Ch.3. Constructions on Sets

cf. constructions on types
in prog. languages.

Sets as extensions of properties

$$\{ x \mid \underbrace{P}_{\text{a property}}(x) \}$$

always a set ?

↯ Russell's paradox (really a contradiction)

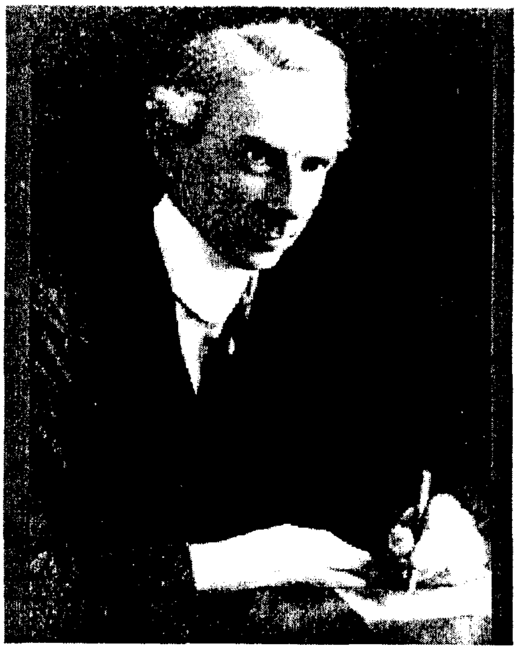
A safe way to build sets :

$$\{ x \in \underbrace{S}_{\text{a set}} \mid \underbrace{P(x)}_{\text{a property}} \}$$

But this needs sufficiently big sets S

By fiat: $\mathbb{N}$ is a set

powerset $\mathcal{P}(X) = \{ A \mid A \subseteq X \}$ is a set



$$R = \{x \mid x \notin x\}$$

$R \in R$?

$R \in R \Rightarrow R \notin R$

$R \notin R \Rightarrow R \in R$

Further constructions on sets

Let A, B be sets

Their product

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

ordered pair:
$(a, b) = (a', b')$ iff $a = a'$ and $b = b'$.

Can be realised as a set:

$$(a, b) = \{ \{a\}, \{a, b\} \}$$

Their disjoint union (or disjoint sum)

$$A \uplus B = \underset{\underset{(1,a)}{\rotatebox{90}{$\in$}}}{(\{1\} \times A)} \cup \underset{\underset{(2,b)}{\rotatebox{90}{$\in$}}}{(\{2\} \times B)}$$

Let I be a set s.t.

A_i is a set for all $i \in I$.

The big union

$$\bigcup_{i \in I} A_i \quad \left[\text{or } \bigcup \{A_i \mid i \in I\}\right]$$

$$= \{x \mid \exists i \in I.\ x \in A_i\} \quad \text{is a set.}$$

The big intersection

$$\bigcap_{i \in I} A_i \quad \left[\text{or } \bigcap \{A_i \mid i \in I\}\right]$$

$j \in I$

$$= \{x \in A_j \mid \forall i \in I.\ x \in A_i\}.$$

[Needs I nonempty, or universe $\mathcal{U}$ so empty intersections are $\mathcal{U}$.]

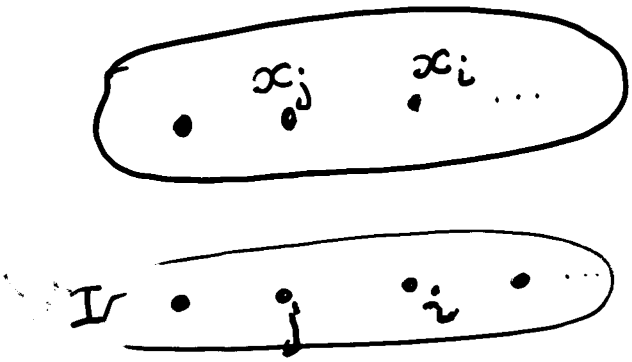
Indexed sets:

Let I be a set s.t.

x_i is an element for all $i \in I$

Then,

$\{x_i \mid i \in I\}$ is a set.



Cantor's diagonal argument revisited

Let X be a set. There is no bijection $\theta : X \to \mathcal{P}(X)$.

$x \qquad \theta(x)$

Proof By contradiction. Suppose

$$\theta : X \to \mathcal{P}(X)$$

is a bijection.

Define
$$Y = \{ x \in X \mid x \notin \theta(x) \}.$$

	$\cdots$	x	$\cdots$	y	$\cdots$
$\vdots$		$\vdots$		$\vdots$	
$\theta(x)$	$\cdots$	$\boxed{F}$	$\cdots$	F	$\cdots$
$\vdots$		$\vdots$		$\vdots$	
$\theta(y)$	$\cdots$	F	$\cdots$	$\boxed{F}$	$\cdots$
$\vdots$		$\vdots$		$\vdots$	

As θ is a bijection, there is $y \in X$ such that $\theta(y) = Y$. But ...

Some Consequences

- set of relations between sets X and Y:

$$\mathcal{P}(X \times Y).$$

- set of partial functions from set X to set Y:

$$(X \rightharpoonup Y) = \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a partial fn.} \}.$$

- set of total functions from set X to set Y:

$$(X \to Y) = \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a function} \}.$$

Higher Order Logic:

If $(x \in X \Rightarrow e \in Y)$, then

$$\lambda x \in Y..\, e \quad \in (X \to Y)$$

$$= \{ (x, e) \mid x \in X \}$$