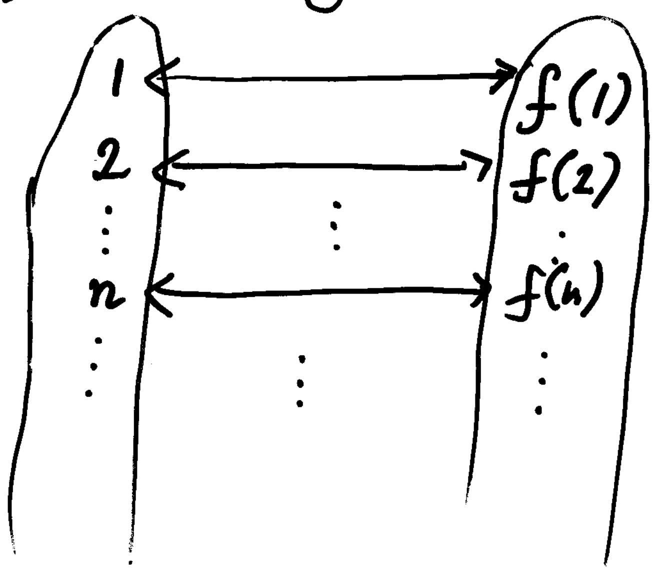
Two sets have the same size (or cardinality) iff there is a bijection between them :



A set A is finite iff there is a bijection $f : \{m \in \mathbb{N} \mid m \le n\} \longrightarrow A$ for some $n \in \mathbb{N}_0$



A set A is countable iff A is finite or there is a bijection $f : \mathbb{N} \longrightarrow A$.



Lemma 2.23 Any subset A of $\mathbb{N}$ is countable.

Proof idea:

Define $f : \mathbb{N} \longrightarrow A$ by mathl. ind.

$f(1)$ is least element of A if $A \neq \emptyset$;
undefined otherwise.

$f(n+1)$ is least element of A above $f(n)$
if $f(n)$ is defined & there is
a member of A above $f(n)$;
undefined otherwise.

Corollary 2.24

$\Rightarrow ?\checkmark$

A set B is countable iff there is a bijection $g : A \longrightarrow B$ where $A \subseteq \mathbb{N}$.

$\Leftarrow ?$

Lemma 2.25

$\Rightarrow ?\checkmark$

A set B is countable iff there is an injection $f : B \to \mathbb{N}$.

$\Leftarrow ?$

Lemma 2.26

$\Rightarrow ?\checkmark$

A set B is countable iff there is an injection $f : B \longrightarrow A$ where A is countable.

$\Leftarrow ?$

In particular, a subset of a countable set is countable.

Theorem 2.34 $\mathbb{R}$ is uncountable.

Via $(0,1] =_{def} \{ r \in \mathbb{R} \mid 0 < r \leq 1 \}$

is uncountable.

But $S =_{def} \{ s \in (0,1] \mid s \text{ can be expressed by a finite decimal} \}$

...

Proof. By contradiction.
Assume $\mathbb{R}$ is countable.
Then $(0, 1]$ is countable.

$]0,1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$

$$
\begin{array}{llllllll}
f(1) = & 0. & \boxed{d_1^1} & d_2^1 & d_3^1 & \cdots & d_i^1 & \cdots \\
f(2) = & 0. & d_1^2 & \boxed{d_2^2} & d_3^2 & \cdots & d_i^2 & \cdots \\
f(3) = & 0. & d_1^3 & d_2^3 & \boxed{d_3^3} & \cdots & d_i^3 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
f(n) = & 0. & d_1^n & d_2^n & d_3^n & \cdots & d_i^n & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots &
\end{array}
$$

$$r = 0.\ r_1\ r_2\ r_3\ \cdots\ r_i\ \cdots$$

$$r_i = \begin{cases} 1 & \text{if } d_i^i \neq 1 \\ 2 & \text{if } d_i^i = 1. \end{cases}$$

Lemma 2.27 The set $\mathbb{N}\times\mathbb{N}$ is countable.

$f: \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N} \qquad f(m,n) = 2^m \times 3^n$

Corollary 2.28 The set $\mathbb{Q}^+$ is countable

$f: \mathbb{Q}^+ \to \mathbb{N}\times\mathbb{N} \qquad f(m/n) = (m, n)$

Lemma 2.29 Suppose $A_1, A_2, \dots, A_n, \dots$ are countable sets. Their union

$$A = \{x \mid \exists n \in \mathbb{N}.\ x \in A_n\}$$ is countable.

Cor $\mathbb{Z} = \{0\} \cup \mathbb{N} \cup \{-n \mid n \in \mathbb{N}\}$ is countable.

An algebraic number is a solution to a polynomial equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

where the coefficients $a_0, \dots, a_n \in \mathbb{Z}$.

A transcendental is a real number which is not algebraic.

There are (uncountably many) transcendental numbers!