Strategies as Profunctors

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Abstract. A new characterization of nondeterministic concurrent strategies exhibits strategies as certain discrete fibrations—or equivalently presheaves—over configurations of the game. This leads to a lax functor from the bicategory of strategies to the bicategory of profunctors. The lax functor expresses how composition of strategies is obtained from that of profunctors by restricting to 'reachable' elements, which gives an alternative formulation of the composition of strategies. It provides a fundamental connection—and helps explain the mismatches—between two generalizations of domain theory to forms of intensional domain theories, one based on games and strategies, and the other on presheaf categories and profunctors. In particular cases, on the sub-bicategory of rigid strategies which includes 'simple games' (underlying AJM and HO games), and stable spans (specializing to Berry's stable functions, in the deterministic case), the lax functor becomes a pseudo functor. More generally, the laxness of the functor suggests what structure is missing from categories and profunctors in order that they can be made to support the operations of games and strategies. By equipping categories with the structure of a 'rooted' factorization system and ensuring all elements of profunctors are 'reachable,' we obtain a pseudo functor embedding the bicategory of strategies in the bicategory of reachable profunctors. This shift illuminates early work on bistructures and bidomains, where the Scott order and Berry's stable order are part of a factorization system, giving a sense in which bidomains are games.

1 Introduction

A very general definition of nondeterministic concurrent strategy between games represented by event structures, has recently been given—see [1] for further background and examples. Building on this work and a new characterization of strategies (Lemma 1) we exhibit a strategy in a game as a presheaf, and a strategy between games as a profunctor. This exposes a lax functor from a bicategory of games and strategies to the bicategory of profunctors. In several well-known sub-bicategories of games the lax functor becomes a pseudo functor.

This somewhat technical result is significant because both strategies and profunctors have been central to generalizations of domain theory to forms of intensional domain theories. Game semantics has been strikingly successful, following the seminal work on game semantics and PCF [2, 3]. Profunctors themselves provide a rich framework in which to generalize domain theory in a way that is arguably closer to that initiated by Dana Scott than game semantics; we refer the reader to Hyland's case for such a generalization [4] and to the relevance of presheaf categories and profunctors to concurrent computation [5]. The lax functor from strategies to profunctors provides a fundamental connection between the two approaches. Indeed it exhibits composition of strategies as essentially composition of profunctors but restricted to those elements which are 'reachable'; roughly, they are 'reachable' in the sense of satisfying the causal-dependency constraints of both components of the composition, whereas profunctor composition allows elements merely when input matches output.

Arguably the concept of strategy is potentially as fundamental as that of relation. But for this potential to be seen and realized the concept needs to be developed in sufficient generality. This is one motivation for grounding strategies in a general model for concurrent computation. Doing so has exposed unexpected characteristics of strategies, which carry the concept of strategy into new terrain.

A surprise in developing this work has been the central role taken by the *reversal*, or undoing, of (compound) moves of Opponent. The idea first appears rather formally in Lemma 1 where, in a strategy, reversals of Opponent moves satisfy the same property as moves of Player. It then takes on a key role in characterizing a concurrent strategy in a game as a discrete fibration which preserves Player moves and reversals of Opponent moves (Theorem 1). Pushing the idea to completion we are led to a view of games as factorization systems in which 'left' maps stand for the reversal of compound Opponent moves while 'right' maps stand for compound Player moves. Section 9 gives a sketch of how concurrent games and strategies essentially form a sub-bicategory within a bicategory of strategies between games as rooted factorization systems.

2 Event structures and stable families

An *event structure* comprises $(E, \mathrm{Con}, \leq)$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$
\begin{aligned}
&\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \mathrm{Con} \text{ for all } e \in E,\\
&Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}\\
&X \in \mathrm{Con}\ \&\ e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.
\end{aligned}
$$

The *configurations*, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X \text{ is finite} \Rightarrow X \in \mathrm{Con}$, and
Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

Often we shall be concerned with just the finite configurations of an event structure. We write $\mathcal{C}(E)$ for the set of *finite* configurations.

We say that events e, e' are *concurrent*, and write $e\, co\, e'$ if $\{e, e'\} \in \mathrm{Con}\ \&\ e \not\leq e'\ \&\ e' \not\leq e$. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e

and e' are distinct with $e \leq e'$ and no event in between, will play a very important role. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X; note if $X \in \mathrm{Con}$, then $[X] \in \mathrm{Con}$.

Operations such as synchronized parallel composition are awkward to define directly on the simple event structures above. It is useful to broaden event structures to stable families, where operations are often carried out more easily, and then turned into event structures by the operation Pr below.

A *stable family* comprises $\mathcal{F}$, a nonempty family of finite subsets, called *configurations*, which satisfy:
Completeness: $\forall Z \subseteq \mathcal{F}.\ Z\uparrow \implies \bigcup Z \in \mathcal{F}$;
Coincidence-freeness: For all $x \in \mathcal{F}$, $e, e' \in x$ with $e \neq e'$,

$$\exists y \in \mathcal{F}.\ y \subseteq x \ \&\ (e \in y \iff e' \notin y)\,;$$

Stability: $\forall x, y \in \mathcal{F}.\ x \uparrow y \implies x \cap y \in \mathcal{F}$.
Above, $Z\uparrow$ means $\exists x \in \mathcal{F} \forall z \in Z.\ z \subseteq x$, and expresses the compatibility of Z in $\mathcal{F}$; we use $x \uparrow y$ for $\{x, y\}\uparrow$. We call elements of $\bigcup \mathcal{F}$ *events* of $\mathcal{F}$.

Proposition 1. *Let x be a configuration of a stable family $\mathcal{F}$. For $e, e' \in x$ define*

$$e' \leq_x e \ \textit{iff}\ \forall y \in \mathcal{F}.\ y \subseteq x \ \&\ e \in y \implies e' \in y.$$

When $e \in x$ define the prime configuration

$$[e]_x = \bigcap \{y \in \mathcal{F} \mid y \subseteq x \ \&\ e \in y\}\,.$$

Then $\leq_x$ is a partial order and $[e]_x$ is a configuration such that

$$[e]_x = \{e' \in x \mid e' \leq_x e\}.$$

Moreover the configurations $y \subseteq x$ are exactly the down-closed subsets of $\leq_x$.

Proposition 2. *Let $\mathcal{F}$ be a stable family. Then,* $\mathrm{Pr}(\mathcal{F}) =_{\mathrm{def}} (P, \mathrm{Con}, \leq)$ *is an event structure where:*

$$\begin{aligned}
&P = \{[e]_x \mid e \in x \ \&\ x \in \mathcal{F}\}\ ,\\
&Z \in \mathrm{Con} \ \textit{iff}\ Z \subseteq P \ \&\ \bigcup Z \in \mathcal{F} \quad \textit{and,}\\
&p \leq p' \ \textit{iff}\ p, p' \in P \ \&\ p \subseteq p'\,.
\end{aligned}$$

A (partial) map of stable families $f : \mathcal{F} \to \mathcal{G}$ is a partial function f from the events of $\mathcal{F}$ to the events of $\mathcal{G}$ such that for all configurations $x \in \mathcal{F}$,

$$fx \in \mathcal{G} \ \&\ (\forall e_1, e_2 \in x.\ f(e_1) = f(e_2) \implies e_1 = e_2)\,.$$

Maps of event structures are maps of their stable families of configurations. Maps compose as functions. We say a map is *total* when it is total as a function.

Pr is the right adjoint of the "inclusion" functor, taking an event structure E to the stable family $\mathcal{C}(E)$. The unit of the adjunction $E \to \mathrm{Pr}(\mathcal{C}(E))$ takes an event e to the prime configuration $[e] =_{\mathrm{def}} \{e' \in E \mid e' \leq e\}$. The counit *max* : $\mathcal{C}(\mathrm{Pr}(\mathcal{F})) \to \mathcal{F}$ takes prime configuration $[e]_x$ to its maximum event e; the image of a configuration $x \in \mathcal{C}(\mathrm{Pr}(\mathcal{F}))$ under the map *max* is $\bigcup x \in \mathcal{F}$.

Definition 1. Let $\mathcal{F}$ be a stable family. We use $x {-\!\!\subset} y$ to mean y covers x in $\mathcal{F}$, *i.e.* $x \subsetneq y$ in $\mathcal{F}$ with nothing in between, and $x \xrightarrow{e}\!\!\subset y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{F}$ and event $e \notin x$. We sometimes use $x \xrightarrow{e}\!\!\subset$, expressing that event e is enabled at configuration x, when $x \xrightarrow{e}\!\!\subset y$ for some y. W.r.t. $x \in \mathcal{F}$, write $[e)_x =_{\text{def}} \{e' \in E \mid e' \leq_x e \ \& \ e' \neq e\}$, so, for example, $[e)_x \xrightarrow{e}\!\!\subset [e]_x$. The relation of *immediate* dependence of event structures generalizes: with respect to $x \in \mathcal{F}$, the relation $e \rightarrowtriangle_x e'$ means $e \leq_x e'$ with $e \neq e'$ and no event in between.

3 Process operations

Products Let $\mathcal{A}$ and $\mathcal{B}$ be stable families with events A and B, respectively. Their product, the stable family $\mathcal{A} \times \mathcal{B}$, has events comprising pairs in $A \times_* B =_{\text{def}} \{(a, *) \mid a \in A\} \cup \{(a, b) \mid a \in A \ \& \ b \in B\} \cup \{(*, b) \mid b \in B\}$, the product of sets with partial functions, with (partial) projections π_1 and π_2—treating $*$ as 'undefined'—with configurations $x \in \mathcal{A} \times \mathcal{B}$ iff

$$
\begin{aligned}
&x \text{ is a finite subset of } A \times_* B \text{ s.t. } \ \pi_1 x \in \mathcal{A} \ \& \ \pi_2 x \in \mathcal{B},\\
&\forall e, e' \in x.\ \pi_1(e) = \pi_1(e') \text{ or } \pi_2(e) = \pi_2(e') \Rightarrow e = e'\,, \&\\
&\forall e, e' \in x.\ e \neq e' \Rightarrow \exists y \subseteq x.\ \pi_1 y \in \mathcal{A} \ \& \ \pi_2 y \in \mathcal{B} \ \& \ (e \in y \iff e' \notin y)\,.
\end{aligned}
$$

Right adjoints preserve products. Consequently we obtain a product of event structures A and B by first regarding them as stable families $\mathcal{C}(A)$ and $\mathcal{C}(B)$, forming their product $\mathcal{C}(A) \times \mathcal{C}(B), \pi_1, \pi_2$, and then constructing the event structure

$$A \times B =_{\text{def}} \Pr(\mathcal{C}(A) \times \mathcal{C}(B))$$

and its projections as $\Pi_1 =_{\text{def}} \pi_1 max$ and $\Pi_2 =_{\text{def}} \pi_2 max$.

Restriction The *restriction* of $\mathcal{F}$ to a subset of events R is the stable family $\mathcal{F} \upharpoonright R =_{\text{def}} \{x \in \mathcal{F} \mid x \subseteq R\}$. Defining $E \upharpoonright R$, the restriction of an event structure E to a subset of events R, to have events $E' = \{e \in E \mid [e] \subseteq R\}$ with causal dependency and consistency induced by E, we obtain $\mathcal{C}(E \upharpoonright R) = \mathcal{C}(E) \upharpoonright R$.

Proposition 3. *Let $\mathcal{F}$ be a stable family and R a subset of its events. Then,* $\Pr(\mathcal{F} \upharpoonright R) = \Pr(\mathcal{F}) \upharpoonright max^{-1} R$.

Synchronized compositions Synchronized parallel compositions are obtained as restrictions of products to those events which are allowed to synchronize or occur asynchronously according to the specific synchronized composition. For example, the synchronized composition of Milner's CCS on stable families $\mathcal{A}$ and $\mathcal{B}$ (with labelled events) is defined as $\mathcal{A} \times \mathcal{B} \upharpoonright R$ where R comprises events which are pairs $(a, *)$, $(*, b)$ and (a, b), where in the latter case the events a of $\mathcal{A}$ and b of $\mathcal{B}$ carry complementary labels. Similarly, synchronized compositions of event structures A and B are obtained as restrictions $A \times B \upharpoonright R$. By Proposition 3, we can equivalently form a synchronized composition of event structures by forming the synchronized composition of their stable families of configurations, and then obtaining the resulting event structure—this has the advantage of eliminating superfluous events earlier.

Projection Event structures support a simple form of hiding. Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$.

4 Event structures with polarities

Both a game and a strategy in a game are to be represented by an event structure with polarity, which comprises (E, pol) where E is an event structure with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity + (Player) or − (Opponent) to its events. The events correspond to (occurrences of) moves. Maps of event structures with polarity are maps of event structures which preserve polarity.

Dual and parallel composition of games The *dual*, $E^\perp$, of an event structure with polarity E comprises a copy of the event structure E but with a reversal of polarities. We write $\overline{e} \in E^\perp$ for the event complementary to $e \in E$ and *vice versa*. The operation $A \| B$—a simple parallel composition of games—simply forms the disjoint juxtaposition of A, B, two event structures with polarity; a finite subset of events is consistent if its intersection with each component is consistent.

5 Pre-strategies

Let A be an event structure with polarity, thought of as a game. A *pre-strategy in* A represents a nondeterministic play of the game and is defined to be a total map $\sigma : S \to A$ from an event structure with polarity S. Two pre-strategies $\sigma : S \to A$ and $\tau : T \to A$ in A will be essentially the same when they are isomorphic, *i.e.* there is an isomorphism $\theta : S \cong T$ such that $\sigma = \tau\theta$; then we write $\sigma \cong \tau$.

Let A and B be event structures with polarity. Following Joyal [6], a pre-strategy *from* A *to* B is a pre-strategy in $A^\perp \| B$, so a total map $\sigma : S \to A^\perp \| B$. It thus determines a span

$$A^\perp \xleftarrow{\sigma_1} S \xrightarrow{\sigma_2} B\,,$$

of event structures with polarity where σ_1, σ_2 are *partial* maps and for all $s \in S$ either, but not both, $\sigma_1(s)$ or $\sigma_2(s)$ is defined. We write $\sigma : A \mathrel{\mathrlap{\rightarrow}\mkern1mu{+}} B$ to express that σ is a pre-strategy from A to B. Note a pre-strategy σ in a game A coincides with a pre-strategy from the empty game $\sigma : \varnothing \mathrel{\mathrlap{\rightarrow}\mkern1mu{+}} A$.

5.1 Composing pre-strategies

Consider two pre-strategies $\sigma : A \mathrel{\mathrlap{\rightarrow}\mkern1mu{+}} B$ and $\tau : B \mathrel{\mathrlap{\rightarrow}\mkern1mu{+}} C$ as spans:

$$A^\perp \xleftarrow{\sigma_1} S \xrightarrow{\sigma_2} B \qquad B^\perp \xleftarrow{\tau_1} T \xrightarrow{\tau_2} C\,.$$

Their composition $\tau\odot\sigma : A \twoheadrightarrow C$ is defined as a synchronized composition, followed by projection to hide internal synchronization events. It is convenient to build the synchronized composition from the product of stable families $\mathcal{C}(S) \times \mathcal{C}(T)$, with projections π_1 and π_2, as

$$\mathcal{C}(T)\odot\mathcal{C}(T) =_{\rm def} \mathcal{C}(S) \times \mathcal{C}(T) \upharpoonright R\,,\ \text{where}$$

$$\begin{aligned} R \ = \ & \{(s,*) \mid s \in S \ \&\ \sigma_1(s) \text{ is defined}\} \cup \{(*,t) \mid t \in T \ \&\ \tau_2(t) \text{ is defined}\} \cup \\ & \{(s,t) \mid s \in S \ \&\ t \in T \ \&\ \sigma_2(s) = \overline{\tau_1(t)} \text{ with both defined}\}\,. \end{aligned}$$

Define $T\odot S =_{\rm def} \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \downarrow V$, where

$$\begin{aligned} V = \ & \{p \in \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \mid \exists s \in S.\ \mathit{max}(p) = (s,*)\} \cup \\ & \{p \in \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \mid \exists t \in T.\ \mathit{max}(p) = (*,t)\}\,. \end{aligned}$$

The span $\tau\odot\sigma$ comprises maps $\upsilon_1 : T\odot S \to A^\perp$ and $\upsilon_2 : T\odot S \to C$, which on events p of $T\odot S$ act so $\upsilon_1(p) = \sigma_1(s)$ when $\mathit{max}(p) = (s,*)$ and $\upsilon_2(p) = \tau_2(t)$ when $\mathit{max}(p) = (*,t)$, and are undefined elsewhere.

5.2 Concurrent copy-cat

Let A be an event structure with polarity. The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$, based on the idea that Player moves, of +ve polarity, always copy previous corresponding moves of Opponent, of −ve polarity. For $c \in A^\perp \| A$ we use $\overline{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component. Define $\mathbb{C}\mathbb{C}_A$ to comprise the event structure with polarity $A^\perp \| A$ together with extra causal dependencies $\overline{c} \leq_{\mathbb{C}\mathbb{C}_A} c$ for all events c with $\mathit{pol}_{A^\perp \| A}(c) = +$.

Proposition 4. *Let A be an event structure with polarity. Then $\mathbb{C}\mathbb{C}_A$ is an event structure with polarity. Moreover,*
$x \in \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ *iff* $x \in \mathcal{C}(A^\perp \| A)$ & $\forall c \in x.\ \mathit{pol}_{A^\perp \| A}(c) = + \implies \overline{c} \in x\,.$

The *copy-cat* pre-strategy $\gamma_A : A \twoheadrightarrow A$ is defined to be the map $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$ where γ_A is the identity on the common set of events.

6 Strategies

The main result of [1] is that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient for copy-cat to behave as identity w.r.t. the composition of pre-strategies. Receptivity ensures an openness to all possible moves of Opponent. A pre-strategy σ is *receptive* iff

$$\sigma x \xrightarrow{a}\!\subset \ \&\ \mathit{pol}_A(a) = - \text{ implies } \exists! s \in S.\ x \xrightarrow{s}\!\subset \ \&\ \sigma(s) = a\,.$$

Innocence restricts the behaviour of Player; Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$ beyond those imposed by the game. A pre-strategy σ is *innocent* when

$$s \rightarrowtriangle s' \text{ and } pol(s) = + \text{ or } pol(s') = - \text{ implies } \sigma(s) \rightarrowtriangle \sigma(s').$$

Copy-cat behaves as identity w.r.t. composition, *i.e.* $\sigma \circ \gamma_A \cong \sigma$ and $\gamma_B \circ \sigma \cong \sigma$, for a pre-strategy $\sigma : A \twoheadrightarrow B$, iff σ is receptive and innocent; copy-cat pre-stategies $\gamma_A : A \twoheadrightarrow A$ are receptive and innocent [1].

This result motivates the definition of a *strategy* as a pre-strategy which is receptive and innocent. We obtain a bicategory, **Games**, in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \twoheadrightarrow B$ and the 2-cells are maps of spans. The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the functoriality of synchronized composition).

6.1 A new characterization of concurrent strategies

Let x and x' be configurations of an event structure with polarity. Write $x \subseteq^- x'$ to mean $x \subseteq x'$ and $pol(x' \smallsetminus x) \subseteq \{-\}$, *i.e.* the configuration x' extends the configuration x solely by events of $-$ve polarity. Similarly, write $x \subseteq^+ x'$ to mean $x \subseteq x'$ and $pol(x' \smallsetminus x) \subseteq \{+\}$. With this notation in place we can give an attractive characterization of concurrent strategies, key to this paper. (Its proof is in the Appendix.)

Lemma 1. *A strategy in a game A comprises $\sigma : S \to A$, a total map of event structures with polarity, such that*
(i) whenever $y \subseteq^+ \sigma x$ in $\mathcal{C}(A)$ there is a (necessarily unique) $x' \in \mathcal{C}(S)$ so that $x' \subseteq x$ & $\sigma x' = y$, i.e.

$$\begin{tikzcd} x' \arrow[r, dotted, phantom, "\subseteq"] \arrow[d, dotted, mapsto, "\sigma"'] & x \arrow[d, mapsto, "\sigma"] \\ y \arrow[r, phantom, "\subseteq^+"] & \sigma x , \end{tikzcd}$$

and
(ii) whenever $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$ there is a unique $x' \in \mathcal{C}(S)$ so that $x \subseteq x'$ & $\sigma x' = y$, i.e.

$$\begin{tikzcd} x \arrow[r, dotted, phantom, "\subseteq"] \arrow[d, mapsto, "\sigma"'] & x' \arrow[d, dotted, mapsto, "\sigma"] \\ \sigma x \arrow[r, phantom, "\subseteq^-"] & y . \end{tikzcd}$$

7 Strategies as discrete fibrations

Condition (i) of Lemma 1, concerning the order $\subseteq^+$, is familiar from discrete fibrations (*cf.* Definition 2 below) while condition (ii), concerning $\subseteq^-$, is dual—the order $\subseteq^-$ simply points in the wrong direction. This suggests building a new relation $\sqsubseteq$ associated with an event structure with polarity out of compositions of $\subseteq^+$ with the *reversed* order $\supseteq^-$. In fact $\sqsubseteq$, the relation so obtained, is a partial order and instances of $\sqsubseteq$ always factor uniquely as an instance of $\supseteq^-$, associated with the reversal or undoing of Opponent moves, followed by $\subseteq^+$, the performance of Player moves (Section 7.1). We call the order $\sqsubseteq$ the *Scott* order because increasing w.r.t. $\sqsubseteq$ is associated with more +ve events (think more output) and less −ve events (think less input)—reminiscent of the pointwise order on functions in domain theory.

The seemingly formal Scott order will be the key to a new understanding of strategies as discrete fibrations (Theorem 1). Discrete fibrations are a reformulation of presheaves so reveal strategies $\sigma : S \to A$ in a game A as certain presheaves over $(\mathcal{C}(A), \sqsubseteq_A)$ (Section 7.2). Through the fortuitous way in which the Scott order interacts with the dual and parallel operations on games a strategy between games $\sigma : A \longrightarrow\!\!\!\!\!+ B$ turns into a presheaf over $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$, *i.e.* a profunctor from $(\mathcal{C}(A), \sqsubseteq_A)$ to $(\mathcal{C}(B), \sqsubseteq_B)$ (Section 7.3).

7.1 The Scott order in games

Let A be an event structure with polarity. The $\subseteq$-order on its configurations decomposes into two more fundamental orders $\subseteq^-$ and $\subseteq^+$. Define the *Scott* order, between configurations $x, y \in \mathcal{C}(A)$, by

$$x \sqsubseteq_A y \iff x \supseteq^- x \cap y \subseteq^+ y\,.$$

We use $\supseteq^-$ as the converse order to $\subseteq^-$. The properties of the Scott order are summarised in the next proposition. In particular,

$$x \sqsubseteq_A y \text{ iff } x \supseteq^- \cdot \subseteq^+ \cdot \supseteq^- \cdots \supseteq^- \cdot \subseteq^+ y\,.$$

Proposition 5. *Let A be an event structure with polarity.*
(i) If $x \subseteq^+ w \supseteq^- y$ in $\mathcal{C}(A)$, then $x \supseteq^- x \cap y \subseteq^+ y$ in $\mathcal{C}(A)$.
(ii) The relation $\sqsubseteq_A$ is the transitive closure of the relation $\supseteq^- \cup \subseteq^+$.
(iii) $(\mathcal{C}(A), \sqsubseteq_A)$ is a partial order for which whenever $x \sqsubseteq_A y$ there is a unique z, viz.. $x \cap y$, for which $x \supseteq^- z \subseteq^+ y$.

Proof. (i) Assume $x \subseteq^+ w \supseteq^- y$ in $\mathcal{C}(A)$. Clearly $x \supseteq x \cap y$. Suppose $a \in x$ and $pol_A(a) = +$. Then $a \in w$, and because only −ve events are lost from w in $w \supseteq^- y$ we obtain $a \in y$, so $a \in x \cap y$. It follows that $x \supseteq^- x \cap y$, as required. Similarly, $x \cap y \subseteq^+ y$. (ii) Directly from (i). (iii) Clearly $\sqsubseteq$ is reflexive. Supposing $x \sqsubseteq y$, *i.e.* $x \supseteq^- x \cap y \subseteq^+ y$ in $\mathcal{C}(A)$ we see that the +ve events of x are included in y, and the −ve events of y are included in x. Hence if $x \sqsubseteq y$ and $y \sqsubseteq x$ in $\mathcal{C}(A)$ then x and y have the same +ve and −ve events and so are equal. Transitivity follows by (ii). Unique-factorization follows from the fact that when $x \supseteq^- z \subseteq^+ y$ necessarily $z = x \cap y$, as is easy to show. □

7.2 Strategies in games as presheaves

Let A be an event structure with polarity. We shall show how strategies in A correspond to cerain fibrations, so presheaves, over the order $(\mathcal{C}(A), \sqsubseteq_A)$. We concentrate on discrete fibrations over partial orders.

Definition 2. A *discrete fibration* over a partial order $(Y, \sqsubseteq_Y)$ is a partial order $(X, \sqsubseteq_X)$ and an order-preserving function $f : X \to Y$ such that

$$\forall x \in X, y' \in Y.\ y' \sqsubseteq_Y f(x) \implies \exists! x' \sqsubseteq_X x.\ f(x') = y' .$$

Via the Scott order we can recast strategies $\sigma : S \to A$ as those discrete fibrations $F : (\mathcal{C}(S), \sqsubseteq_S) \to (\mathcal{C}(A), \sqsubseteq_A)$ which preserve $\varnothing$, $\supseteq^-$ and $\subseteq^+$ in the sense that $F(\varnothing) = \varnothing$ while $x \supseteq^- y$ implies $F(x) \supseteq^- F(y)$, and $x \subseteq^+ y$ implies $F(x) \subseteq^+ F(y)$, for $x, y \in \mathcal{C}(S)$:

Theorem 1. *(i) Let $\sigma : S \to A$ be a strategy in game A. The map σ“ taking a finite configuration $x \in \mathcal{C}(S)$ to $\sigma x \in \mathcal{C}(A)$ is a discrete fibration from $(\mathcal{C}(S), \sqsubseteq_S)$ to $(\mathcal{C}(A), \sqsubseteq_A)$ which preserves $\varnothing$, $\supseteq^-$ and $\subseteq^+$.*
(ii) Suppose $F : (\mathcal{C}(S), \sqsubseteq_S) \to (\mathcal{C}(A), \sqsubseteq_A)$ is a discrete fibration which preserves $\varnothing$, $\supseteq^-$ and $\subseteq^+$. There is a unique strategy $\sigma : S \to A$ such that $F = \sigma$“.

Proof. (i) That σ“ forms a discrete fibration is a direct corollary of Lemma 1. As a map of event structures with polarity, σ“ automatically preserves $\varnothing$, $\supseteq^-$ and $\subseteq^+$. (ii) Assume F is a discrete fibration preserving $\varnothing$, $\supseteq^-$ and $\subseteq^+$. First observe a consequence, that if $x \subseteq^+ x'$ in $\mathcal{C}(S)$ and $F(x) \subseteq^+ y'' \subseteq F(x')$ in $\mathcal{C}(A)$, then there is a unique $x'' \in \mathcal{C}(S)$ such that $x \subseteq^+ x'' \subseteq x'$ and $F(x'') = y''$. (An analogous observation holds with $+$ replaced by $-$.) Suppose now $x \overset{+}{\longrightarrow\!\!\subset} x'$ in $\mathcal{C}(S)$—where we write $x \overset{+}{\longrightarrow\!\!\subset} x'$ to abbreviate $x \overset{s}{\longrightarrow\!\!\subset} x'$ for some +ve $s \in S$. As F preserves $\subseteq^+$, $F(x) \subseteq^+ F(x')$. The observation implies $F(x) \overset{+}{\longrightarrow\!\!\subset} F(x')$ in $\mathcal{C}(A)$. Similarly, $x \overset{-}{\longrightarrow\!\!\subset} x'$ implies $F(x) \overset{-}{\longrightarrow\!\!\subset} F(x')$.

Define the relation $\approx$ between prime intervals $[x, x']$, where $x {-\!\!\subset} x'$, as the least equivalence relation such that $[x, x'] \approx [y, y']$ if $x {-\!\!\subset} y$ and $x' {-\!\!\subset} y'$ with $y \neq x'$. For configurations of an event structure, $[x, x'] \approx [y, y']$ iff $x \overset{e}{\longrightarrow\!\!\subset} x'$ and $y \overset{e}{\longrightarrow\!\!\subset} y'$ for some common event e. As F preserves coverings it preserves $\approx$. Consequently we obtain a well-defined function $\sigma : S \to A$ by taking s to a if an instance $x \overset{s}{\longrightarrow\!\!\subset} x'$ is sent to $F(x) \overset{a}{\longrightarrow\!\!\subset} F(x')$. Clearly σ preserves polarities.

By induction on the length of covering chains $\varnothing \overset{s_1}{\longrightarrow\!\!\subset} x_1 \overset{s_2}{\longrightarrow\!\!\subset} \cdots \overset{s_n}{\longrightarrow\!\!\subset} x_n = x$ and the fact that F preserves $\varnothing$ and coverings, $\varnothing = F(\varnothing) \overset{\sigma(s_1)}{\longrightarrow\!\!\subset} F(x_1) \overset{\sigma(s_2)}{\longrightarrow\!\!\subset} \cdots \overset{\sigma(s_n)}{\longrightarrow\!\!\subset} F(x_n) = F(x)$ with $\sigma x = F(x) \in \mathcal{C}(A)$. Moreover we cannot have $\sigma(s_i) = \sigma(s_j)$ for distinct i, j without contradicting F preserving coverings. This establishes $\sigma : S \to A$ as a total map of event structures with polarity. The assumed properties of F directly ensure that σ satisfies the two conditions of Lemma 1 required of strategy. ◻

As discrete fibrations correspond to presheaves, Theorem 1 entails that strategies $\sigma : S \to A$ correspond to (certain) presheaves over $(\mathcal{C}(A), \sqsubseteq_A)$—the presheaf for σ is a functor $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \to \mathbf{Set}$ sending y to the fibre $\{x \in \mathcal{C}(S) \mid \sigma x = y\}$.

7.3 Strategies between games as profunctors

A strategy $\sigma : A \twoheadrightarrow B$ determines a discrete fibration $\sigma``$ over $(\mathcal{C}(A^\perp \| B), \sqsubseteq_{A^\perp \| B})$. But

$$\begin{aligned}(\mathcal{C}(A^\perp \| B), \sqsubseteq_{A^\perp \| B}) &\cong (\mathcal{C}(A^\perp), \sqsubseteq_{A^\perp}) \times (\mathcal{C}(B), \sqsubseteq_B) && (1)\\ &\cong (\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)\,. && (2)\end{aligned}$$

The first step (1) relies on the correspondence between a configuration of $A^\perp \| B$ and a pair, with left component a configuration of $A^\perp$ and right component a configuration of B. In the last step (2) we are using the correspondence between configurations of $A^\perp$ and A induced by the correspondence $a \leftrightarrow \overline{a}$ between their events: a configuration x of $A^\perp$ corresponds to a configuration $\overline{x} =_{\mathrm{def}} \{\overline{a} \mid a \in x\}$ of A. Because $A^\perp$ reverses the roles of $+$ and $-$ in A, the order $x \sqsubseteq_{A^\perp} y$, *i.e.* $x \supseteq^- x \cap y \subseteq^+ y$ in $\mathcal{C}(A^\perp)$, corresponds to the order $\overline{y} \sqsubseteq_A \overline{x}$, *i.e.* $\overline{y} \supseteq^- \overline{x} \cap \overline{y} \subseteq^+ \overline{x}$ in $\mathcal{C}(A)$, so $\overline{x} \sqsubseteq_A^{\mathrm{op}} \overline{y}$.

It follows that a strategy $\sigma : S \to A^\perp \| B$ determines a discrete fibration

$$\sigma`` : (\mathcal{C}(S), \sqsubseteq_S) \to (\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$$

where $\sigma``(x) = (\overline{\sigma_1 x},\ \sigma_2 x)$, for $x \in \mathcal{C}(S)$. One way to define a *profunctor* from $(\mathcal{C}(A), \sqsubseteq_A)$ to $(\mathcal{C}(B), \sqsubseteq_B)$ is as a discrete fibration over $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$. Hence the strategy σ determines a profunctor[1] $\sigma`` : (\mathcal{C}(A), \sqsubseteq_A) \twoheadrightarrow (\mathcal{C}(B), \sqsubseteq_B)$.

8 A lax functor from strategies to profunctors

We now study how the operation from strategies σ to profunctors $\sigma``$ preserves identities and composition.

8.1 Identity

The operation $(-)``$ preserves identities:

Lemma 2. *Let A be an event structure with polarity. For $x \in \mathcal{C}(A^\perp \| A)$,*

$$x \in \mathcal{C}(\mathbb{C}_A) \quad \textit{iff} \quad x_2 \sqsubseteq_A \overline{x}_1\,,$$

where $x_1 \in \mathcal{C}(A^\perp)$ and $x_2 \in \mathcal{C}(A)$ are the projections of x to its components.

Proof. From Proposition 4, we deduce: $x \in \mathcal{C}(CC_A)$ iff (i) $\overline{x}_1^+ \supseteq x_2^+$ and (ii) $\overline{x}_1^- \subseteq x_2^-$, where $z^+ = \{a \in z \mid pol_A(a) = +\}$ and $z^- = \{a \in z \mid pol_A(a) = -\}$ for $z \in \mathcal{C}(A)$. It remains to argue that (i) and (ii) iff $x_2 \supseteq^- \overline{x}_1 \cap x_2 \subseteq^+ \overline{x}_1$. □

Corollary 1. *Let A be an event structure with polarity. The profunctor $\gamma_A``$ of the copy-cat strategy γ_A is an identity profunctor on $(\mathcal{C}(A), \sqsubseteq_A)$.*

Proof. The profunctor $\gamma_A`` : (\mathcal{C}(A), \sqsubseteq_A) \twoheadrightarrow (\mathcal{C}(A), \sqsubseteq_A)$ sends $x \in \mathcal{C}(\mathbb{C}_A)$ to $(\overline{x}_1, x_2) \in (\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(A), \sqsubseteq_A)$ precisely when $x_2 \sqsubseteq_A \overline{x}_1$. It is thus an identity on $(\mathcal{C}(A), \sqsubseteq_A)$. □

[1] Most often a profunctor from $(\mathcal{C}(A), \sqsubseteq_A)$ to $(\mathcal{C}(B), \sqsubseteq_B)$ is defined as a functor $(\mathcal{C}(A), \sqsubseteq_A) \times (\mathcal{C}(B), \sqsubseteq_B)^{\mathrm{op}} \to \mathbf{Set}$, *i.e.*, as a presheaf over $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$, and as such corresponds to a discrete fibration.

8.2 Composition

We need to relate the composition of strategies to the standard composition of profunctors. Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be strategies, so $\sigma : A \rightarrow\!\!\!\!+\!\!\!> B$ and $\tau : B \rightarrow\!\!\!\!+\!\!\!> C$. Abbreviating, for instance, $(\mathcal{C}(A), \sqsubseteq_A)$ to $\mathcal{C}(A)$, strategies σ and τ give rise to profunctors $\sigma“ : \mathcal{C}(A) \rightarrow\!\!\!\!+\!\!\!> \mathcal{C}(B)$ and $\tau“ : \mathcal{C}(B) \rightarrow\!\!\!\!+\!\!\!> \mathcal{C}(C)$. Their composition is the profunctor $\tau“ \circ \sigma“ : \mathcal{C}(A) \rightarrow\!\!\!\!+\!\!\!> \mathcal{C}(C)$ built, as now described, as a discrete fibration from the discrete fibrations $\sigma“ : \mathcal{C}(S) \to \mathcal{C}(A)^{\mathrm{op}} \times \mathcal{C}(B)$ and $\tau“ : \mathcal{C}(T) \to \mathcal{C}(B)^{\mathrm{op}} \times \mathcal{C}(C)$.

First, we define the set of *matching pairs*,

$$M =_{\mathrm{def}} \{(x, y) \in \mathcal{C}(S) \times \mathcal{C}(T) \mid \sigma_2 x = \overline{\tau_1 y}\},$$

on which we define $\sim$ as the least equivalence relation for which

$$(x, y) \sim (x', y') \quad \text{if} \quad x \sqsubseteq_S x' \ \&\ y' \sqsubseteq_T y \ \&\ \sigma_1 x = \sigma_1 x' \ \&\ \tau_2 y' = \tau_2 y\,.$$

Define an order on equivalence classes $M/\sim$ by:

$$\begin{aligned} m \sqsubseteq m' \ \text{ iff } \ & m = \{(x, y)\}_\sim \ \&\ m' = \{(x', y')\}_\sim \ \&\ x \sqsubseteq_S x' \ \&\ y \sqsubseteq_T y' \ \& \\ & \sigma_2 x = \sigma_2 x' \ \&\ \tau_1 y = \tau_1 y'\,, \end{aligned}$$

for some matching pairs $(x, y), (x', y')$—so then $\sigma_2 x = \sigma_2 x' = \overline{\tau_1 y} = \overline{\tau_1 y'}$. The relation $\sqsubseteq$ above is easily seen to be a partial order on $M/\sim$. The profunctor composition $\tau“ \circ \sigma“$ is given as

$$\tau“ \circ \sigma“ : M/\sim \ \to \ \mathcal{C}(A)^{\mathrm{op}} \times \mathcal{C}(C)\,, \text{ acting so } \{(x, y)\}_\sim \mapsto (\overline{\sigma_1 x}, \tau_2 y)$$

—it inherits from $\sigma“$ and $\tau“$ the property of being a discrete fibration.

It is *not* the case that $(\tau \odot \sigma)“$ and $\tau“ \circ \sigma“$ coincide up to isomorphism. The profunctor composition $\tau“ \circ \sigma“$ will generally contain extra equivalence classes $\{(x, y)\}_\sim$ for matching pairs (x, y) which are "unreachable." Although $\sigma_2 x = \overline{\tau_1 y}$, equals z say, automatically for a matching pair (x, y), the configurations x and y may impose incompatible causal dependencies on their 'interface' z so never be realized as a configuration in the synchronized composition $\mathcal{C}(T) \odot \mathcal{C}(S)$ used in building the composition of strategies $\tau \odot \sigma$.

Example 1. Let A and C both be the empty event structure $\varnothing$. Let B be the event structure consisting of the two concurrent events b_1, assumed $-$ve, and b_2, assumed +ve in B . Let the strategy $\sigma : \varnothing \rightarrow\!\!\!\!+\!\!\!> B$ comprise the event structure $s_1 \rightarrowtriangle s_2$ with s_1 $-$ve and s_2 +ve, $\sigma(s_1) = b_1$ and $\sigma(s_2) = b_2$. In $B^\perp$ the polarities are reversed so there is a strategy $\tau : B \rightarrow\!\!\!\!+\!\!\!> \varnothing$ comprising the event structure $t_2 \rightarrowtriangle t_1$ with t_2 $-$ve and t_1 +ve yet with $\tau(t_1) = \overline{b}_1$ and $\tau(t_2) = \overline{b}_2$. The equivalence class $\{(x, y)\}_\sim$, where $x = \{s_1, s_2\}$ and $y = \{t_1, t_2\}$, would be present in the profunctor composition $\tau“ \circ \sigma“$, in addition to $\{(\varnothing, \varnothing)\}_\sim$, whereas $\tau \odot \sigma$ would be the empty strategy and accordingly the profunctor $(\tau \odot \sigma)“$ only has a single element, $\varnothing$. $\square$

8.3 Laxness

This section establishes the exact relation between the two compositions $(\tau\odot\sigma)``$ and $\tau``\circ\sigma``$. The proofs use that the equivalence relation $\sim$ between matching pairs is generated by a single-step relation:

Lemma 3. *On matching pairs, define*

$$(x,y) \rightsquigarrow_1 (x',y') \quad \textit{iff} \quad \exists s \in S, t \in T.\ x \stackrel{s}{-\!\!\!\subset} x' \ \&\ y \stackrel{t}{-\!\!\!\subset} y' \ \&\ \sigma_2(s) = \overline{\tau_1(t)}\,.$$

The smallest equivalence relation including $\rightsquigarrow_1$ coincides with the relation $\sim$.

Now we make precise what it means for a matching pair to be reachable.

Definition 3. For (x,y) a matching pair, define

$$\begin{aligned} x\cdot y =_{\text{def}} &\{(s,*) \mid s \in x \ \&\ \sigma_1(s) \text{ is defined}\} \cup \{(*,t) \mid t \in y \ \&\ \tau_2(t) \text{ is defined}\} \cup \\ &\{(s,t) \mid s \in x \ \&\ t \in y \ \&\ \sigma_2(s) = \overline{\tau_1(t)}\}\,. \end{aligned}$$

Say (x,y) is *reachable* if $x\cdot y \in \mathcal{C}(T)\odot\mathcal{C}(S)$, and *unreachable* otherwise.

For $z \in \mathcal{C}(T)\odot\mathcal{C}(S)$ say a *visible prime* of z is a prime of the form $[(s,*)]_z$, for $(s,*) \in z$, or $[(*,t)]_z$, for $(*,t) \in z$.

We can specify when a matching pair is reachable without invoking the composition of strategies, important for the generalization in Section 9:

Proposition 6. *A matching pair (x,y) is reachable iff there is a sequence of matching pairs $(\varnothing,\varnothing) = (x_0,y_0), \cdots, (x_i,y_i), (x_{i+1},y_{i+1}), \cdots, (x_n,y_n) = (x,y)$ such that for all i, either $(x_i,y_i) \rightsquigarrow_1 (x_{i+1},y_{i+1})$ or $\exists s \in S.\ x_i \stackrel{s}{-\!\!\!\subset} x_{i+1} \ \&\ y_i = y_{i+1} \ \&\ \sigma_1(s)$ is defined or $\exists t \in T.\ y_i \stackrel{t}{-\!\!\!\subset} y_{i+1} \ \&\ x_i = x_{i+1} \ \&\ \tau_2(t)$ is defined. (The relation $\rightsquigarrow_1$ is that introduced in Lemma 3.)*

Theorem 2 below provides the precise relation between $(\tau\odot\sigma)``$ and $\tau``\circ\sigma``$. Its proof requires that reachable matching pairs are $\sim$-equivalent iff they are associated with the same configuration in $T\odot S$, the import of (ii) in the next lemma.

Lemma 4. *(i) If (x,y) is a reachable matching pair and $(x,y)\sim(x',y')$, then (x',y') is a reachable matching pair. (ii) Whenever (x,y), (x',y') are reachable matching pairs, $(x,y)\sim(x',y')$ iff $x\cdot y$ and $x'\cdot y'$ have the same visible primes.*

Proof. We use Lemma 3 characterizing $\sim$ in terms of $\rightsquigarrow_1$.
(i) Suppose $(x,y) \rightsquigarrow_1 (x',y')$ or $(x',y') \rightsquigarrow_1 (x,y)$. By inspection of the construction of the product of stable families in Section 3, if $x\cdot y \in \mathcal{C}(T)\odot\mathcal{C}(S)$ then $x'\cdot y' \in \mathcal{C}(T)\odot\mathcal{C}(S)$.

(ii) *"If":* Suppose $x \cdot y$ and $x' \cdot y'$ have the same visible primes, forming the set Q. Then $z =_{\mathrm{def}} \bigcup Q \in \mathcal{C}(T) \odot \mathcal{C}(S)$, being the union of a compatible set of configurations in $\mathcal{C}(T) \odot \mathcal{C}(S)$. Moreover, $z \subseteq x \cdot y, x' \cdot y'$. Take a covering chain

$$z \overset{e_1}{-\!\!\subset} \cdots z_i \overset{e_i}{-\!\!\subset} z_{i+1} \cdots \overset{e_n}{-\!\!\subset} x \cdot y$$

in $\mathcal{C}(T) \odot \mathcal{C}(S)$. Each $(\pi_1 z_i, \pi_2 z_i)$ is a matching pair. Necessarily, $e_i = (s_i, t_i)$ for some $s_i \in S$, $t_i \in T$, with $\sigma_2(s_i) = \overline{\tau_1(t_i)}$, again by the definition of $\mathcal{C}(T) \odot \mathcal{C}(S)$. Thus

$$(\pi_1 z_i, \pi_2 z_i) \rightsquigarrow_1 (\pi_1 z_{i+1}, \pi_2 z_{i+1}) \,.$$

Hence $(\pi_1 z, \pi_2 z) \sim (x, y)$, and similarly $(\pi_1 z, \pi_2 z) \sim (x', y')$, so $(x, y) \sim (x', y')$.
"Only if": It suffices to observe that if $(x, y) \rightsquigarrow_1 (x', y')$, then $x \cdot y$ and $x' \cdot y'$ have the same visible primes. But if $(x, y) \rightsquigarrow_1 (x', y')$ then $x \cdot y \overset{(s,t)}{-\!\!\subset} x' \cdot y'$, for some $s \in S, t \in T$, and no visible prime of $x' \cdot y'$ contains (s, t). □

Theorem 2. *Let $\sigma : A \longrightarrow\!\!\!\!\!\!\shortmid\;\; B$ and $\tau : B \longrightarrow\!\!\!\!\!\!\shortmid\;\; C$ be strategies. Defining*

$$\varphi_{\sigma,\tau} : \mathcal{C}(T \circledcirc S) \to M/\sim \quad by \quad \varphi_{\sigma,\tau}(z) = \{(\mathit{\Pi}_1 z, \mathit{\Pi}_2 z)\}_\sim \,,$$

where $\mathit{\Pi}_1 z = \pi_1 \bigcup z$ and $\mathit{\Pi}_2 z = \pi_2 \bigcup z$, yields an injective, order-preserving function from $(\mathcal{C}(T \circledcirc S), \sqsubseteq_{T \circledcirc S})$ to $(M/\sim, \sqsubseteq)$—its range is precisely the equivalence classes $\{(x, y)\}_\sim$ for reachable matching pairs (x, y). The diagram

$$\begin{tikzcd}
(\mathcal{C}(T \circledcirc S), \sqsubseteq_{T \circledcirc S}) \arrow[r, "\varphi_{\sigma,\tau}"] \arrow[d, "(\tau \circledcirc \sigma)``"'] & (M/\sim, \sqsubseteq) \arrow[dl, "\tau``\circ\sigma``"] \\
(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(C), \sqsubseteq_C) &
\end{tikzcd}$$

commutes.

Proof. For $z \in \mathcal{C}(T \circledcirc S)$, we obtain that $\varphi_{\sigma,\tau}(z) = (\mathit{\Pi}_1 z, \mathit{\Pi}_2 z) = (\pi_1 \bigcup z, \pi_2 \bigcup z)$ is a matching pair, from the definition of $\mathcal{C}(T) \odot \mathcal{C}(S)$; it is clearly reachable as $\pi_1 \bigcup z \cdot \pi_2 \bigcup z = \bigcup z \in \mathcal{C}(T) \odot \mathcal{C}(S)$. For any reachable matching pair (x, y) let z be the set of visible primes of $x \cdot y$. Then, $z \in \mathcal{C}(T \circledcirc S)$ and, by Lemma 4(ii), $(\mathit{\Pi}_1 z, \mathit{\Pi}_2 z) \sim (x, y)$ so $\varphi_{\sigma,\tau}(z) = \{(x, y)\}_\sim$. Injectivity of $\varphi_{\sigma,\tau}$ follows directly from Lemma 4(ii).

To show that $\varphi_{\sigma,\tau}$ is order-preserving it suffices to show if $z \mathrel{-\!\!\!\subset} z'$ in $(\mathcal{C}(T \circledcirc S), \sqsubseteq)$ then $\varphi_{\sigma,\tau}(z) \sqsubseteq \varphi_{\sigma,\tau}(z')$ in $(M/\sim, \sqsubseteq)$. (The covering relation $\mathrel{-\!\!\!\subset}$ is w.r.t. $\sqsubseteq$.) If $z \mathrel{-\!\!\!\subset} z'$ then either $z \overset{p}{-\!\!\subset} z'$, with p +ve, or $z' \overset{p}{-\!\!\subset} z$, with p −ve, for p a visible prime of $\mathcal{C}(T) \odot \mathcal{C}(S)$, *i.e.* with $max(p)$ of the form $(s, *)$ or $(*, t)$. We concentrate on the case where p is +ve (the proof when p is −ve is similar). In the case where p is +ve,

$$\mathit{\Pi}_1 z \cdot \mathit{\Pi}_2 z = \bigcup z \subseteq \bigcup z' = \mathit{\Pi}_1 z' \cdot \mathit{\Pi}_2 z'$$

in $\mathcal{C}(T) \odot \mathcal{C}(S)$ and there is a covering chain

$$\bigcup z = w_0 \overset{(s_1,t_1)}{-\!\!\subset} w_1 \cdots \overset{(s_n,t_n)}{-\!\!\subset} w_n \overset{max(p)}{-\!\!\subset} \bigcup z'$$

in $\mathcal{C}(T) \odot \mathcal{C}(S)$. Each w_i, for $0 \le i \le m$, is associated with a reachable matching pair $(\pi_1 w_i, \pi_2 w_i)$ where $\pi_1 w_i \cdot \pi_2 w_i = w_i$. Also $(\pi_1 w_i, \pi_2 w_i) \rightsquigarrow_1 (\pi_1 w_{i+1}, \pi_2 w_{i+1})$, for $0 \le i < m$. Hence $(\Pi_1 z, \Pi_2 z) \sim (\pi_1 w_n, \pi_2 w_n)$, by Lemma 3. If $max(p) = (s, *)$ then $\pi_1 w_n \overset{s}{\longrightarrow\!\!\subset} \Pi_1 z'$, with s +ve, and $\pi_2 w_n = \Pi_2 z'$. If $max(p) = (*, t)$ then $\pi_1 w_n = \Pi_1 z'$ and $\pi_2 w_n \overset{t}{\longrightarrow\!\!\subset} \Pi_2 z'$, with t +ve. In either case $\pi_1 w_n \sqsubseteq_S \Pi_1 z'$ and $\pi_2 w_n \sqsubseteq_T \Pi_2 z'$ with $\sigma_2 \pi_1 w_n = \sigma_2 \Pi_1 z'$ and $\tau_1 \pi_2 w_n = \tau_1 \Pi_2 z'$. Hence, from the definition of $\sqsubseteq$ on $M/\sim$,

$$\varphi_{\sigma,\tau}(z) = \{(\Pi_1 z, \Pi_2 z)\}_\sim = \{(\pi_1 w_n, \pi_2 w_n)\}_\sim \sqsubseteq \{(\Pi_1 z', \Pi_2 z')\}_\sim = \varphi_{\sigma,\tau}(z')\,.$$

It remains to show commutativity of the diagram. Let $z \in \mathcal{C}(T \odot S)$. Then,

$$(\tau\text{``} \circ \sigma\text{``})(\varphi_{\sigma,\tau}(z)) = (\tau\text{``} \circ \sigma\text{``})(\{(\Pi_1 z, \Pi_2 z)\}_\sim) = (\overline{\sigma_1 \Pi_1 z},\ \tau_2 \Pi_2 z) = (\tau \odot \sigma)\text{``}(z)\,,$$

via the definition of $\tau \odot \sigma$—as required. □

Because $(-)$“ does not preserve composition up to isomorphism but only up to the transformation φ of Theorem 2:

Corollary 2. *The operation $(-)$“ forms a* lax *functor from the bicategory of strategies to that of profunctors; identities are preserved up to the isomorphism of Corollary 1 while composition is preserved up to φ of Theorem 2.*

Despite laxness, the relation between strategy composition and profunctor composition is surprisingly straightforward: *the composition of strategies, viewed as a profunctor, is given by restricting the composition of profunctors to reachable matching pairs.*

In special cases composition is preserved up to isomorphism because all the relevant matching pairs are reachable. Say a strategy σ is *rigid* when the components σ_1, σ_2 preserve causal dependency when defined. In fact, rigid strategies form a sub-bicategory of **Games**. For composable rigid strategies σ and τ we do have $(\tau \odot \sigma)\text{``} \cong \tau\text{``} \circ \sigma\text{``}$. Stable spans (including Berry's stable functions), those strategies between games where all moves are +ve [1], and simple games [7, 8] lie within the bicategory of rigid strategies.

9 Games as factorization systems

The results of Section 7.1 show an event structure with polarity determines a factorization system [9]; the 'left' maps are given by $\supseteq^-$ and the 'right' maps by $\subseteq^+$. More specifically they form an instance of a *rooted* factorization system $(\mathbb{X}, \to_L, \to_R, 0)$ where maps $f : x \to_L x'$ are the 'left' maps and $g : x \to_R x'$ the 'right' maps of a factorization system on a small category $\mathbb{X}$, with distinguished object 0, such that any object x of $\mathbb{X}$ is reachable by a chain of maps

$$0 \leftarrow_L \cdot \to_R \cdots \leftarrow_L \cdot \to_R x\,.$$

Think of objects of $\mathbb{X}$ as configurations, the R-maps as standing for (compound) Player moves and L-maps for the reverse, or undoing, of (compound) Opponent moves in a game.

The characterization of strategy, Lemma 1, exhibits a strategy as a discrete fibration w.r.t. $\sqsubseteq$ whose functor preserves $\varnothing$, $\supseteq^-$ and $\subseteq^+$. This generalizes. Define a strategy in a rooted factorization system to be a functor from another rooted factorization system preserving 0, L-maps, R-maps and forming a discrete fibration. To obtain strategies *between* rooted factorization systems we again follow the methodology of Joyal [6], and take a strategy from $\mathbb{X}$ to $\mathbb{Y}$ to be a strategy in the dual of $\mathbb{X}$ in parallel composition with $\mathbb{Y}$. Now the dual operation becomes the opposite construction on a factorization system, reversing the roles and directions of the 'left' and 'right' maps. The parallel composition of factorization systems is given by their product. Composition of strategies is given essentially as that of profunctors, but restricting to reachable elements—the definition of reachable element is a direct generalization of Proposition 6. The bicategory of concurrent strategies is equivalent to the sub-bicategory in which the objects and strategies are on rooted factorization systems of the form of $((\mathcal{C}(A), \sqsubseteq_A), \supseteq^-, \subseteq^+, \varnothing)$ for an event structure with polarity A. One pay-off of the increased generality is that bistructures, a way to present Berry's bidomains as factorization systems [10], inherit a reading as games. The new view also allows us to formalize strategies in some games based on moves as vectors, as in some games of chase, in which moves of Player (as hunter) and Opponent (as prey) may be translations in space or changes in velocity.

Acknowledgments Thanks to Pierre Clairambault, Marcelo Fiore, Julian Gutierrez, Thomas Hildebrandt, Martin Hyland, Alex Katovsky, Samuel Mimram, Gordon Plotkin, Silvain Rideau and Sam Staton for helpful remarks. The support of Advanced Grant ECSYM of the ERC is acknowledged with gratitude.

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Appendix: Additional proofs

Lemma 1. *A strategy in a game A comprises $\sigma : S \to A$, a total map of event structures with polarity, such that*
(i) whenever $y \subseteq^+ \sigma x$ in $\mathcal{C}(A)$ there is a (necessarily unique) $x' \in \mathcal{C}(S)$ so that $x' \subseteq x$ & $\sigma x' = y$, i.e.

$$\begin{tikzcd} x' \arrow[r, dotted, no head, "\subseteq" description] \arrow[d, mapsto, dotted, "\sigma"'] & x \arrow[d, mapsto, "\sigma"] \\ y \arrow[r, phantom, "\subseteq^+"] & \sigma x \end{tikzcd},$$

and
(ii) whenever $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$ there is a unique $x' \in \mathcal{C}(S)$ so that $x \subseteq x'$ & $\sigma x' = y$, i.e.

$$\begin{tikzcd} x \arrow[r, dotted, no head, "\subseteq" description] \arrow[d, mapsto, "\sigma"'] & x' \arrow[d, mapsto, dotted, "\sigma"] \\ \sigma x \arrow[r, phantom, "\subseteq^-"] & y \end{tikzcd}.$$

Proof. Let $\sigma : S \to A$ be a total map of event structures with polarity. We show σ is a strategy iff (i) and (ii).
"Only if": (i) It suffices to show the seemingly weaker property (i)$'$ that

$$y \stackrel{a}{-\!\!\!\subset} \sigma x \ \& \ pol(a) = + \implies \exists x' \in \mathcal{C}(S).\, x' -\!\!\!\subset x \ \& \ \sigma x' = y$$

for $a \in A, x \in \mathcal{C}(S), y \in \mathcal{C}(A)$. Then (i), with $y \subseteq^+ \sigma x$, follows by considering a covering chain $y -\!\!\!\subset \cdots -\!\!\!\subset \sigma x$. (The uniqueness of x is a direct consequence of σ being a total map of event structures.) To show (i)$'$, suppose $y \stackrel{a}{-\!\!\!\subset} \sigma x$ with a +ve. Then $\sigma(s) = a$ for some unique $s \in x$ with s +ve. Supposing s were not $\le$-maximal in x, then $s \rightarrowtriangle s'$ for some $s' \in x$. By +-innocence $a = \sigma(s) \rightarrowtriangle \sigma(s') \in \sigma x$ implying a is not $\le$-maximal in σx. This contradicts $y \stackrel{a}{-\!\!\!\subset} \sigma x$. Hence s is $\le$-maximal and $x' =_{\text{def}} x \setminus \{s\} \in \mathcal{C}(S)$ with $x' -\!\!\!\subset x$ and $\sigma x' = y$.
(ii) Assuming $\sigma x \subseteq^- y$ we can form a covering chain

$$\sigma x \stackrel{a_1}{-\!\!\!\subset} y_1 \cdots \stackrel{a_n}{-\!\!\!\subset} y_n = y\,.$$

By repeated use of receptivity we obtain the existence of x' where $x \subseteq x'$ and $\sigma x' = y$. To show the uniqueness of x' suppose $x \subseteq z, z'$ and $\sigma z = \sigma z' = y$. Suppose that $z \neq z'$. Then, without loss of generality, suppose there is a $\le_S$-minimal $s' \in z'$ with $s' \notin z$. Then $[s') \subseteq z$, with s of −ve polarity. Now $\sigma(s') \in y$ so there is $s \in z$ for which $\sigma(s) = \sigma(s')$. We have $[s), [s') \subseteq z$ so $[s) \uparrow [s')$. We show $[s) = [s')$. Suppose $s_1 \rightarrowtriangle s$. Then by −-innocence, $\sigma(s_1) \rightarrowtriangle \sigma(s)$. As $\sigma(s') = \sigma(s)$ and σ is a map of event structures there is $s_2 < s'$ such that $\sigma(s_2) = \sigma(s_1)$. But s_1, s_2 both belong to the configuration $[s) \cup [s')$ so $s_1 = s_2$, as σ is a map, and $s_1 < s'$. Symmetrically, if $s_1 \rightarrowtriangle s'$ then $s_1 < s$. It follows that $[s) = [s')$. Now both $[s) \stackrel{s}{-\!\!\!\subset}$ and $[s) \stackrel{s'}{-\!\!\!\subset}$ with $\sigma(s) = \sigma(s')$ where both s, s' have −ve polarity. As σ is

receptive, $s = s'$. This implies $s' \in z$, a contradiction. Hence, $z = z'$ and we have established uniqueness of x'.
"If": Assume σ satisfies (i) and (ii). Clearly σ is receptive by (ii). We establish innocence via an observation, that in any event structure E,

$$(\exists x, x_1 \in \mathcal{C}(E).\ x \overset{s}{\longrightarrow\!\!\subset} x_1 \overset{s'}{\longrightarrow\!\!\subset}) \iff s \rightarrowtriangle s' \text{ or } s\ co\ s' .$$

Suppose $s \rightarrowtriangle_S s'$ and $pol(s) = +$. Then $x \overset{s}{\longrightarrow\!\!\subset} x_1 \overset{s'}{\longrightarrow\!\!\subset} x'$ for some $x, x_1, x' \in \mathcal{C}(S)$. Hence $\sigma x \overset{s}{\longrightarrow\!\!\subset} \sigma x_1 \overset{s'}{\longrightarrow\!\!\subset} \sigma x'$. Either, as required, $\sigma(s) \rightarrowtriangle_S \sigma(s')$ or $\sigma(s)\, co\, \sigma(s')$. Assume the latter. Then $\sigma x \overset{\sigma(s')}{\longrightarrow\!\!\subset} y_2 \overset{\sigma(s)}{\longrightarrow\!\!\subset} \sigma x'$ where $y_2 = x \cup \{\sigma(s')\}$, with $pol(\sigma(s)) = +$. From (i) we obtain a unique $x_2 \in \mathcal{C}(S)$ such that $x_2 \subseteq x'$ and $\sigma x_2 = y_2$. As σ is a total map of event structures, we obtain $x_2 \overset{s}{\longrightarrow\!\!\subset} x'$ and subsequently $x \overset{s'}{\longrightarrow\!\!\subset} x_2$, contradicting $s \rightarrowtriangle_S s'$.

Suppose $s \rightarrowtriangle_S s'$ and $pol(s') = -$. The case where $pol(s) = +$ is covered by the previous argument. Suppose $pol(s) = -$. Then $x \overset{s}{\longrightarrow\!\!\subset} x_1 \overset{s'}{\longrightarrow\!\!\subset} x'$ for some $x, x_1, x' \in \mathcal{C}(S)$. Again, $\sigma x \overset{s}{\longrightarrow\!\!\subset} \sigma x_1 \overset{s'}{\longrightarrow\!\!\subset} \sigma x'$. Assume, to obtain a contradiction, that $\sigma(s)\, co\, \sigma(s')$. Then $\sigma x \overset{\sigma(s')}{\longrightarrow\!\!\subset} y_2 \overset{\sigma(s)}{\longrightarrow\!\!\subset} \sigma x'$, where $y_2 = x \cup \{\sigma(s')\}$. As σ is already known to be receptive, we obtain

$$x \overset{e'}{\longrightarrow\!\!\subset} x_2 \overset{e}{\longrightarrow\!\!\subset} x'' \ \&\ \sigma x_2 = y_2 \ \&\ \sigma x'' = \sigma x' .$$

From the uniqueness part of (ii) we deduce $x'' = x'$. As σ is a total map of event structures, $e = s$ and $e' = s'$. Thus $x \overset{s'}{\longrightarrow\!\!\subset}$, which contradicts $s \rightarrowtriangle_S s'$. Via the observation we conclude that $\sigma(s) \rightarrowtriangle_S \sigma(s')$. □

Lemma 3. On matching pairs, define

$$(x, y) \rightsquigarrow_1 (x', y') \ \text{ iff } \ \exists s \in S, t \in T.\ x \overset{s}{\longrightarrow\!\!\subset} x' \ \&\ y \overset{t}{\longrightarrow\!\!\subset} y' \ \&\ \sigma_2(s) = \overline{\tau_1(t)} .$$

The smallest equivalence relation including $\rightsquigarrow_1$ coincides with the relation $\sim$.
Proof. From their definitions, $\rightsquigarrow_1$ is included in $\sim$. To prove the converse, it suffices to show that matching pairs (x, y), (x', y') satisfying

$$x \sqsubseteq_S x' \ \&\ y' \sqsubseteq_T y \ \&\ \sigma_1 x = \sigma_1 x' \ \&\ \tau_2 y' = \tau_2 y ,$$

—the clause used in the definition $\sim$ —are in the equivalence relation generated by $\rightsquigarrow_1$. Take a covering chain

$$x {-\!\!\sqsubset}_S x_1 {-\!\!\sqsubset}_S \cdots x_m {-\!\!\sqsubset}_S x'$$

in $(\mathcal{C}(S), \sqsubseteq_S)$. Here ${-\!\!\sqsubset}_S$ is the covering relation w.r.t. the order $\sqsubseteq_s$, so $x {-\!\!\sqsubset}_S x_1$ means x, x_1 are distinct and $x \sqsubseteq_S x_1$ with nothing strictly in between. Via the map σ we obtain

$$\sigma_2 x {-\!\!\sqsubset}_B \sigma_2 x_1 {-\!\!\sqsubset}_B \cdots \sigma_2 x_m {-\!\!\sqsubset}_B \sigma_2 x'$$

in $\mathcal{C}(B)$ where $\sigma_2 x = \overline{\tau_1 y}$ and $\sigma_2 x' = \overline{\tau_1 y'}$. Via the discrete fibration τ" we obtain a covering chain in the reverse direction,

$$y \sqsupset\!\!-_T y_1 \sqsupset\!\!-_T \cdots y_m \sqsupset\!\!-_T y'$$

in $(\mathcal{C}(T), \sqsubseteq_T)$, where each each (x_i, y_i), for $1 \le i \le m$, is a matching pair. Moreover, $(x_i, y_i) \rightsquigarrow_1 (x_{i+1}, y_{i+1})$ or $(x_{i+1}, y_{i+1}) \rightsquigarrow_1 (x_i, y_i)$ at each i with $1 \le i \le m$. Hence (x, y) and (x', y') are in the equivalence relation generated by $\rightsquigarrow_1$. □

Proposition 6. A matching pair (x, y) is reachable iff there is a sequence of matching pairs $(\varnothing, \varnothing) = (x_0, y_0), \cdots, (x_i, y_i), (x_{i+1}, y_{i+1}), \cdots, (x_n, y_n) = (x, y)$ such that for all i, either $(x_i, y_i) \rightsquigarrow_1 (x_{i+1}, y_{i+1})$ or $\exists s \in S.\ x_i \xrightarrow{s}\!\!\subset x_{i+1}$ & $y_i = y_{i+1}$ & $\sigma_1(s)$ is defined or $\exists t \in T.\ y_i \xrightarrow{t}\!\!\subset y_{i+1}$ & $x_i = x_{i+1}$ & $\tau_2(t)$ is defined. (The relation $\rightsquigarrow_1$ is that introduced in Lemma 3.)

Proof. "Only if": Assuming (x, y) is reachable, $x \cdot y \in \mathcal{C}(T) \odot \mathcal{C}(S)$, so there is a covering chain

$$\varnothing = z_0 \xrightarrow{e_1}\!\!\subset \cdots z_i \xrightarrow{e_{i+1}}\!\!\subset z_{i+1} \cdots \xrightarrow{e_n}\!\!\subset z_n = x \cdot y$$

in $\mathcal{C}(T) \odot \mathcal{C}(S)$. Each $(x_i, y_i) =_{\text{def}} (\pi_1 z_i, \pi_2 z_i)$ is a matching pair, from the definition of $\mathcal{C}(T) \odot \mathcal{C}(S)$. Moreover e_{i+1} has one of the forms (s, t), $(s, *)$ or $(*, t)$ for events $s \in S, t \in T$, which accord with the three cases in the proposition. *"If":* By induction along a sequence of matching pairs described above, each $x_i \cdot y_i \in \mathcal{C}(T) \odot \mathcal{C}(S)$ from the definition of such configurations in Section 5.1. □