# The True Concurrency of Herbrand's Theorem

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**Abstract**

Herbrand's theorem, widely regarded as a cornerstone of proof theory, exposes some of the constructive content of classical logic. In its simplest form, it reduces the validity of a first-order purely existential formula to that of a finite disjunction. In the general case, it reduces first-order validity to propositional validity, by understanding the structure of the assignment of first-order terms to existential quantifiers, and the causal dependency between quantifiers.

In this paper, we show that Herbrand's theorem in its general form can be elegantly stated and proved as a theorem in the framework of concurrent games, a denotational semantics designed to faithfully represent causality and independence in concurrent systems, thereby exposing the concurrency underlying the computational content of classical proofs. The causal structure of concurrent strategies, paired with annotations by first-order terms, is used to specify the dependency between quantifiers implicit in proofs. Furthermore concurrent strategies can be composed, yielding a compositional proof of Herbrand's theorem, simply by interpreting classical sequent proofs in a well-chosen denotational model.

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## 1 Introduction

> "What more do we know when we have proved a theorem
> by restricted means than if we merely know it is true?"

Kreisel's question is the driving force for much modern Proof Theory. This paper is concerned with Herbrand's Theorem, perhaps the earliest result in that direction. It is a simple consequence of completeness and compactness in first-order logic. So it is an example of information being extracted from the bare fact of provability. Usually by contrast one thinks in terms of extracting information from the proofs themselves, typically - as in Kohlenbach's proof mining - via some form of functional interpretation. This has the advantage that information is extracted compositionally in the spirit of functional programming. Specifically information for $\vdash A$ and $\vdash A \to B$ can be composed to give information for $\vdash B$; or, in terms of the sequent calculus, we can interpret the cut rule.

It seems to be folklore that there is a problem for Herbrand's Theorem. That is made precise in Kohlenbach [17] which shows that one cannot hope directly to use collections of

Herbrand terms for $\vdash A$ and $\vdash A \to B$ to give a collection for $\vdash B$. That leaves the possibility of making some richer data compositional, realised indirectly in Gerhardy and Kohlenbach [11] with data provided by Shoenfield's version [30] of Gödel's Dialectica Interpretation [14]. Now functional interpretations make no pretence to be faithful to the structure of proofs as encapsulated in systems like the sequent calculus: they explore in a sequential order terms proposed by a proof as witnesses for existential quantifiers, but this order is certainly not intrinsic to the proof. Thus it is compelling to seek some compositional form of Herbrand's Theorem faithful to the structure of proofs and to the dependency between terms; for cut-free proofs, Miller's *expansion trees* [24] capture precisely this "Herbrand content" (the information pertaining to quantifier instantiations), but not compositionally.

In this paper, we provide such a compositional form of Herbrand's theorem, presented as a game semantics for first-order classical logic. Our games have two players, both playing on the quantifiers of a formula $\varphi$. ∃loïse, playing the existential quantifiers, defends the validity of $\varphi$. ∀bélard, playing the universal quantifiers, attemps to falsify it. This understanding of formulas as games is folklore in mathematical logic and computer science. However, like functional interpretations, such games are usually sequential [7, 19]. In contrast, our model captures the exact dependence and independence between quantifiers. To achieve that we build on *concurrent/asynchronous* games [23, 27, 4], which marry game semantics with the so-called *true concurrency* approach to models of concurrent systems, and avoid interleavings. So in a formal sense, our model highlights a parallelism inherent to classical proofs. In essence, our strategies are close to expansion trees enriched with an explicit acyclicity witness.

The computational content of classical logic is a longstanding active topic, with a wealth of related works, and it is hard to give it justice in this short introduction. There are, roughly speaking, two families of approaches. On the one hand, some (including the functional interpretations mentioned above) extract from proofs a sequential procedure, *e.g.* via translation to sequential calculi or by annotating a proof to sequentialize or determinize its behaviour under cut reduction [13, 8]. Other than the cited above, influencial developments in this "polarized" approach include work by Berardi [2], Coquand [7], Parigot [26], Krivine [18], and others. Polarization yields better-behaved dynamics and a non-degenerate equational theory, but distorts the intent of the proof by an added unintended sequentiality. On the other hand, some works avoid polarization – including, of course, Gentzen's *Hauptsatz* [10]. This causes issues, notably unrestricted cut reduction yields a degenerate equational theory [13] and enjoys only *weak*, rather than *strong*, normalization [8]. Nevertheless, witness extraction remains possible (though it is non-deterministic). Particularly relevant to our endeavour is a recent activity around the matter of enriching expansion trees so as to support cuts. This includes Heijltjes' *proof forests* [15], McKinley's *Herbrand nets* [21], and Hetzl and Weller's recent *expansion trees with cuts* [16]. In all three cases, a generalization of expansion trees allowing cuts is given along with a weakly normalizing cut reduction procedure. Intuitions from games are often mentioned, but the methods used are syntactic and based on rewriting.

Other related works include Laurent's model for the first-order $\lambda\mu$-calculus [19], whose annotation of moves via first-order terms is similar to ours; and Mimram's categorical presentation of a games model for a linear first-order logic without propositional connectives [25].

Since our model avoids polarization, some phenomena from the proof theory of classical logic reflect in it: our semantics does not preserve cut reduction – if it did, it would be a boolean algebra [13]. Yet it preserves it in a sense for *first-order MLL* [12]. Likewise, just as classical proofs can lead to arbitrary large cut-free proofs [8], our semantics may yield *infinite* strategies, from which *finite* sub-strategies can nonetheless always be extracted. This reflects that non-polarized proof systems for classical logic are often only weakly normalizing.

In Section 2 we recall Herbrand's theorem, and introduce the game-theoretic language leading to our compositional reformulation of it. The rest of the paper describes the interpretation of proofs as winning strategies: in Section 3 we give the interpretation of propositional MLL, in Section 4 we deal with quantifiers, and finally, in Section 5, we add contraction and weakening and complete the interpretation.

Details of the constructions are given as an appendix.

## 2 From Herbrand to winning $\Sigma$-strategies

A **signature** is $\Sigma = (\Sigma_f, \Sigma_p)$, with $\Sigma_f$ a countable set of **function symbols** ($\mathsf{f}, \mathsf{g}, \mathsf{h}$, *etc.* range over function symbols), and $\Sigma_p$ a countable set of **predicate symbols** ($\mathsf{P}, \mathsf{Q}$, *etc.* range over predicate symbols). There is an **arity function** $\mathrm{ar} : \Sigma_f \uplus \Sigma_p \to \mathbb{N}$ where $\uplus$ is the usual set-theoretic union, where argument sets are disjoint. For a relative gain in simplicity in some arguments and examples, we assume that $\Sigma$ has at least one constant symbol, *i.e.* a function symbol of arity 0. We use $\mathsf{a}, \mathsf{b}, \mathsf{c}, \dots$ to range over constant symbols.

If $\mathcal{V}$ is a set of **variable names**, we write $\mathsf{Tm}_\Sigma(\mathcal{V})$ for the set of first-order terms on $\Sigma$ with free variables in $\mathcal{V}$. We use variables $t, s, u, v, \dots$ to range over terms. **Literals** have the form $\mathsf{P}(t_1, \dots, t_n)$ or $\neg\mathsf{P}(t_1, \dots, t_n)$, where $\mathsf{P}$ is a $n$-ary predicate symbol and the $t_i$s are terms. **Formulas** are also closed under quantifiers, and the connectives $\vee$ and $\wedge$. **Negation** is not considered a logical connective: the negation $\varphi^\perp$ of $\varphi$ is obtained by De Morgan rules. We write $\mathsf{Form}_\Sigma(\mathcal{V})$ for the set of **first-order formulas** on $\Sigma$ with free variables in $\mathcal{V}$, and use $\varphi, \psi, \dots$ to range over them. We also write $\mathsf{QF}_\Sigma(\mathcal{V})$ for the set of **quantifier-free** formulas. Finally, we write $\mathrm{fv}(\varphi)$ or $\mathrm{fv}(t)$ for the set of free variables in a formula $\varphi$ or a term $t$. Formulas are considered up to $\alpha$-conversion and satisfy Barendregt's convention.

### 2.1 Herbrand's theorem

Intuitionistic logic has the *witness* property: if $\exists\mathsf{x}\,\varphi$ holds intuitionistically, then there is some term $t$ such that $\varphi(t)$ holds. While this fails in classical logic, Herbrand's theorem, in its popular form, gives a weakened classical version, a *finite disjunction property.*

▶ **Theorem 1.** *Let $\mathscr{T}$ be a theory finitely axiomatized by universal formulas. Let $\psi = \exists\mathsf{x}_1 \dots \exists\mathsf{x}_n \varphi(\mathsf{x}_1, \dots, \mathsf{x}_n)$ be a purely existential formula ($\varphi \in \mathsf{QF}_\Sigma$). Then, $\mathscr{T} \models \psi$ iff there are closed terms $(t_{i,j})_{1\leq i\leq p, 1\leq j\leq n}$ such that $\mathscr{T} \models \bigvee_{i=1}^{p} \varphi(t_{i,1}, \dots, t_{i,n})$.*

▶ **Example 2.** Consider the formula $\psi = \exists\mathsf{x}\neg\mathsf{P}(\mathsf{x}) \vee \mathsf{P}(\mathsf{f}(\mathsf{x}))$ (where $\mathsf{f} \in \Sigma_f$). A valid Herbrand disjunction for $\psi$ is $(\neg\mathsf{P}(c) \vee \mathsf{P}(\mathsf{f}(\mathsf{c}))) \vee (\neg\mathsf{P}(\mathsf{f}(\mathsf{c})) \vee \mathsf{P}(\mathsf{f}(\mathsf{f}(\mathsf{c}))))$ where $\mathsf{c}$ is some constant symbol.

A similar disjunction property holds for general formulas, though it is harder to state. A common way to do so is by reduction to the above: a formula $\varphi$ is converted to prenex normal form and universally quantified variables are replaced with new function symbols added to $\Sigma$, in a process called *Herbrandization* (dual to Skolemization). For instance, the *drinker's formula* (DF): $\exists\mathsf{x}\forall\mathsf{y}\neg\mathsf{P}(\mathsf{x}) \vee \mathsf{P}(\mathsf{y})$, yields by Herbrandization the formula $\psi$ of Example 2.

Instead, to avoid prenexification and Skolemization and the corresponding distortion of the formula, one may adopt a representation of proofs that displays the instantiation of existential quantifiers with finitely many witnesses while staying structurally faithful to the original formula. To that end Miller proposes **expansion trees** [24]. They can be introduced via a game-theoretic metaphor, reminiscent of [7]. Two players, ∃loïse and ∀bélard, debate the validity of a formula. On a formula $\forall\mathsf{x}\varphi$, ∀bélard provides a fresh variable $\mathsf{x}$ and the game keeps going on $\varphi$. On $\exists\mathsf{x}\varphi$, ∃loïse provides a *term* $t$, possibly containing variables previously

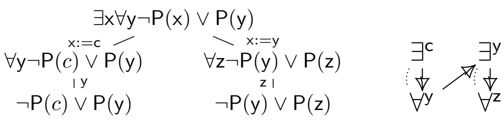



**Figure 1** An expansion tree and winning Σ-strategy for DF

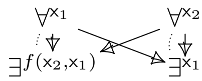



**Figure 2** A partially ordered winning Σ-strategy

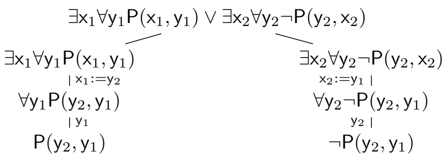



**Figure 3** An incorrect expansion tree

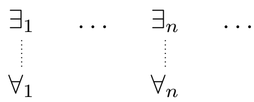



**Figure 4** The arena $[\![DF]\!]^\exists$

introduced by ∀bélard. ∃loïse, though, has a special power: at any time she can *backtrack* to a previous existential position, and propose a new term. Figure 1 (left) shows an expansion tree for DF. It may be read from top to bottom, and from left to right: ∃loïse plays $c$, then ∀bélard introduces y, then ∃loïse *backtracks* (we jump to the right branch) and plays y, and finally ∀bélard introduces z. ∃loïse wins: the disjunction of the leaves is a tautology.

However the metaphor has limits, it suggests a sequential ordering between branches, which expansion trees do not have in reality: the order is only implicit in the term annotations. Besides, the natural ordering between quantifiers induced by terms is not always sequential. It is, of course, always acyclic – on expansion trees this is ensured by an *acyclicity correctness criterion*, whose necessity is made obvious by the (incorrect) expansion tree of Figure 3 "proving" a falsehood. This acyclicity entails the existence of a sequentialization, but commiting to one is an arbitrary choice not forced by the proof.

A partial order is much more faithful to the proof. In this paper, we show that expansion trees can be made compositional modulo a change of perspective: rather than derived we consider this order primitive, and only later decorate it with term annotations. For instance, we display in Figure 2 the formal object, called a (sequential) *winning* Σ*-strategy*, matching in our framework the expansion tree for DF. Another winning Σ-strategy, displayed in Figure 2, illustrates that this order is not always naturally sequential. By lack of space we do not define expansion trees here, though they are captured in essence by our strategies.

## 2.2 Expansion trees as winning Σ-strategies

We now introduce our formulation of expansion trees as Σ*-strategies*. Although our definitions look superficially very different from Miller's, the only fundamental difference is the explicit display of the dependency between quantifiers. Σ-strategies will be certain partial orders, with elements either "∀ events" or "∃ events". Events will carry terms, in a way that respects causal dependency. Σ-strategies will play on *games* representing the formulas. The first component of a game is its *arena*, that specifies the causal ordering between quantifiers.

▶ **Definition 3.** An **arena** is $A = (|A|, \leq_A, \mathrm{pol}_A)$ where $|A|$ is a set of **events**, $\leq_A$ is a partial order that is *forest-shaped*:

*(1)* if $a_1 \leq_A a$ and $a_2 \leq_A a$, then either $a_1 \leq_A a_2$ or $a_2 \leq_A a_1$, and

*(2)* for all $a \in |A|$, the branch $[a]_A = \{a' \in A \mid a' \leq_A a\}$ is finite. Finally, $\mathrm{pol}_A : |A| \to \{\forall, \exists\}$ is a **polarity function** which expresses if a move belongs to ∃loïse or ∀bélard.

A **configuration** of an arena (or any partial order) is a down-closed set of events. We write $\mathscr{C}^\infty(A)$ for the set of configurations of $A$, and $\mathscr{C}(A)$ for the set of *finite* configurations.

The arena only describes the moves available to both players; it says nothing about terms or winning. Similarly to expansion trees where only ∃loïse can replicate her moves, our arenas will at first be biased towards ∃loïse: each ∃ move exists in as many copies as she might desire, whereas ∀ events are *a priori* not copied. Figure 4 shows the ∃-biased arena $[\![DF]\!]^\exists$ for DF. The order is drawn from top to bottom. Although only ∃loïse can replicate her moves, the universal quantifier is also copied as it depends on the existential quantifier.

*Strategies* on an arena $A$ will be certain *augmentations* of prefixes of $A$. They carry causal dependency between quantifiers induced by term annotations, but not the terms themselves.

For any partial order $A$ and $a_1, a_2 \in |A|$, we write $a_1 \rightarrowtriangle_A a_2$ (or $a_1 \rightarrowtriangle a_2$ if $A$ is clear from the context) if $a_1 <_A a_2$ with no other event in between – this notation was used implicitly in Figures 1 and 2. We call $\rightarrowtriangle$ **immediate causal dependency**.

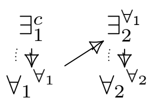
▶ **Definition 4.** A **strategy** $\sigma$ on arena $A$, written $\sigma : A$, is a partial order $(|\sigma|, \leq_\sigma)$ with $|\sigma| \subseteq |A|$, such that for all $a \in |\sigma|$, $[a]_\sigma$ is finite (an *elementary event structure*); subject to:
*(1) Arena-respecting.* We have $\mathscr{C}^\infty(\sigma) \subseteq \mathscr{C}^\infty(A)$,
*(2) Receptivity.* If $x \in \mathscr{C}(\sigma)$ *s.t.* $x \cup \{a^\forall\} \in \mathscr{C}(A)$, then $a \in |\sigma|$,
*(3) Courtesy.* If $a_1 \rightarrowtriangle_\sigma a_2$ and ($\mathrm{pol}(a_1) = \exists$ or $\mathrm{pol}(a_2) = \forall$), then $a_1 \rightarrowtriangle_A a_2$.

These strategies are essentially the *receptive ingenuous strategies* of Melliès and Mimram [23], though their formulation, with a direct handle on causality, is closer to Rideau and Winskel's later *concurrent strategies* [27]. Receptivity means that ∃loïse cannot refuse to acknowledge a move by ∀bélard, and courtesy that the only new causal constraints that she can enforce with respect to the game is that some existential quantifiers depend on some universal quantifiers. Ignoring terms, Figure 2 (right) displays a strategy on the arena of Figure 4 – in Figure 2 we also show via dotted lines the immediate dependency of the arena.

Let us now add terms, and define Σ-*strategies*.

▶ **Definition 5.** A **Σ-strategy** on arena $A$ is a strategy $\sigma : A$, with a **labelling function** $\lambda_\sigma : |\sigma| \to \mathsf{Tm}_\Sigma(|\sigma|)$, satisfying (with $[a]_\sigma^\forall = \{a' \in |\sigma| \mid a' \leq_\sigma a \ \&\ \mathrm{pol}_A(a') = \forall\}$):
*(1)* Σ*-receptivity:* $\forall a^\forall \in |\sigma|, \lambda_\sigma(a) = a$,
*(2)* Σ*-courtesy:* $\forall a^\exists \in |\sigma|, \lambda_\sigma(a) \in \mathsf{Tm}_\Sigma([a]_\sigma^\forall)$.

Rather than having ∀ moves introduce fresh variables, we consider them as *variables* themselves. Hence, the ∃ moves carry terms having as free variables the ∀ moves in their causal history. For instance the diagram of Figure 1 (right) is meant formally to denote the one on the right (where superscripts are the terms given by $\lambda$). In the sequel we omit the (redundant) annotation of ∀bélard's events.



Besides the fact that they are not assumed finite, Σ-strategies are more general than expansion trees: they have an explicit causal ordering, which may be more constraining than that given by the terms. A Σ-strategy $\sigma : A$ is **minimal** iff whenever $a_1 \rightarrowtriangle_\sigma a_2$ such that $a_1 \notin \mathrm{fv}(\lambda_\sigma(a_2))$, then $a_1 \rightarrowtriangle_A a_2$ as well. In a minimal Σ-strategy $\sigma : A$, the ordering $\leq_\sigma$ is actually redundant and can be uniquely recovered from $\lambda_\sigma$ and $\leq_A$.

Now, we adjoin *winning conditions* to arenas and define *winning* Σ-*strategies*. As in expansion trees, we aim to capture that the substitution (by terms from the strategies) of the expansion of the original formula is a tautology.

▶ **Definition 6.** A **game** $\mathcal{A}$ is an arena $A$, with $\mathcal{W}_\mathcal{A} : (x \in \mathscr{C}^\infty(A)) \to \mathsf{QF}^\infty_\Sigma(x)$ expressing **winning conditions**, where $\mathsf{QF}^\infty_\Sigma(x)$ denotes the **infinitary quantifier-free formulas** – obtained from $\mathsf{QF}_\Sigma(x)$ by adding infinitary connectives $\bigvee_{i\in I} \varphi_i$ and $\bigwedge_{i\in I} \varphi_i$, with $I$ countable.

For a game interpreting a formula $\varphi$, the winning conditions associate configurations of the arena $[\![\varphi]\!]$ with the propositional part of the corresponding *expansion* of $\varphi$. For instance:

$$\begin{array}{rcl}\mathcal{W}_{[\![DF]\!]^\exists}(\{\exists_3, \forall_3, \exists_6, \forall_6\}) & = & (\neg \mathsf{P}(\exists_3) \vee \mathsf{P}(\forall_3)) \vee (\neg \mathsf{P}(\exists_6) \vee \mathsf{P}(\forall_6)) \\ \mathcal{W}_{[\![DF]\!]^\exists}(\{\exists_3, \forall_3, \exists_6\}) & = & (\neg \mathsf{P}(\exists_3) \vee \mathsf{P}(\forall_3)) \vee \top\end{array}$$

recalling that the arena for DF appears in Figure 4. In the second clause, $\top$ (the true formula) comes from ∀bélard not having played $\forall_6$ yet, yielding victory to ∃loïse on that copy. The winning conditions yield syntactic, uninterpreted formulas: we keep the second formula as-is although it is equivalent to $\top$. Finally, we can define *winning strategies.*

▶ **Definition 7.** If $\sigma : A$ is a $\Sigma$-strategy and $x \in \mathscr{C}^\infty(\sigma)$, we say that $x$ is **tautological** in $\sigma$ if the formula $\mathcal{W}_\mathcal{A}(x)[\lambda_\sigma]$ corresponding to the substitution of $\mathcal{W}_\mathcal{A}(x) \in \mathsf{QF}^\infty_\Sigma(x)$ by $\lambda_\sigma : x \to \mathsf{Tm}_\Sigma(x)$, is a (possibly infinite) tautology.

Then, a $\Sigma$-strategy $\sigma : A$ is **winning** if for any $x \in \mathscr{C}^\infty(\sigma)$ that is **∃-maximal** (*i.e.* such taht for all $a \in |\sigma|$ with $x \cup \{a\} \in \mathscr{C}^\infty(\sigma)$, $\mathrm{pol}_A(a) = \forall$), x is tautological.

Finally, a $\Sigma$-strategy $\sigma : A$ is **top-winning** if $|\sigma| \in \mathscr{C}^\infty(\sigma)$ is tautological.

## 2.3 Constructions on games and Herbrand's theorem

To complete our statement of Herbrand's theorem with $\Sigma$-strategies, it remains to set the interpretation of formulas as games. To that end we introduce a few constructions on games, first at the level of arenas and then enriched with winning conditions. We write $\emptyset$ for the **empty arena**. If $A$ is an arena, $A^\perp$ is its **dual**, with same events and causality but polarity reversed. We review some other constructions.

▶ **Definition 8.** The **simple parallel composition** $A_1 \parallel A_2$ of $A_1$ and $A_2$ has as events the tagged disjoint union $\{1\} \times |A_1| \uplus \{2\} \times |A_2|$, as causal order that given by $(i, a) \leq_{A_1\parallel A_2} (j, a')$ iff $i = j$ and $a \leq_{A_i} a'$, and, as polarity $\mathrm{pol}_{A_1\parallel A_2}((i, a)) = \mathrm{pol}_{A_i}(a)$.

Configurations $x \in \mathscr{C}^\infty(A \parallel B)$ have the form $\{1\} \times x_A \cup \{2\} \times x_B$ with $x_A \in \mathscr{C}^\infty(A)$ and $x_B \in \mathscr{C}^\infty(B)$, which we write $x = x_A \parallel x_B$. This construction has a general counterpart $\parallel_{i\in I} A_i$ with $I$ at most countable, defined likewise. In particular we will later use the uniform countably infinite parallel composition $\parallel_\omega A$. Another important construction is *prefixing.*

▶ **Definition 9.** For $\alpha \in \{\forall, \exists\}$ and $A$ an arena, $\alpha.A$ has events $\{(1, \alpha)\} \cup \{2\} \times |A|$ and causality $(i, a) \leq (j, a')$ iff $i = j = 2$ and $a \leq_A a'$, or $(i, a) = (1, \alpha)$; *i.e.* $(1, \alpha)$ is the unique minimal event. Its polarity is $\mathrm{pol}_{\alpha.A}((1, \alpha)) = \alpha$ and $\mathrm{pol}_{\alpha.A}((2, a)) = \mathrm{pol}_A(a)$.

*Configurations* $x \in \mathscr{C}^\infty(\alpha.A)$ are $\emptyset$, or $\{(1, \alpha)\} \cup \{2\} \times x_A$ ($x_A \in \mathscr{C}^\infty(A)$), written $\alpha.x_A$.

Now, let us enrich these with winning, yielding the constructions on games used for interpreting formulas. Importantly, the inductive interpretation of formulas requires us to consider formulas with free variables. For $\mathcal{V}$ a finite set, a **$\mathcal{V}$-game** is defined as a game $\mathcal{A}$ (Def. 6), except that winning may also depend on $\mathcal{V}$: for $x \in \mathscr{C}^\infty(A)$, $\mathcal{W}_\mathcal{A}(x) \in \mathsf{QF}^\infty_{\Sigma\uplus\mathcal{V}}(x)$.

We now define all our constructions, on $\mathcal{V}$-games rather than games. The duality $(-)^\perp$ extends to $\mathcal{V}$-games, simply by negating the winning conditions: for all $x \in \mathscr{C}^\infty(A)$, $\mathcal{W}_{\mathcal{A}^\perp}(x) = \mathcal{W}_\mathcal{A}(x)^\perp$. The $\parallel$ of arenas gives rise to *two* constructions, $\otimes$ and $⅋$, on $\mathcal{V}$-games:

$$\llbracket \top \rrbracket^{\exists}_{\mathcal{V}} = 1 \qquad \llbracket \mathsf{P}(t_1,\ldots,t_n)\rrbracket^{\exists}_{\mathcal{V}} = \mathsf{P}(t_1,\ldots,t_n) \qquad \llbracket \exists \mathsf{x}\varphi\rrbracket^{\exists}_{\mathcal{V}} = ?\exists \mathsf{x}.\llbracket\varphi\rrbracket_{\mathcal{V}\uplus\{\mathsf{x}\}} \qquad \llbracket \varphi_1 \vee \varphi_2\rrbracket^{\exists}_{\mathcal{V}} = \llbracket\varphi_1\rrbracket^{\exists}_{\mathcal{V}} \parr \llbracket\varphi_2\rrbracket^{\exists}_{\mathcal{V}}$$
$$\llbracket \bot \rrbracket^{\exists}_{\mathcal{V}} = \bot \qquad \llbracket \neg\mathsf{P}(t_1,\ldots,t_n)\rrbracket^{\exists}_{\mathcal{V}} = \neg\mathsf{P}(t_1,\ldots,t_n) \qquad \llbracket \forall \mathsf{x}\varphi\rrbracket^{\exists}_{\mathcal{V}} = \forall \mathsf{x}.\llbracket\varphi\rrbracket_{\mathcal{V}\uplus\{\mathsf{x}\}} \qquad \llbracket \varphi_1 \wedge \varphi_2\rrbracket^{\exists}_{\mathcal{V}} = \llbracket\varphi_1\rrbracket^{\exists}_{\mathcal{V}} \otimes \llbracket\varphi_2\rrbracket^{\exists}_{\mathcal{V}}$$

**Figure 5** $\exists$-biased interpretation of formulas

▶ **Definition 10.** For $\mathcal{A}$ and $\mathcal{B}$ $\mathcal{V}$-games, we define two $\mathcal{V}$-games with arena $A \parallel B$ and winning conditions $\mathcal{W}_{\mathcal{A}\otimes\mathcal{B}}(x_A \parallel x_B) = \mathcal{W}_{\mathcal{A}}(x_A) \wedge \mathcal{W}_{\mathcal{B}}(x_B)$ and $\mathcal{W}_{\mathcal{A}\parr\mathcal{B}}(x_A \parallel x_B) = \mathcal{W}_{\mathcal{A}}(x_A) \vee \mathcal{W}_{\mathcal{B}}(x_B)$.

Note the implicit renaming so that $\mathcal{W}_{\mathcal{A}}(x_A), \mathcal{W}_{\mathcal{B}}(x_B)$ are in $\mathsf{QF}^{\infty}_{\Sigma\uplus\mathcal{V}}(x_A \parallel x_B)$ rather than $\mathsf{QF}^{\infty}_{\Sigma\uplus\mathcal{V}}(x_A), \mathsf{QF}^{\infty}_{\Sigma\uplus\mathcal{V}}(x_B)$ respectively – we will often keep such renamings implicit.

Observe that $\otimes$ and $\parr$ are De Morgan duals, *i.e.* $(\mathcal{A}\otimes\mathcal{B})^{\perp} = \mathcal{A}^{\perp} \parr \mathcal{B}^{\perp}$. We write these operations $\otimes$ and $\parr$ rather than $\wedge$ and $\vee$, because they behave more like the connectives of linear logic [12] than those of classical logic; for each $\mathcal{V}$ the $\otimes$ and $\parr$ will form the basis of a $*$-autonomous structure and hence a model of multiplicative linear logic (see Section 3).

To interpret classical logic however, we will need *replication.*

▶ **Definition 11.** For $\mathcal{V}$-game $\mathcal{A}$, we define the $\mathcal{V}$-games $!\mathcal{A}, ?\mathcal{A}$ with arena $\parallel_{\omega} A$ and winning:

$$\mathcal{W}_{!\mathcal{A}}(\parallel_{i\in\omega} x_i) = \bigwedge_{i\in\omega} \mathcal{W}_{\mathcal{A}}(x_i) \qquad \mathcal{W}_{?\mathcal{A}}(\parallel_{i\in\omega} x_i) = \bigvee_{i\in\omega} \mathcal{W}_{\mathcal{A}}(x_i)$$

Though $\mathcal{W}_{!\mathcal{A}}(x)$ (resp. $\mathcal{W}_{?\mathcal{A}}(x)$) is an infinite conjunction (resp. disjunction), it simplifies to a finite one when $x$ visits finitely many copies (with cofinitely many copies of $\mathcal{W}_{\mathcal{A}}(\emptyset)$).

Next we show how $\mathcal{V}$-games support quantifiers.

▶ **Definition 12.** Let $\mathcal{A}$ a $(\mathcal{V} \uplus \{\mathsf{x}\})$-game, we define the $\mathcal{V}$-game $\forall \mathsf{x}.\mathcal{A}$ and its dual $\exists \mathsf{x}.\mathcal{A}$ with arenas $\forall.A$ and $\exists.A$ respectively, with $\mathcal{W}_{\forall\mathsf{x}.\mathcal{A}}(\emptyset) = \top$, $\mathcal{W}_{\exists\mathsf{x}.\mathcal{A}}(\emptyset) = \bot$, and:

$$\mathcal{W}_{\forall\mathsf{x}.\mathcal{A}}(\forall.x_A) = \mathcal{W}_{\mathcal{A}}(x_A)[\forall/\mathsf{x}] \qquad \mathcal{W}_{\exists\mathsf{x}.\mathcal{A}}(\exists.x_A) = \mathcal{W}_{\mathcal{A}}(x_A)[\exists/\mathsf{x}]$$

Finally, we regard a literal $\varphi$ as a $\mathcal{V}$-game on arena $\emptyset$, with $\mathcal{W}_{\varphi}(\emptyset) = \varphi$. We write 1 and $\bot$ for the unit $\mathcal{V}$-games on arena $\emptyset$ with winning conditions respectively $\top$ and $\bot$.

Putting these together, we give in Figure 5 the $\exists$-biased interpretation of a formula $\varphi \in \mathsf{Form}_{\Sigma}(\mathcal{V})$ as a $\mathcal{V}$-game. Note the difference between the case of existential and universal formulas, reflecting the bias towards ∃loïse. This is indeed compatible with the examples given previously. We can now state our concurrent version of Herbrand's theorem.

▶ **Theorem 13.** *For any $\varphi \in \mathsf{Form}_{\Sigma}$, $\models \varphi$ iff there exists a finite, top-winning $\sigma : \llbracket\varphi\rrbracket^{\exists}$.*

Besides the game-theoretic language, the difference with expansion trees is superficial: on $\varphi$, expansion trees essentially coincide with the *minimal* top-winning $\Sigma$-strategies $\sigma : \llbracket\varphi\rrbracket^{\exists}$. The effort to change view point, from a syntactic construction to a (game) semantic one, will however pay off now, when we show how to *compose* $\Sigma$-strategies.

## 2.4 Compositional Herbrand's theorem

Unlike expansion trees, strategies can be *composed.* Whereas Theorem 13 above could be deduced via the connection with expansion trees, that proof would intrinsically rely on the admissibility of cut in the sequent calculus. Instead, we will give an alternative proof of Herbrand's theorem where the witnesses are obtained truly *compositionally* from any sequent proof, without first eliminating cuts. In other words, strategies will come naturally from the interpretation of the classical sequent calculus in a semantic model.

**$\mathcal{V}$-MLL**

$$\textsc{Ax}\ \frac{}{\vdash^{\mathcal{V}} \varphi^{\perp}, \varphi}\ \mathrm{fv}(\varphi) \subseteq \mathcal{V} \qquad \textsc{Cut}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \varphi^{\perp}, \Delta}{\vdash^{\mathcal{V}} \Gamma, \Delta} \qquad \textsc{Ex}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi, \Delta}{\vdash^{\mathcal{V}} \Gamma, \psi, \varphi, \Delta}$$

$$\top\mathrm{I}\ \frac{}{\vdash^{\mathcal{V}} \top} \qquad \bot\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V}} \Gamma, \bot} \qquad \wedge\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \psi, \Delta}{\vdash^{\mathcal{V}} \Gamma, \varphi \wedge \psi, \Delta} \qquad \vee\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi, \Delta}{\vdash^{\mathcal{V}} \Gamma, \varphi \vee \psi, \Delta}$$

**First-order MLL (MLL$_1$)**

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V} \uplus \{\mathsf{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathsf{x}.\, \varphi}\ \mathsf{x} \notin \mathrm{fv}(\Gamma) \qquad \exists\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi[t/\mathsf{x}]}{\vdash^{\mathcal{V}} \Gamma, \exists \mathsf{x}.\, \varphi}\ t \in \mathsf{Tm}_{\Sigma}(\mathcal{V})$$

**LK**

$$\mathrm{C}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \varphi}{\vdash^{\mathcal{V}} \Gamma, \varphi} \qquad \mathrm{W}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V}} \Gamma, \varphi}$$

**Figure 6** Rules for the sequent calculus LK

To compose $\Sigma$-strategies, we must restore the symmetry between $\exists$loïse and $\forall$bélard in the interpretation of formulas. The *non-biased* interpretation $[\![\varphi]\!]_{\mathcal{V}}$ of $\varphi \in \mathsf{Form}_{\Sigma}(\mathcal{V})$ is defined as for $[\![\varphi]\!]^{\exists}_{\mathcal{V}}$, except for $[\![\forall \mathsf{x} \varphi]\!]_{\mathcal{V}} = {!\forall \mathsf{x}}.[\![\varphi]\!]_{\mathcal{V} \uplus \{\mathsf{x}\}}$. Thus we lose finiteness: $\exists$loïse must be reactive to the infinite number of copies potentially opened by $\forall$bélard. But we can now state:

▶ **Theorem 14.** *For $\varphi$ closed, the following are equivalent:* (1) $\models \varphi$, (2) *there exists a finite, top-winning $\Sigma$-strategy $\sigma : [\![\varphi]\!]^{\exists}$,* (3) *there exists a* winning *$\Sigma$-strategy $\sigma : [\![\varphi]\!]$.*

**Proof.** That *(2)* implies *(1)* is easy, as a finite top-winning $\sigma : [\![\varphi]\!]^{\exists}$ directly informs a proof.

That *(3)* implies *(2)* is more subtle: first, one may restrict a winning $\sigma : [\![\varphi]\!]$ to $[\![\varphi]\!]^{\exists}$ to obtain a finite top-winning strategy. However, this top-winning strategy *may not be finite*. Yet, it follows by compactness that there is always a finite top-winning sub-strategy that may be effectively computed from $\sigma$. See the Appendix E for details.

The proof that *(1)* implies *(3)* is our main contribution: a winning strategy will be computed from a proof using our denotational model of classical proofs. ◀

Our source sequent calculus (Figure 6) is fairly standard, one-sided, with rules presented in the multiplicative style. A notable variation is that sequents carry a set $\mathcal{V}$ of free variables, that may appear freely in formulas. The introduction rule for $\forall$ introduces a fresh variable, whereas the introduction rule for $\exists$ provides a term whose free variables must be in $\mathcal{V}$.

What mathematical structure is required to interpret this sequent calculus? Ignoring the $\mathcal{V}$ annotations, the first group is nothing but Multiplicative Linear Logic (MLL). Propositional ($\mathcal{V}$-)MLL can be interpreted in a $*$-autonomous category [3]. Accordingly, in Section 3, we first construct a $*$-autonomous category Ga of games and winning $\Sigma$-strategies. Then, in Section 4, we build the structure required for the interpretation of quantifiers, still ignoring contraction and weakening. For each set of variables $\mathcal{V}$ we construct a $*$-autonomous category $\mathcal{V}$-Ga, with a fibred structure to link the $\mathcal{V}$-Ga together for distinct $\mathcal{V}$s and suitable structure to deal with quantifiers, obtaining a model of first-order MLL. Finally in Section 5 we complete the interpretation by adding the exponential modalities from linear logic to the interpretation of quantifiers, and get from that an interpretation of contraction and weakening.

## 3 A $*$-autonomous category

The following theorem, on cut reduction for MLL, is folklore.

▶ **Theorem 15.** *There is a set of reduction rules on MLL sequent proofs, written $\rightsquigarrow_{\mathrm{MLL}}$, such that for any proof $\pi$ of a sequent $\vdash \Gamma$, there is a cut-free $\pi'$ of $\Gamma$ such that $\pi \rightsquigarrow^*_{\mathrm{MLL}} \pi'$.*

The reduction $\rightsquigarrow_{\mathrm{MLL}}$ comprises *logical* reductions, reducing a cut on a formula $\varphi/\varphi^\perp$, between two proofs starting with the introduction rule for the main connective of $\varphi/\varphi^\perp$; and *structural* reductions, consisting in commutations between rules so as to reach the logical steps. We assume some familiarity with this process.

In this section we aim to give an interpretation of MLL proofs, which should be invariant under cut-elimination. Categorical logic tells us that this is essentially the same as producing a $*$-*autonomous category*. We opt here for the equivalent formulation by Cockett and Seely as a *symmetric linearly distributive category with negation* [6].

▶ **Definition 16.** A **symmetric linearly distributive category** is a category $\mathcal{C}$ with two symmetric monoidal structures $(\otimes, 1)$ and $(⅋, \perp)$ which *distribute*: there is a natural $\delta_{A,B,C} : A \otimes (B ⅋ C) \stackrel{\mathcal{C}}{\to} (A \otimes B) ⅋ C$, the *linear distribution*, subject to coherence conditions [6].

A symmetric linearly distributive category **with negation** also has a function $(-)^\perp$ on objects and families of maps $\eta_A : 1 \stackrel{\mathcal{C}}{\to} A^\perp ⅋ A$ and $\epsilon_A : A \otimes A^\perp \stackrel{\mathcal{C}}{\to} \perp$ such that the canonical composition $A \to A \otimes (A^\perp ⅋ A) \to (A \otimes A^\perp) ⅋ A \to A$, and its dual $A^\perp \to A^\perp$, are identities.

Note also the degenerate case of a **compact closed category**, which is a symmetric linearly distributive category where the monoidal structures $(\otimes, 1)$ and $(⅋, \perp)$ coincide.

Abusing terminology, we will refer to *symmetric linearly distributive categories with negation* by the shorter $*$-*autonomous categories*. This should not create any confusion in the light of their equivalence [6]. If $\mathcal{C}$ a $*$-autonomous category comes with a choice of $[\![\mathsf{P}(t_1, \dots, t_n)]\!]$ (an object of $\mathcal{C}$) for all closed literal, then this interpretation can be extended to all closed quantifier-free formulas following Figure 5. For all such $\varphi$, we have $[\![\varphi^\perp]\!] = [\![\varphi]\!]^\perp$.

The interpretation of MLL proofs in a $*$-autonomous category $\mathcal{C}$ is standard [29]: a proof $\pi$ of a *MLL sequent* $\vdash \varphi_1, \dots, \varphi_n$ is interpreted as a morphism $[\![\pi]\!] : 1 \stackrel{\mathcal{C}}{\to} [\![\varphi_1]\!] ⅋ \cdots ⅋ [\![\varphi_n]\!]$. This interpretation is sound *w.r.t.* provability: if $\varphi$ is provable, then $1 \to_{\mathcal{C}} [\![\varphi]\!]$ is inhabited. Furthermore, the categorical laws make this interpretation invariant under cut reduction.

▶ **Theorem 17.** *If $\pi \rightsquigarrow_{\mathrm{MLL}} \pi'$ are proofs of $\vdash \Gamma$, $[\![\pi]\!] = [\![\pi']\!]$.*

So a proof has the same denotation as its cut-free form obtained by Theorem 15. In the rest of this section we construct a concrete $*$-autonomous category of games and winning $\Sigma$-strategies; supporting the interpretation of MLL. This is done in three stages: first we focus on composition of $\Sigma$-strategies (without winning), then we extend this to a compact closed category. Finally, adding back winning, we split $\parallel$ into two $\otimes$ and $⅋$, and prove $*$-autonomy.

## 3.1 Composition of $\Sigma$-strategies

We construct a category $\mathrm{Ar}_\Sigma$ having arenas as objects, and as morphisms from $A$ to $B$ the $\Sigma$-strategies $\sigma : A^\perp \parallel B$, also written $\sigma : A \stackrel{\mathrm{Ar}_\Sigma}{\to\!\!\!\!\!+} B$. The composition of $\sigma : A \stackrel{\mathrm{Ar}_\Sigma}{\to\!\!\!\!\!+} B$ and $\tau : B \stackrel{\mathrm{Ar}_\Sigma}{\to\!\!\!\!\!+} C$ will be computed in two stages: first, the *interaction* $\tau \circledast \sigma$ is obtained as the most general partial-order-with-terms satisfying the constraints given by both $\sigma$ and $\tau$ – Figure 7 displays such an interaction. Then, we will obtain the *composition* $\tau \odot \sigma$ by hiding events in $B$. In the example of Figure 7 we get the single annotated event $\exists_5^{\mathsf{f}(\mathsf{g}(\mathsf{c}),\mathsf{h}(\mathsf{c}))}$.

We fix some definitions on terms and substitutions. If $\mathcal{V}_1, \mathcal{V}_2$ are sets, a **substitution** $\gamma : \mathcal{V}_1 \stackrel{\mathcal{S}}{\to} \mathcal{V}_2$ is a function $\gamma : \mathcal{V}_2 \to \mathsf{Tm}_\Sigma(\mathcal{V}_1)$. For $t \in \mathsf{Tm}_\Sigma(\mathcal{V}_2)$, we write $t[\gamma] \in \mathsf{Tm}_\Sigma(\mathcal{V}_1)$ for the substitution operation. Substitutions form a category $\mathcal{S}$, which is *cartesian*: the empty set $\emptyset$ is terminal, and the product of $\mathcal{V}_1$ and $\mathcal{V}_2$ is their disjoint union $\mathcal{V}_1 + \mathcal{V}_2$. From

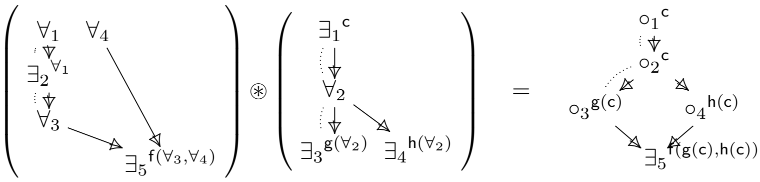



**Figure 7** Interaction of $\sigma : 1^\perp \parallel (\exists_1\forall_2\exists_3 \parallel \exists_4)$ and $\tau : (\exists_1\forall_2\exists_3 \parallel \exists_4)^\perp \parallel \exists_5$

$\gamma : \mathcal{V}_1 \overset{\mathcal{S}}{\to} \mathcal{V}_2$ and $\gamma' : \mathcal{V}'_1 \overset{\mathcal{S}}{\to} \mathcal{V}_2$, we say that $\gamma$ **subsumes** $\gamma'$, written $\gamma' \preccurlyeq \gamma$, if there is $\alpha : \mathcal{V}'_1 \overset{\mathcal{S}}{\to} \mathcal{V}_2$ *s.t.* $\gamma \circ \alpha = \gamma'$ – giving a preorder on substitutions with codomain $\mathcal{V}_2$.

Consider first the *closed* interaction of two $\Sigma$-strategies $\sigma : A$ and $\tau : A^\perp$. As they disagree on the polarities on $A$ we drop them – $\tau \circledast \sigma$ will be a *neutral* $\Sigma$*-strategy* on a *neutral arena*:

▶ **Definition 18.** A **neutral arena** is an arena, without polarities. **Neutral strategies** $\sigma : A$, are defined as in Definition 4 without *(2), (3)*. **Neutral $\Sigma$-strategies** additionally have $\lambda_\sigma : (s \in |\sigma|) \to \mathsf{Tm}_\Sigma([s]_\sigma)$, and are **idempotent**: for all $a \in |a|, \lambda_\sigma(a)[\lambda_\sigma] = \lambda_\sigma(a)$.

Forgetting polarities, every $\Sigma$-strategy is a neutral one. Given $\sigma$ and $\tau$, $\tau \circledast \sigma$ is a *minimal strengthening* of $\sigma$ and $\tau$, regarding both the causal structure and term annotations, *i.e.* a *meet* for the partial order (*idempotence* above is required for it to be antisymmetric):

▶ **Definition 19.** For $\sigma, \tau : A$ neutral $\Sigma$-strategies, we write $\sigma \preccurlyeq \tau$ iff $|\sigma| \subseteq |\tau|$, $\mathscr{C}^\infty(\sigma) \subseteq \mathscr{C}^\infty(\tau)$, and for all $x \in \mathscr{C}(|\sigma|)$, $\lambda_\tau \upharpoonright x$ subsumes $\lambda_\sigma \upharpoonright x$ (regarded as substitutions $x \overset{\mathcal{S}}{\to} x$).

Ignoring terms, any two $\sigma$ and $\tau$ have a meet $\sigma \wedge \tau$; this is a simplification of the *pullback* in the category of event structures, exploiting the absence of conflict [31]. The partial order $(|\sigma \wedge \tau|, \leq_{\sigma\wedge\tau})$ has events all common moves of $\sigma$ and $\tau$ with a causal history compatible with both $\leq_\sigma$ and $\leq_\tau$, and for $\leq_{\sigma\wedge\tau}$ the minimal causal order compatible with both.

However, two neutral $\Sigma$-strategies do not necessarily have a meet for $\preccurlyeq$ (see Appendix 112). Hence, we focus on the meets occurring from compositions of $\Sigma$-strategies and show that for $\sigma : A$ and $\tau : A^\perp$ dual $\Sigma$-strategies the meet *does* exists:

▶ **Lemma 20.** *Any two $\Sigma$-strategies $\sigma : A$ and $\tau : A^\perp$ have a meet $\sigma \wedge \tau$.*

**Proof.** We start with the causal meet $\sigma \wedge \tau$, which we enrich with $\lambda_{\sigma\wedge\tau}$ the *most general unifier* of $\lambda_\sigma \upharpoonright |\sigma \wedge \tau|$ and $\lambda_\tau \upharpoonright |\sigma \wedge \tau|$, obtained by well-founded induction on $\leq_{\sigma\wedge\tau}$:

$$\lambda_{\sigma\wedge\tau}(a) = \begin{cases} \lambda_\sigma(a)[\lambda_{\sigma\wedge\tau} \upharpoonright [a]] & \text{if } \mathrm{pol}_A(a) = \exists \\ \lambda_\tau(a)[\lambda_{\sigma\wedge\tau} \upharpoonright [a]] & \text{if } \mathrm{pol}_A(a) = \forall \end{cases}$$

where $[a] = \{a' \in A \mid a' <_{\sigma\wedge\tau} a\}$. It follows that this is indeed the *m.g.u.* – in particular, we exploit that from $\Sigma$-courtesy, if $a^\exists \in |\sigma|$ then $\lambda_\sigma(a) \in \mathsf{Tm}_\Sigma([a)_\sigma)$. ◀

However this is not sufficient: for composable $\sigma : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$, the games are not purely dual; we need to "pad out" $\sigma$ and $\tau$ and compute instead $(\sigma \parallel C^\perp) \wedge (A \parallel \tau)$, where the parallel composition of Definition 8 is extended with terms in the obvious way, and where $\lambda_A(a) = a$ for all $a \in |A|$. Now $\sigma \parallel C^\perp : A^\perp \parallel B \parallel C^\perp$ and $A \parallel \tau : A \parallel B^\perp \parallel C$ are dual, but $\Sigma$-courtesy from $\Sigma$-strategies is relaxed to idempotence. Yet, Lemma 20 still holds since, from idempotence, if $a^\exists \in |\sigma|$ then either $\lambda_\sigma(a) \in \mathsf{Tm}_\Sigma([a)_\sigma)$ or $\lambda_\sigma(a) = a$. Hence, we can define $\tau \circledast \sigma = (\sigma \parallel C^\perp) \wedge (A \parallel \tau) : A \parallel B \parallel C$.

Variables appearing in $\lambda_{\tau\circledast\sigma}$ cannot be events in $B$ – they must be negative in $A^\perp \parallel C$. So we can define $\tau \odot \sigma = (\tau \circledast \sigma) \cap (A \parallel C)$ the restriction of $\tau \circledast \sigma$ to $A \parallel C$, with same causal order and term annotation. The pair $(|\tau \odot \sigma|, \leq_{\tau\odot\sigma})$ is a strategy, as an instance of the constructions in [4], and this extends to terms so that $\tau \odot \sigma : A^\perp \parallel C$ is a $\Sigma$-strategy, the **composition** of $\sigma$ and $\tau$. Because interaction is defined as a meet for $\preccurlyeq$, it follows that it is compatible with it, *i.e.* if $\sigma \preccurlyeq \sigma'$, then $\tau \circledast \sigma \preccurlyeq \tau \circledast \sigma'$. This is preserved by projection, and hence $\tau \odot \sigma \preccurlyeq \tau \odot \sigma'$ as well. This compatibility of composition with $\preccurlyeq$ will be used later on, together with the easy fact that $\preccurlyeq$ is more constrained on $\Sigma$-strategies:

▶ **Lemma 21.** *For $\sigma, \sigma' : A$ $\Sigma$-strategies, if $\sigma \preccurlyeq \sigma'$, then $\lambda_\sigma(s) = \lambda_{\sigma'}(s)$ for all $s \in |\sigma|$.*

To complete our category, we also define the *copycat strategy.*

▶ **Definition 22.** For an arena $A$, the **copycat $\Sigma$-strategy** $\text{æ}_A : A^\perp \parallel A$ has events $|\text{æ}_A| = A^\perp \parallel A$. Writing $\overline{(i,a)} = (3-i, a)$, its partial order $\leq_{\text{æ}_A}$ is the transitive closure of $\leq_{A^\perp \parallel A} \cup \{(c, \overline{c}) \mid c^\forall \in |A^\perp \parallel A|\}$ and its labelling function is $\lambda_{\text{æ}_A}(c^\forall) = c$, $\lambda_{\text{æ}_A}(c^\exists) = \overline{c}$.

The proof of categorical laws are variations on construction of the bicategory in [4].

▶ Proposition 23. There is a poset-enriched category $\text{Ar}_\Sigma$ with arenas as objects, and $\Sigma$-strategies as morphisms.

## 3.2 Compact closed structure

We show that $\text{Ar}_\Sigma$ is compact closed. The **tensor product** of arenas $A$ and $B$ is $A \parallel B$. For $\Sigma$-strategies $\sigma_1 : A_1^\perp \parallel B_1$ and $\sigma_2 : A_2^\perp \parallel B_2$, we have $\sigma_1 \parallel \sigma_2 : (A_1^\perp \parallel B_1) \parallel (A_2^\perp \parallel B_2)$, which is isomorphic to $(A_1 \parallel A_2)^\perp \parallel (B_1 \parallel B_2)$ – overloading notations, we also write $\sigma_1 \parallel \sigma_2 : (A_1 \parallel A_2)^\perp \parallel (B_1 \parallel B_2)$ for the obvious renaming. It is not difficult to prove:

▶ Proposition 24. Simple parallel composition yields an enriched functor $\parallel : \text{Ar}_\Sigma \times \text{Ar}_\Sigma \to \text{Ar}_\Sigma$.

For the compact closed structure, we elaborate the renaming used above. We write $f : A \cong B$ for an **isomorphism of arenas**, preserving and reflecting all structure.

▶ **Definition 25.** For $f : A \cong B$ and $\sigma : A$ a $\Sigma$-strategy, the **renaming** $f * \sigma : B$ has components $|f * \sigma| = f|\sigma|$, $\leq_{f*\sigma} = \{(f\,a_1, f\,a_2) \mid a_1 \leq_\sigma a_2\}$ and $\lambda_{f*\sigma}(f\,a) = \lambda_\sigma(a)[f]$.

In particular, if $f : A \cong B$, then the corresponding **copycat strategy** is $\text{æ}_f = (A^\perp \parallel f) * \text{æ}_A : A^\perp \parallel B$. We use this to define the structural morphisms for the symmetric monoidal structure of $\text{Ar}_\Sigma$. For instance, the iso $\alpha_{A,B,C} : (A \parallel B) \parallel C \cong A \parallel (B \parallel C)$ yields $\text{æ}_{\alpha_{A,B,C}} : (A \parallel B) \parallel C \overset{\text{Ar}_\Sigma}{\twoheadrightarrow} A \parallel (B \parallel C)$. The other structural morphisms arise similarly. Coherence and naturality then follows from the key *copycat lemma*:

▶ **Lemma 26.** *For $\sigma : A^\perp \parallel B$ a $\Sigma$-strategy and $f : B \cong C$, $\text{æ}_f \odot \sigma = (A^\perp \parallel f) * \sigma : A^\perp \parallel C$.*

As a corollary we get coherence for the structural morphisms (following from those on isomorphisms), and naturality. For all $A$ we get $\eta_A : \emptyset \overset{\text{Ar}_\Sigma}{\twoheadrightarrow} A^\perp \parallel A$ and $\epsilon_A : A \parallel A^\perp \overset{\text{Ar}_\Sigma}{\twoheadrightarrow} \emptyset$ as the obvious renamings of copycat. Checking the law for compact closed categories is a variation of the idempotence of copycat. Overall:

▶ Proposition 27. $\text{Ar}_\Sigma$ is a poset-enriched compact closed category.

## 3.3 A linearly distributive category with negation

Finally, we reinstate winning conditions. We first note:

▶ Proposition 28. There is a (poset-enriched) category $\text{Ga}_\Sigma$ with objects the games (Definition 6) on $\Sigma$, and morphisms $\Sigma$-strategies $\sigma : \mathcal{A}^\perp \parr \mathcal{B}$, also written $\sigma : \mathcal{A} \overset{\text{Ga}_\Sigma}{\twoheadrightarrow} \mathcal{B}$.

That copycat is winning boils down to the excluded middle. That $\tau \odot \sigma : \mathcal{A}^\perp \parr \mathcal{C}$ is winning if $\sigma : \mathcal{A}^\perp \parr \mathcal{B}$ and $\tau : \mathcal{B}^\perp \parr \mathcal{C}$ are, is as in [5]: for $x \in \mathscr{C}(\tau \odot \sigma)$ $\exists$-maximal we find a witness $y \in \mathscr{C}(\tau \circledast \sigma)$ (*i.e.* $y \cap (A \parallel C) = x$) *s.t.* $y \cap (A \parallel B) \in \sigma$, $y \cap (B \parallel C) \in \tau$ are $\exists$-maximal; and apply transitivity of implication. The equations follow from $\mathrm{Ar}_\Sigma$. Likewise:

▶ Proposition 29. The functor $\parallel : \mathrm{Ar}_\Sigma \times \mathrm{Ar}_\Sigma \to \mathrm{Ar}_\Sigma$ splits into $\otimes, \parr : \mathrm{Ga}_\Sigma \times \mathrm{Ga}_\Sigma \to \mathrm{Ga}_\Sigma$.

It suffices to check winning, which is straightforward. It remains to prove that all structural morphisms from $\mathrm{Ar}_\Sigma$ (copycat strategies) are winning, which boils down to the following sufficient conditions to hold: For $\mathcal{A}, \mathcal{B}$ games, a **win-iso** $f : \mathcal{A} \to \mathcal{B}$ is an iso $f : A \cong B$ such that $(\mathcal{W}_\mathcal{A}(x))^\perp \vee \mathcal{W}_B(f\,x)$ is a tautology, for all $x \in \mathscr{C}^\infty(A)$.

▶ **Lemma 30.** *If $f : \mathcal{A} \to \mathcal{B}$ is a win-iso, then $\mathrm{æ}_f : \mathcal{A}^\perp \parr \mathcal{B}$ is a winning $\Sigma$-strategy.*

This easily entails that all structural morphisms (including linear distributivity) are winning. Finally $\eta_\mathcal{A} : 1 \overset{\mathrm{Ga}_\Sigma}{\twoheadrightarrow} \mathcal{A}^\perp \parr \mathcal{A}$ and $\epsilon_\mathcal{A} : \mathcal{A} \otimes \mathcal{A}^\perp \overset{\mathrm{Ga}_\Sigma}{\twoheadrightarrow} \perp$ are winning, which concludes:

▶ Proposition 31. $\mathrm{Ga}_\Sigma$ is a poset-enriched $*$-autonomous category.

# 4 A model of first-order MLL

We move on to $\mathrm{MLL}_1$, *i.e.* all rules except for contraction and weakening. Before developing the interpretation, we discuss cut elimination. There are three new cut reduction rules, displayed in Figure 8: the new *logical* reduction ($\forall/\exists$), and two for the propagation of cuts past introduction rules for $\forall$ and $\exists$. Writing $\pi \rightsquigarrow_{\mathrm{MLL}_1} \pi'$ for the reduction obtained with these new rules together with $\rightsquigarrow_{\mathrm{MLL}}$:

▶ Proposition 32. Let $\pi$ be any $\mathrm{MLL}_1$ proof of $\vdash^\mathcal{V} \Gamma$. Then, there is a cut-free proof $\pi'$ of $\vdash^\mathcal{V} \Gamma$ s.t. $\pi \rightsquigarrow^*_{\mathrm{MLL}_1} \pi'$.

$$\textsc{Cut}\ \dfrac{\forall\mathrm{I}\ \dfrac{\dfrac{\pi_1}{\vdash^{\mathcal{V}\uplus\{\mathsf{x}\}} \Gamma, \varphi}}{\vdash^{\mathcal{V}} \Gamma, \forall \mathsf{x}.\,\varphi} \qquad \exists\mathrm{I}\ \dfrac{\dfrac{\pi_2}{\vdash^{\mathcal{V}} \varphi^\perp[t/\mathsf{x}], \Delta}}{\vdash^{\mathcal{V}} \exists \mathsf{x}.\,\varphi^\perp, \Delta}}{\vdash^{\mathcal{V}} \Gamma, \Delta} \rightsquigarrow_{\forall/\exists} \textsc{Cut}\ \dfrac{\dfrac{\pi_1[t/\mathsf{x}]}{\vdash^{\mathcal{V}} \Gamma, \varphi[t/\mathsf{x}]} \qquad \dfrac{\pi_2}{\vdash^{\mathcal{V}} \varphi^\perp[t/\mathsf{x}], \Delta}}{\vdash^{\mathcal{V}} \Gamma, \Delta}$$

$$\textsc{Cut}\ \dfrac{\dfrac{\pi_1}{\vdash^{\mathcal{V}} \Gamma, \psi} \qquad \forall\mathrm{I}\ \dfrac{\dfrac{\pi_2}{\vdash^{\mathcal{V}\uplus\{\mathsf{x}\}} \psi^\perp, \Delta, \varphi}}{\vdash^{\mathcal{V}} \psi^\perp, \Delta, \forall \mathsf{x}.\,\varphi}}{\vdash^{\mathcal{V}} \Gamma, \Delta, \forall \mathsf{x}.\,\varphi} \rightsquigarrow_{\textsc{Cut}/\forall} \forall\mathrm{I}\ \dfrac{\textsc{Cut}\ \dfrac{\dfrac{\pi_1}{\vdash^{\mathcal{V}\uplus\{\mathsf{x}\}} \Gamma, \psi} \qquad \dfrac{\pi_2}{\vdash^{\mathcal{V}\uplus\{\mathsf{x}\}} \psi^\perp, \Delta, \varphi}}{\vdash^{\mathcal{V}\uplus\{\mathsf{x}\}} \Gamma, \Delta, \varphi}}{\vdash^{\mathcal{V}} \Gamma, \Delta, \forall \mathsf{x}.\,\varphi}$$

$$\textsc{Cut}\ \dfrac{\dfrac{\pi_1}{\vdash^{\mathcal{V}} \Gamma, \psi} \qquad \exists\mathrm{I}\ \dfrac{\dfrac{\pi_2}{\vdash^{\mathcal{V}} \psi^\perp, \Delta, \varphi[t/\mathsf{x}]}}{\vdash^{\mathcal{V}} \psi^\perp, \Delta, \exists \mathsf{x}.\,\varphi}}{\vdash^{\mathcal{V}} \Gamma, \Delta, \exists \mathsf{x}.\,\varphi} \rightsquigarrow_{\textsc{Cut}/\exists} \exists\mathrm{I}\ \dfrac{\textsc{Cut}\ \dfrac{\dfrac{\pi_1}{\vdash^{\mathcal{V}} \Gamma, \psi} \qquad \dfrac{\pi_2}{\vdash^{\mathcal{V}} \psi^\perp, \Delta, \varphi[t/\mathsf{x}]}}{\vdash^{\mathcal{V}} \Gamma, \Delta, \varphi[t/\mathsf{x}]}}{\vdash^{\mathcal{V}} \Gamma, \Delta, \exists \mathsf{x}.\,\varphi}$$

**Figure 8** Additional cut elimination rules for $\mathrm{MLL}_1$

The first rule of Figure 8 requires the introduction of *substitution* on proofs. In general, for a proof $\pi$ of $\vdash^{\mathcal{V}_2} \Gamma$ and $\gamma : \mathcal{V}_1 \to \mathcal{V}_2$ we obtain $\pi[\gamma]$ a proof of $\vdash^{\mathcal{V}_1} \Gamma[\gamma]$ by propagating $\gamma$ through $\pi$, substituting formulas and terms. A degenerate case of this is the substitution of a proof $\pi$ of $\vdash^\mathcal{V} \Gamma$ by *weakening* $\mathrm{w}_{\mathcal{V},\mathsf{x}} : \mathcal{V} \uplus \{\mathsf{x}\} \to \mathcal{V}$, obtaining $\pi_1[\mathrm{w}_{\mathcal{V},\mathsf{x}}]$, a proof of $\vdash^{\mathcal{V}\uplus\mathsf{x}} \Gamma$. As this leaves the formulas and terms unchanged we leave it implicit in the reduction rules – it is used for instance implicitly in the commutation $\textsc{Cut}/\forall$.

Substitution is key in the cut reduction of quantifiers. However it is best studied independently of quantifiers, in a model of $\mathcal{V}$-MLL (see Figure 6). This is the topic of the next subsection, prior to the interpretation of the introduction rules for quantifiers.

## 4.1 A fibred model of $\mathcal{V}$-MLL

Following [20, 28], we expect to model $\mathcal{V}$-MLL and substitution in:

▶ **Definition 33.** Let $*$-Aut be the category of $*$-autonomous categories and functors preserving the structure on the nose. A **strict $\mathcal{S}$-indexed $*$-autonomous category** is a functor $\mathcal{T} : \mathcal{S}^{\mathrm{op}} \to *\text{-Aut}$.

Such definitions (*e.g. hyperdoctrines* [28]) are usually phrased only up to isomorphism; for simplicity we opt here for a lighter definition. Writing $\mathcal{V}_n = \{\mathsf{x}_1, \dots, \mathsf{x}_n\}$, we say that $\mathcal{T}$ **supports** $\Sigma$ if for every predicate symbol $\mathsf{P}$ of arity $n$ there is $[\![\mathsf{P}]\!]_{\mathcal{V}_n}$ a chosen object of $\mathcal{T}(\mathcal{V}_n)$. For $t_1, \dots, t_n \in \mathsf{Tm}_\Sigma(\mathcal{V})$ we can then set $[\![\mathsf{P}(t_1, \dots, t_n)]\!] = \mathcal{T}([t_1/\mathsf{x}_1, \dots, t_n/\mathsf{x}_n])([\![\mathsf{P}]\!]_{\mathcal{V}_n})$ an object of $\mathcal{T}(\mathcal{V})$, also written $[\![\mathsf{P}]\!]_{\mathcal{V}_n}[t_1/\mathsf{x}_1, \dots, t_n/\mathsf{x}_n]$.

For any finite $\mathcal{V}$, this lets us interpret $\mathcal{V}$-MLL in $\mathcal{T}(\mathcal{V})$ as in Section 3. Besides $\mathcal{V}$-MLL in isolation, this also models substitutions. In games the functorial action of $\mathcal{T}$ on $\gamma : \mathcal{V}_1 \to \mathcal{V}_2$ will correspond to substitution on games $A[\gamma] = \mathcal{T}(\gamma)(A)$ and strategies $\sigma[\gamma] = \mathcal{T}(\gamma)(\sigma)$. This matches syntactic substitution, as $\mathcal{T}(\gamma)$ preserves the $*$-autonomous structure.

Let us now introduce the concrete structure. For any finite $\mathcal{V}$, the *fibre* $\mathcal{T}(\mathcal{V})$ is the category $\mathrm{Ga}_{\Sigma \uplus \mathcal{V}}$ built in Section 3, on the extended signature $\Sigma \uplus \mathcal{V}$. Recall that its *objects* are games on the signature $\Sigma \uplus \mathcal{V}$, *i.e.* the $\mathcal{V}$*-games* of Section 2.3. *Morphisms* between $\mathcal{V}$-games $\mathcal{A}$ and $\mathcal{B}$ are winning $(\Sigma \uplus \mathcal{V})$-strategies on $\mathcal{A}^\perp \parr \mathcal{B}$ regarded as a game on signature $\Sigma \uplus \mathcal{V}$ – also called **winning $\Sigma$-strategies on the $\mathcal{V}$-game $\mathcal{A}^\perp \parr \mathcal{B}$.**

Finally, for $\mathcal{A}$ a $\mathcal{V}_2$-game and $\gamma : \mathcal{V}_1 \to \mathcal{V}_2$ a substitution, the game $\mathcal{T}(\gamma)(\mathcal{A}) = \mathcal{A}[\gamma]$ is defined as having arena $A$, and, for $x \in \mathscr{C}^\infty(A)$, $\mathcal{W}_{\mathcal{A}[\gamma]}(x) = \mathcal{W}_\mathcal{A}(x)[\gamma] \in \mathsf{QF}^\infty_{\Sigma \uplus \mathcal{V}_1}(x)$. Likewise, given $\mathcal{A}$ and $\mathcal{B}$ two $\mathcal{V}$-games and $\sigma : \mathcal{A}^\perp \parr \mathcal{B}$, $\sigma[\gamma]$ has the same components as $\sigma$, but term annotations $\lambda_{\sigma[\gamma]}(s) = \lambda(s)[\gamma] \in \mathsf{Tm}_{\Sigma \uplus \mathcal{V}_1}(x)$. It is a simple verification to prove:

▶ Proposition 34. For any $\gamma : \mathcal{V}_1 \xrightarrow{\mathcal{S}} \mathcal{V}_2$, $\mathcal{T}(\gamma) : \mathcal{T}(\mathcal{V}_2) \to \mathcal{T}(\mathcal{V}_1)$ is a strict $*$-autonomous functor preserving the order.

## 4.2 Quantifiers

Finally, we give the interpretation of ∀I and ∃I. For now, we consider a *linear* interpretation $[\![-]\!]^\ell$ of formulas defined like $[\![-]\!]^\exists_\mathcal{V}$ except for $[\![\exists \mathsf{x} \varphi]\!]^\ell_\mathcal{V} = \exists \mathsf{x}. [\![\varphi]\!]^\ell_\mathcal{V}$.

Besides preserving the $*$-autonomous structure, substitution also propagates through quantifiers, from which we have:

▶ **Lemma 35.** *Let $\varphi \in \mathsf{Form}_\Sigma(\mathcal{V}_2)$ and $\gamma : \mathcal{V}_1 \to \mathcal{V}_2$ a substitution, then $[\![\varphi[\gamma]]\!]^\ell_{\mathcal{V}_1} = [\![\varphi]\!]^\ell_{\mathcal{V}_2}[\gamma]$.*

This will be used implicitly from now on. The definition of quantifiers on games of Definition 12 extends to functors $\forall_{\mathcal{V},\mathsf{x}}, \exists_{\mathcal{V},\mathsf{x}} : \mathcal{T}(\mathcal{V} \uplus \{\mathsf{x}\}) \to \mathcal{T}(\mathcal{V})$. From $\sigma : \mathcal{A}^\perp \parr \mathcal{B}$, $\forall_{\mathcal{V},\mathsf{x}}(\sigma) : (\forall \mathsf{x}. \mathcal{A})^\perp \parr \forall \mathsf{x}. \mathcal{B}$ plays copycat on the initial ∀, then plays as $\sigma$ (similarly for $\exists_{\mathcal{V},\mathsf{x}}(\sigma)$). Following Lawvere [20], one expects adjunctions $\exists_{\mathcal{V},\mathsf{x}} \dashv \mathcal{T}(\mathrm{w}_{\mathcal{V},\mathsf{x}}) \dashv \forall_{\mathcal{V},\mathsf{x}}$. Unfortunately, this fails – we present this failure later as the non-preservation of $\rightsquigarrow_{\textsc{Cut}/\forall}$.

We now interpret ∀I and ∃I. First, we give a strategy introducing a witness $t$.

▶ **Definition 36.** The $(\Sigma \uplus \mathcal{V})$-strategy $\exists^t_A : A^\perp \parallel \exists. A$ is $(|A^\perp \parallel \exists. A|, \leq_{\exists^t_A}, \lambda_{\exists^t_A})$ where $\leq_{\exists^t_A}$ includes $\leq_{\text{æ}_A}$, plus dependencies $\{((2, \exists), (2, a)) \mid a \in A\} \uplus \{((2, \exists), (1, a)) \mid \exists a_0^\forall \in A.\ a_0 \leq_A a\}$ and term assignment that of $\text{æ}_A$ plus $\lambda_{\exists^t_A}((2, \exists)) = t$.

In other words, $\exists^t_A$ plays ∃ annotated with $t$, then proceeds as copycat on $A$. We have:

▶ Proposition 37. Let $\mathcal{A}$ be a $\mathcal{V}$-game, and $t \in \mathsf{Tm}_\Sigma(\mathcal{V})$. Then, $\exists^t_A : \mathcal{A}[t/\mathsf{x}] \overset{\mathcal{V}\text{-}\mathrm{Ga}_\Sigma}{\twoheadrightarrow} \exists \mathsf{x}. \mathcal{A}$.

Indeed, any ∃-maximal $x_A \parallel \exists. x_A \in \mathscr{C}^\infty(\exists^t_A)$ corresponds to a tautology $\mathcal{W}_{\mathcal{A}[t/\mathsf{x}]}(x_A)^\perp \vee \mathcal{W}_\mathcal{A}(x_A)[t/\mathsf{x}]$. We interpret ∃I by post-composing with $\exists^t_A$ (as in Figure 10 without the last step). This validates $\rightsquigarrow_{\textsc{Cut}/\exists}$, by associativity of composition.

To a strategy $\sigma$, the operation interpreting ∀I adds ∀ as new minimal event, and sets it as a dependency for all events whose annotation comprise the distinguished variable $\mathsf{x}$.

▶ **Definition 38.** For $\sigma$ a $(\Sigma \uplus \mathcal{V} \uplus \{\mathsf{x}\})$-strategy on $A^\perp \parallel B$, the $(\Sigma \uplus \mathcal{V})$-strategy $\forall\mathrm{I}^{\mathsf{x}}_{A,B}(\sigma) : A^\perp \parallel \forall.B$ has events $|\sigma| \uplus \{(2,\forall)\}$, term assignment $\lambda((2,\forall)) = (2,\forall)$ and causality $\lambda(s) = \lambda_\sigma(s)[(2,\forall)/\mathsf{x}]$ $(s \in |\sigma|)$, and $\leq = \leq_\sigma \cup \{((2,\forall), s) \mid s \in \forall.B \quad \vee \quad \exists s' \leq_\sigma s,\ \mathsf{x} \in \mathrm{fv}(\lambda_\sigma(s'))\}$.

▶ Proposition 39. If $\sigma$ is winning on a $(\mathcal{V} \uplus \{\mathsf{x}\})$-game $\mathcal{A}[\mathrm{w}_{\mathcal{V},\mathsf{x}}] \parr \mathcal{B}$, $\forall\mathrm{I}^{\mathsf{x}}_{A,B}(\sigma)$ is winning on the $\mathcal{V}$-game $\mathcal{A} \parr \forall\mathsf{x}.\,\mathcal{B}$.

Indeed, if $\forall$bélard does not play $(2,\forall)$ we get a tautology, otherwise the remaining configuration is in $\sigma$ and so is tautological. This completes the interpretation of $\mathrm{MLL}_1$. It leaves $\rightsquigarrow_{\forall/\exists}$ invariant, but fails $\rightsquigarrow_{\mathrm{CUT}/\forall}$. This stems from the fact that the *minimal* $\Sigma$-strategies are not stable under composition (see Example 156 in Appendix). The interpretation of cut-free proofs yield minimal $\Sigma$-strategies. In contrast, in compositions interpreting cuts, causality may flow through the syntax tree of the cut formula, and create causal dependencies not reflected in the variables. Hence, cut reduction may weaken the causal structure.

▶ **Lemma 40.** *For $\sigma : A \overset{\mathrm{Ar}_\Sigma}{\multimap} B$ and $\tau : B \overset{\mathrm{Ar}_{\Sigma \uplus \{x\}}}{\multimap} C$, we have $\forall I^{\mathsf{x}}_{A,C}(\tau \odot \sigma) \preccurlyeq \forall I^{\mathsf{x}}_{B,C}(\tau) \odot \sigma$.*

By Lemma 21 these two have the same terms on common events. In fact, $\forall\mathrm{I}^{\mathsf{x}}_{A,C}(\tau \odot \sigma)$ and $\forall\mathrm{I}^{\mathsf{x}}_{B,C}(\tau) \odot \sigma$ also have the same *events* – they correspond to the same *expansion tree*, only the acyclicity witness differs. But the variant of $\preccurlyeq$ with $|\sigma_1| = |\sigma_2|$ is not a congruence: relaxing causality of $\sigma$ in $\tau \odot \sigma$ may unlock new events, previously part of causal loops.

As $\preccurlyeq$ is preserved by all operations on $\Sigma$-strategies, we deduce:

▶ **Theorem 41.** *If $\pi \rightsquigarrow_{\mathrm{MLL}_1} \pi'$, then $[\![\pi']\!] \preccurlyeq [\![\pi]\!]$.*

For $\mathrm{MLL}_1$, we conjecture that "having the same expansion tree" (*i.e.* same events and term annotations) is actually a congruence, yielding a $*$-*autonomous hyperdoctrine*. As this would not hold in the presence of contraction and weakening, we leave this for future work.

## 5 Contraction and weakening

In this section we reinstate ! and ? in the interpretation of quantifiers, *i.e.* $[\![\forall\mathsf{x}.\,\varphi]\!]_{\mathcal{V}} = {!}\forall\mathsf{x}.\,[\![\varphi]\!]_{\mathcal{V}\uplus\{\mathsf{x}\}}$ and $[\![\exists\mathsf{x}\,\varphi]\!]_{\mathcal{V}} = {?}\exists\mathsf{x}\,[\![\varphi]\!]_{\mathcal{V}\uplus\mathsf{x}}$ – this is reminiscent of Melliès' discussion on the interaction between quantifiers and exponential modalities in a polarized setting [22].

Unlike for $\mathrm{MLL}_1$, we only aim to map proofs to $\Sigma$-strategies on the appropriate game, with no preservation of reduction. We must interpret contraction and weakening, but also revisit the interpretation of rules for quantifiers as the interpretation of formulas has changed.

*Weakening* is easy: for any game $\mathcal{A}$, any $\Sigma$-strategy $\sigma : \mathcal{A} \multimap 1$ is winning; for definiteness, we use the minimal $e_{\mathcal{A}} : \mathcal{A} \multimap 1$, only closed under receptivity. *Contraction* is much more subtle. To illustrate the difficulty, we present in Figure 9 two simple instances of the contraction $\Sigma$-strategy (without term annotations). The first looks like the usual contraction of AJM games [1]. It can be used to interpret the contraction rule on existential formulas, where it has the effect of taking the union of the different witnesses proposed. But in LK, one can also use contraction on a universal formula, which will appeal to a strategy like the second. Any witness proposed by $\forall$bélard will then have to be propagated to both branches to ensure that we are winning (mimicing the effect of cut reduction).

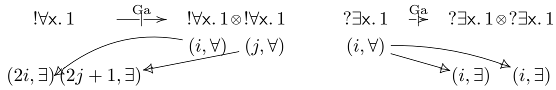



**Figure 9** Two examples of contraction

In order to define this contraction $\Sigma$-strategy along with the tools to revisit the introduction rules for quantifiers, we will first study some properties of the exponential modalities.

$$\left[\!\!\left[ \mathrm{C}\,\frac{\dfrac{\pi}{\vdash^{\mathcal{V}} \Gamma, \varphi, \varphi}}{\vdash^{\mathcal{V}} \Gamma, \varphi} \right]\!\!\right] = \Gamma^{\perp} \overset{[\![\pi]\!]}{\underset{}{\overset{\mathcal{T}(\mathcal{V})}{\to}}} \varphi \parr \varphi \overset{\delta^{\perp}_{\varphi^{\perp}}}{\overset{\mathcal{T}(\mathcal{V})}{\to}} \varphi \qquad \left[\!\!\left[ \forall\mathrm{I}\,\frac{\dfrac{\pi}{\vdash^{\mathcal{V} \uplus \{\mathsf{x}\}} \Gamma, \varphi}}{\vdash^{\mathcal{V}} \Gamma, \forall \mathsf{x}.\, \varphi} \right]\!\!\right] = \Gamma^{\perp} \overset{co_{\Gamma^{\perp}}}{\overset{\mathcal{T}(\mathcal{V})}{\to}} !\Gamma^{\perp} \overset{!(\forall\!\mathrm{I}\,([\![\pi]\!]))}{\overset{\mathcal{T}(\mathcal{V})}{\to}} !\forall \mathsf{x}.\, \varphi \qquad \left[\!\!\left[ \forall\mathrm{I}\,\frac{\dfrac{\pi}{\vdash^{\mathcal{V}} \Gamma, \varphi[t/\mathsf{x}]}}{\vdash^{\mathcal{V}} \Gamma, \exists \mathsf{x}.\, \varphi} \right]\!\!\right] = \Gamma^{\perp} \overset{[\![\pi]\!]}{\overset{\mathcal{T}(\mathcal{V})}{\to}} \varphi[t/\mathsf{x}] \overset{\exists^{t}_{\varphi}}{\overset{\mathcal{T}(\mathcal{V})}{\to}} \exists \mathsf{x}.\varphi \overset{\mathcal{T}(\mathcal{V})}{\to} ?\exists \mathsf{x}.\varphi$$

**Figure 10** Interpretation of the remaining rules of LK

| | | | | |
|---|---|---|---|---|
| $!\mathcal{A} \to\ !!\mathcal{A}$ | $!\mathcal{A} \to\ !\mathcal{A} \otimes !\mathcal{A}$ | $?!\mathcal{A} \to\ !?\mathcal{A}$ | $!\mathcal{A} \otimes !\mathcal{B} \to\ !(\mathcal{A} \otimes \mathcal{B})$ | $!\mathcal{A} \parr !\mathcal{B} \to\ !(\mathcal{A} \parr \mathcal{B})$ |
| $(\langle i, j\rangle, a) \mapsto (i, (j, a))$ | $(2i, a) \mapsto (1, (i, a))$ | $(i, (j, a)) \mapsto (j, (i, a))$ | $(j, (i, a)) \mapsto (i, (j, a))$ | $(j, (i, a)) \mapsto (i, (j, a))$ |
| | $(2i+1, a) \mapsto (2, (i, a))$ | | | |

**Figure 11** Some win-isos with exponentials whose lifting are used in the interpretation

Recall ! and ? from Definition 11, both based on arena $\|_\omega A$. First, we examine their functorial action. Let $\sigma : A \overset{\mathrm{Ar}_\Sigma}{\twoheadrightarrow} B$. Then, $\|_\omega \sigma : \|_\omega (A^\perp \parallel B)$ which is isomorphic to $(\|_\omega A)^\perp \parallel (\|_\omega B)$; overloading notion we still write $\|_\omega \sigma : \|_\omega A \overset{\mathrm{Ar}_\Sigma}{\twoheadrightarrow} \|_\omega B$.

▶ **Lemma 42.** *Let* $\sigma : \mathcal{A} \overset{\mathrm{Ga}_\Sigma}{\twoheadrightarrow} \mathcal{B}$. *Then, we have* $!\sigma = \|_\omega \sigma : !\mathcal{A} \overset{\mathrm{Ga}_\Sigma}{\twoheadrightarrow} !\mathcal{B}$ *and* $?\sigma = \|_\omega \sigma : ?\mathcal{A} \overset{\mathrm{Ga}_\Sigma}{\twoheadrightarrow} ?\mathcal{B}$.

Rather than defining directly the contraction, we build $co_\varphi : [\![\varphi]\!]_{\mathcal{V}} \overset{\mathrm{Ga}_{\Sigma \uplus \mathcal{V}}}{\twoheadrightarrow} ![\![\varphi]\!]_{\mathcal{V}}$ by induction on $\varphi \in \mathsf{Form}_\Sigma(\mathcal{V})$. For $\varphi$ quantifier-free, the empty $co_\varphi : [\![\varphi]\!]_{\mathcal{V}} \twoheadrightarrow ![\![\varphi]\!]_{\mathcal{V}}$ is winning. We get $co_{\forall \mathsf{x}.\, \varphi} : !\forall \mathsf{x}.\, [\![\varphi]\!]_{\mathcal{V}} \twoheadrightarrow !!\forall \mathsf{x}.\, [\![\varphi]\!]_{\mathcal{V}}$ as a particular case of $!\mathcal{A} \twoheadrightarrow !!\mathcal{A}$ from Figure 11. We get $co_{\varphi \wedge \psi}$ and $co_{\varphi \vee \psi}$ by induction and composition with $!\mathcal{A} \otimes !\mathcal{B} \twoheadrightarrow !(\mathcal{A} \otimes \mathcal{B})$, $!\mathcal{A} \parr !\mathcal{B} \twoheadrightarrow !(\mathcal{A} \parr \mathcal{B})$. Finally, $co_{?\exists \mathsf{x}.\, [\![\varphi]\!]_{\mathsf{x}}}$ is obtained analogously to the contraction on the right of Figure 9.

▶ **Lemma 43.** *For any* $(\mathcal{V} \uplus \{\mathsf{x}\})$*-game* $\mathcal{A}$, *there is a winning* $\mu_{\mathcal{A},\mathsf{x}} : \exists \mathsf{x}.\, !\mathcal{A} \overset{\mathcal{V}\text{-Ga}}{\twoheadrightarrow} !\exists \mathsf{x}.\, \mathcal{A}$.

**Proof.** After the unique minimal $\forall$ move (on the left hand side), the strategy simultaneously plays all the $(i, \exists)$ (on the right hand side) with annotation $\forall$; then proceeds as $\mathrm{æ}_{!\mathcal{A}}$. ◀

We get $co_{?\exists \mathsf{x}.\, [\![\varphi]\!]_{\mathsf{x}}}$ by induction, post-composition with $?\mu_{[\![\varphi]\!],\mathsf{x}}$ and distribution of ? over !.

▶ Proposition 44. For any $\varphi \in \mathsf{Form}_\Sigma(\mathcal{V})$, there is a winning $co_{[\![\varphi]\!]_{\mathcal{V}}} : [\![\varphi]\!]_{\mathcal{V}} \overset{\mathcal{V}\text{-Ga}}{\twoheadrightarrow} ![\![\varphi]\!]_{\mathcal{V}}$.

Combining Proposition 44 with other primitives (including $!\mathcal{A} \twoheadrightarrow \mathcal{A}$, playing copycat between $\mathcal{A}$ and the $0^{\text{th}}$ copy on the left, closed under receptivity), we get $\delta_{[\![\varphi]\!]_{\mathcal{V}}} : [\![\varphi]\!]_{\mathcal{V}} \twoheadrightarrow [\![\varphi]\!]_{\mathcal{V}} \otimes [\![\varphi]\!]_{\mathcal{V}}$ for $\varphi \in \mathsf{Form}_\Sigma(\mathcal{V})$. We complete the interpretation in Figure 10, omitting W, which is by post-composition with $e_{\mathcal{A}}$ and silently using the isomorphism between winning $\Sigma$-strategies from 1 to $\Gamma \parr \mathcal{A}$ and from $\Gamma^\perp$ to $\mathcal{A}$. This concludes the proof of Theorem 14.

## 6 Conclusion

For LK there is no hope of preserving unrestricted cut reduction without collapsing to a boolean algebra [13]. There are non-degenerate models for classical logic with an involutive negation, *e.g.* Führman and Pym's *classical categories* [9] with cut reduction only preserved in a lax sense; but our model does not preserve cut reduction even in this weaker sense. Besides this our semantics is infinitary: from the *structural dilemma* in [8] we obtained a proof of some $\exists \mathsf{x}.\, \varphi$ with $\varphi$ quantifier-free (no ∀bélard moves) yielding an infinite $\Sigma$-strategy.

Both phenomena could be avoideed by adopting a polarized model, abandonning however our faithfulness to the raw Herbrand content of proofs. It is a fascinating open question whether one can find a non-polarized model of classical first-order logic that remains finitary – this is strongly related to the actively investigated question of finding a strongly normalizing reduction strategy on syntaxes for expansion trees [15, 21, 16].

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This appendix contains detailed constructions of the model and of the interpretation of first-order classical logic. It is written to be self-contained and independent of the main text of the paper.

In Section A, we develop the basic category of deterministic concurrent strategies that underlies the composition of acyclicity witnesses in our interpretation. In B, we enrich these acyclicity witnesses with term annotations and obtain the compact closed category of arenas and Σ-strategies. In Section C, we add winning conditions and prove $*$-autonomy (along with further structure to interpret quantifiers and structural rules). Finally in Section D, we give the interpretation of a classical sequent calculus and prove our main result.

# A Deterministic concurrent games

In this first section, we introduce the basic compact closed category of deterministic concurrent strategies. It is a simplification of Rideau and Winskel's concurrent games, in the case where both games and strategies do not have conflict. It is essentially an alternative formulation to Melliès and Mimram's asynchronous games.

For the purposes of our paper (a compositional account of Herbrand's theorem), it will provide the mechanism for composing the acyclicity witnesses of expansion trees. The full strategies will comprise a causal strategy as introduced now, accompanied with a labelling associating to each Player event a first-order term. These annotations will be introduced and handled after the basic framework is constructed, in Section B.

## A.1 Preliminary notions

We introduce basic notions on elementary event structures and augmentations.

▶ **Definition 45.** An **elementary event structure (ees)** is a pair $\mathsf{q} = (|\mathsf{q}|, \leq_\mathsf{q})$ where $|\mathsf{q}|$ is a set of *events* and $\leq_\mathsf{q}$ is a partial order on $|\mathsf{q}|$ referred to as the *causal order*, such that for all $e \in |\mathsf{q}|$,

$$[e]_\mathsf{q} = \{e' \in |\mathsf{q}| \mid e' \leq_\mathsf{q} e\}$$

is finite.

In the sequel, we will use some standard notions and notations on event structures. The **configurations** of an ees $\mathsf{q}$ will be written $\mathscr{C}^\infty(\mathsf{q})$, and the **finite configurations** $\mathscr{C}(\mathsf{q})$. We write $\rightarrowtriangle_\mathsf{q}$ for the **immediate causality** relation on $\mathsf{q}$. We also write $-\!\!\subset$ for the **covering relation** between configurations, and $x \overset{e}{-\!\!\!\subset}$ to mean that $x$ extends with event $e$. As above, $[e]_\mathsf{q} = \{e' \in |\mathsf{q}| \mid e' \leq_\mathsf{q} e\}$ is the **prime configuration** of $e$, and $[e)_\mathsf{q} = [e]_\mathsf{q} \setminus \{e\}$. We will also sometimes use the notation $[X]_\mathsf{q}$ where $X$ is a *finite set* of events, rather than a single event, to mean the down-closure of $X$. As usual, we will sometimes omit the $\mathsf{q}$ subscript in $\leq_\mathsf{q}, \rightarrowtriangle_\mathsf{q}, [e]_\mathsf{q}$, etc. when they are clear from the context.

The following partial order on elementary event structures will play a key role in our developments.

▶ **Definition 46.** Let $\mathsf{q}, \mathsf{p}$ be two ees. We write $\mathsf{q} \preccurlyeq \mathsf{p}$ iff $\mathscr{C}(\mathsf{q}) \subseteq \mathscr{C}(\mathsf{p})$.

Clearly, the condition implies that $|\mathsf{q}| \subseteq |\mathsf{p}|$ as well. The condition on configurations amounts to the identity map being a map of event structures in the usual sense. Intuitively, $\mathsf{q} \preccurlyeq \mathsf{p}$ means that $\mathsf{q}$ has fewer events, and those are more causally constrained than in $\mathsf{p}$. It is easy to check that this is a partial order.

## A.2 Meet the meet

In fact, we shall now establish that it is a meet-semilattice; this meet operation will play a crucial role in our development as it will be the basis for the interaction of strategies.

First, we need this preliminary definition.

▶ **Definition 47.** If $\mathsf{q}, \mathsf{p}$ are two ees and $e \in |\mathsf{q}| \cap |\mathsf{p}|$, we say that it is **secured** if there is a covering chain, *i.e.*:

$$\emptyset = x_0 \stackrel{e_1}{\longrightarrow\!\!\!\subset} \dots \stackrel{e_n}{\longrightarrow\!\!\!\subset} x_n$$

where $e_n = e$, and for all $0 \leq i \leq n$, $x_i \in \mathscr{C}(\mathsf{q}) \cap \mathscr{C}(\mathsf{p})$.

Relying on that, we can now define the meet of two ees.

▶ **Definition 48.** Let $\mathsf{p}, \mathsf{q}$ be two ees. We define $\mathsf{p} \wedge \mathsf{q}$ as having:

- *Events:* those of $|\mathsf{q}| \cap |\mathsf{p}|$ that are secured,
- *Causality:* defined as $\leq_{\mathsf{q}\wedge\mathsf{p}} = (\leq_{\mathsf{q}} \cup \leq_{\mathsf{p}})^*$.

There are a few steps towards proving that this is a partial order. First we prove the following basic lemma.

▶ **Lemma 49.** *Let* $\mathsf{p}, \mathsf{q}$ *be two ees. If* $e \in |\mathsf{p} \wedge \mathsf{q}|$*, then there is a covering chain for* $e$

$$\emptyset = x_0 \stackrel{e_1}{\longrightarrow\!\!\!\subset} \dots \stackrel{e_n}{\longrightarrow\!\!\!\subset} x_n$$

*(with* $e_n = e$, $x_i \in \mathscr{C}(\mathsf{p}) \cap \mathscr{C}(\mathsf{q})$*) which is minimal,* i.e. $x_n = [e]_{\mathsf{p}\wedge\mathsf{q}} = \{e' \in |\mathsf{p} \wedge \mathsf{q}| \mid e' \leq_{\mathsf{p}\wedge\mathsf{q}} e\}$.

**Proof.** Since $e \in |\mathsf{p} \wedge \mathsf{q}|$ it is secured by definition, so there is a covering chain

$$\emptyset = x_0 \stackrel{e_1}{\longrightarrow\!\!\!\subset} \dots \stackrel{e_n}{\longrightarrow\!\!\!\subset} x_n$$

(with $e_n = e$, $x_i \in \mathscr{C}(\mathsf{p}) \cap \mathscr{C}(\mathsf{q})$) which is not in general minimal.

First, we notice that we necessarily have $[e]_{\mathsf{p}\wedge\mathsf{q}} \subseteq x_n$. More generally, any $x_i$ is down-closed for $\leq_{\mathsf{p}\wedge\mathsf{q}}$. This is obvious from the fact that as configurations of both $\mathsf{p}$ and $\mathsf{q}$ they are donw-closed for $\leq_{\mathsf{p}}$ and $\leq_{\mathsf{q}}$, so they are down-closed for the transitive closure of their union as well. Since $e \in x_n$, it follows that $[e]_{\mathsf{p}\wedge\mathsf{q}}$. We do not necessarily have $x_n = [e]_{\mathsf{p}\wedge\mathsf{q}}$ though, $x_n$ may comprise more events.

But for all $0 \leq i \leq n$, we observe that

$$x'_i = x_i \cap [e]_{\mathsf{p}\wedge\mathsf{q}} \in \mathscr{C}(\mathsf{p}) \cap \mathscr{C}(\mathsf{q})$$

Let us argue why $x'_i \in \mathscr{C}(\mathsf{p})$. We need to prove that it is down-closed. Let $e' \in x'_i$, and $e'' \leq_{\mathsf{p}} e'$. But then $e' \in [e]_{\mathsf{p}\wedge\mathsf{q}}$, and by definition we have $e'' \leq_{\mathsf{p}\wedge\mathsf{q}} e'$, so $e'' \in [e]_{\mathsf{p}\wedge\mathsf{q}}$ as well. Of course we also have $e'' \in x_i$ since $x_i \in \mathscr{C}(\mathsf{p})$, so $e'' \in x'_i$ which down-closed; hence $x'_i \in \mathscr{C}(\mathsf{p})$. The same holds for $\mathscr{C}(\mathsf{q})$, so $x'_i \in \mathscr{C}(\mathsf{p}) \cap \mathscr{C}(\mathsf{q})$.

Hence, the covering chain above restricts to:

$$\emptyset = x'_0, \dots, x'_n$$

where for all $0 \leq i \leq n-1$, $x'_i = x'_{i+1}$ or $x'_i \stackrel{e_i}{\longrightarrow\!\!\!\subset} x'_{i+1}$ with $e_i \in [e]_{\mathsf{p}\wedge\mathsf{q}}$. From our first observation (that $[e]_{\mathsf{p}\wedge\mathsf{q}} \subseteq x_n$) we have that $x'_n = [e]_{\mathsf{p}\wedge\mathsf{q}}$, yielding a minimal chain as required. ◀

▶ **Lemma 50.** *Let $e, e' \in |\mathsf{p} \wedge \mathsf{q}|$. Then $e \leq_{\mathsf{p}\wedge\mathsf{q}} e'$ iff $e$ appears in any covering chain leading to $e'$ – more formally, iff for all covering chain*

$$\emptyset = x_0 \overset{e_1}{\longrightarrow\!\!\!\subset} \ldots \overset{e_n}{\longrightarrow\!\!\!\subset} x_n$$

*where $e_n = e'$ and for all $0 \leq i \leq n$, $x_i \in \mathscr{C}(\mathsf{p}) \cap \mathscr{C}(\mathsf{q})$, there exists $1 \leq j \leq n$ such that $e = e_j$.*

**Proof.** *If.* Immediate consequence of Lemma 49.

*Only if.* Since $x_n \in \mathscr{C}(\mathsf{p}) \cap \mathscr{C}(\mathsf{q})$ it is down-closed for $\leq_{\mathsf{p}}$ and $\leq_{\mathsf{q}}$, hence for $\leq_{\mathsf{p}\wedge\mathsf{q}}$ as well. Hence $e \in x_n$, so there must be $1 \leq i \leq n$ such that $e_i = e$. ◀

Using this, we can prove:

▶ **Lemma 51.** *For any $\mathsf{q}, \mathsf{p}$, $\leq_{\mathsf{p}\wedge\mathsf{q}}$ is an ees.*

**Proof.** Reflexivity and transitivity are trivial. For antisymmetry, assume $e \leq_{\mathsf{p}\wedge\mathsf{q}} e'$ and $e' \leq_{\mathsf{p}\wedge\mathsf{q}} e$. Take a covering chain for $e'$:

$$\emptyset = x_0 \overset{e_1}{\longrightarrow\!\!\!\subset} \ldots \overset{e_n}{\longrightarrow\!\!\!\subset} x_n$$

with $e_n = e'$. Since $e \leq_{\mathsf{p}\wedge\mathsf{q}} e'$, by Lemma 50 there is $1 \leq j \leq n$ such that $e_j = e$. But then $x_0 \longrightarrow\!\!\!\subset \ldots \longrightarrow\!\!\!\subset x_j$ is a covering chain for $e$, so since $e' \leq_{\mathsf{p}\wedge\mathsf{q}}$ by Lemma 50 again there is $1 \leq k \leq j$ such that $e_k = e'$. But that implies that $k = n$ with $k \leq j \leq n$, so $k = j$ and $e = e'$.

It only remains to prove that for all $e \in |\mathsf{p} \wedge \mathsf{q}|$, $[e]_{\mathsf{p}\wedge\mathsf{q}}$ is finite. But we have already observed that for any covering chain $x_0 \longrightarrow\!\!\!\subset \ldots \longrightarrow\!\!\!\subset x_n$ for $e$, we have $[e]_{\mathsf{p}\wedge\mathsf{q}} \subseteq x_n$, so it must indeed be finite. ◀

Finally, we prove:

▶ Proposition 52. For any $\mathsf{p}, \mathsf{q}$ ees, $\mathsf{p} \wedge \mathsf{q}$ is the greatest lower bound (for $\preccurlyeq$) of $\mathsf{p}$ and $\mathsf{q}$.

**Proof.** By construction, it is obvious that $\mathsf{p} \wedge \mathsf{q} \preccurlyeq \mathsf{p}$ and $\mathsf{p} \wedge \mathsf{q} \preccurlyeq \mathsf{q}$. Assume that we have $\mathsf{r}$ an ees such that $\mathsf{r} \preccurlyeq \mathsf{p}$ and $\mathsf{r} \preccurlyeq \mathsf{q}$.

First, we clearly have $|\mathsf{r}| \subseteq |\mathsf{p}| \cap |\mathsf{q}|$. To make sure that $|\mathsf{r}| \subseteq |\mathsf{p} \wedge \mathsf{q}|$, we need to make sure that any $e \in |\mathsf{r}|$ is secured, *i.e.* has a covering chain. Consider

$$\emptyset = x_0 \overset{e_1}{\longrightarrow\!\!\!\subset} \ldots \overset{e_n}{\longrightarrow\!\!\!\subset} x_n$$

any covering chain of $[e]_{\mathsf{r}}$ in $\mathsf{r}$, *i.e.* $x_n = [e]_{\mathsf{r}}$, $e_n = e$ and for any $0 \leq i \leq n$, $x_i \in \mathscr{C}(\mathsf{r})$. Clearly such a covering chain exists, following any linearization of $\leq_{\mathsf{r}}$ on $[e]_{\mathsf{r}}$. But by hypothesis we have $x_i \in \mathscr{C}(\mathsf{p}) \cap \mathscr{C}(\mathsf{q})$ for all $0 \leq i \leq n$, so this is actually a covering chain for $e$ witnessing its securedness. Hence, $|\mathsf{r}| \subseteq |\mathsf{p} \wedge \mathsf{q}|$.

We also need to prove that $\mathscr{C}(\mathsf{r}) \subseteq \mathscr{C}(\mathsf{p} \wedge \mathsf{q})$. Take $x \in \mathscr{C}(\mathsf{r})$. By hypothesis, $x \in \mathscr{C}(\mathsf{p})$ and $x \in \mathscr{C}(\mathsf{q})$. That means that $x$ is down-closed for both $\leq_{\mathsf{p}}$ and $\leq_{\mathsf{q}}$, so it is down-closed for the transitive closure of their union $\leq_{\mathsf{p}\wedge\mathsf{q}}$; hence $x \in \mathscr{C}(\mathsf{p} \wedge \mathsf{q})$ as required. ◀

It will be convenient in the future to have the following characterization of configurations of the meet.

▶ **Lemma 53.** *Let* $\mathsf{q}, \mathsf{q}'$ *be two ees. The configurations of* $\mathsf{q} \wedge \mathsf{q}'$ *are exactly those* $x \in \mathscr{C}(\mathsf{q}) \cap \mathscr{C}(\mathsf{q}')$ *such that there is a* ***covering chain***

$$\emptyset = x_0 {-\!\!\subset} x_1 {-\!\!\subset} \ldots {-\!\!\subset} x_n = x$$

*where, for all* $0 \leq i \leq n$, $x_i \in \mathscr{C}(\mathsf{q}) \cap \mathscr{C}(\mathsf{q}')$ *– the events of* $\mathsf{q} \wedge \mathsf{q}'$ *are exactly those that appear in a* $x \in \mathscr{C}(\mathsf{q}) \cap \mathscr{C}(\mathsf{q}')$ *reachable in this way.*

**Proof.** Straightforward. ◀

## A.3 Arenas and strategies

Now, we define our notions of games and strategies. For now we are only concerned about the causal aspect, we leave for later the composition of the term annotations.

### A.3.1 Basic definitions

First, we introduce our notion of games.

▶ **Definition 54.** An **uncovered arena** is $A = (|A|, \leq_A, \mathrm{pol}_A)$ with $(|A|, \leq_A)$ an ees, and

$$\mathrm{pol}_A : |A| \to \{\exists, 0, \forall\}$$

a **polarity function**. An event labelled 0 is **neutral** and we additionally require that neutral events are incomparable with non-neutral events. Finally $A$ is an **arena** if it has no neutral events – we will sometimes say *uncovered* arena to give particular emphasis the absence of neutral events.

In the paper, the polarities $\forall$ and $\exists$ will be used sometimes interchangeably with, respectively, $-$ and $+$. We will rarely consider uncovered arenas – only when defining composition and proving its properties. In particular, the objects of our category will be normal arenas with no neutral events. When introducing an event in an (uncovered) arena $A$, we might annotate it with $\forall, 0$ or $\exists$ to convey information on its polarity. The arenas interpreting formulas are always forest-shaped – but we omit this assumption in the definitions as it is not actually needed in the technical development.

▶ **Definition 55.** A **(uncovered) strategy** on (uncovered) arena $A$ is an ees $\sigma = (|\sigma|, \leq_\sigma)$ such that $\sigma \preccurlyeq A$, and verifying the two following additional conditions.

- *Receptivity.* If $x \in \mathscr{C}(\sigma)$ and $x \overset{a^\forall}{-\!\!\!\subset}$ in $A$ (meaning $x \cup \{a\} \in \mathscr{C}(A)$), then $a \in |\sigma|$.
- *Courtesy.* If $a_1 \rightarrowtriangle_\sigma a_2$ and ($\mathrm{pol}_A(a_1) = \exists$ or $\mathrm{pol}_A(a_2) = \forall$), then $a_1 \rightarrowtriangle_A a_2$ as well.

We write $\sigma : A$ to mean that $\sigma$ is a strategy on arena $A$.

One can regard $\sigma : A$ as a concurrent strategy in the usual sense through the identity-on-events map of event structures $\mathrm{id} : \sigma \to A$. The conditions above almost exactly match the standard one, except receptivity which is slightly "optimized". For completeness, the lemma below relates the receptivity condition above to the usual one.

▶ **Lemma 56.** *Take* $\sigma \preccurlyeq A$ *satisfying courtesy. Then, it is receptive iff for all* $x \in \mathscr{C}(A)$ *such that* $x \overset{a^\forall}{-\!\!\!\subset}$ *in* $A$, $x \cup \{a\} \in \mathscr{C}(A)$.

**Proof.** *If.* Obvious.

*Only if.* If $x \overset{a^\forall}{-\!\!\!\subset}$, then by receptivity we know that $a \in |\sigma|$. Take $a' \rightarrowtriangle_\sigma a$. By courtesy, we have $a' \rightarrowtriangle_A a$ as well. Since $x \cup \{a\} \in \mathscr{C}(A)$, then we know that for all $a' \rightarrowtriangle_A a$, we have $a' \in x$. Therefore, for all $a' \rightarrowtriangle_\sigma a$, we have $a' \in x$. It follows that $x \cup \{a\} \in \mathscr{C}(\sigma)$ as well. ◀

### A.3.2 Constructions on ees / arenas

We recall briefly the standard operations on arenas.

▶ **Definition 57.** If $A$ is an arena, its **dual** $A^\perp$ has events $|A^\perp| = |A|$ and causality $\leq_{A^\perp} = \leq_A$, but polarity $\mathrm{pol}_{A^\perp}$ reversed (neutral events remain neutral). We write $A^0$ for the uncovered arena with events and causality that of $A$, and polarities all neutral.

▶ **Definition 58.** If $\mathsf{q}_1, \mathsf{q}_2$ are ees, their **parallel composition** $q_1 \parallel q_2$ has components:

- *Events,* $|\mathsf{q}_1 \parallel \mathsf{q}_2| = \{1\} \times |\mathsf{q}_1| \uplus \{2\} \times |\mathsf{q}_2|$
- *Causality,* given by $(i, e) \leq_{\mathsf{q}_1 \parallel \mathsf{q}_2} (i', e')$ iff $i = j$ and $e \leq_{\mathsf{q}_i} e'$.

If $A_1, A_2$ are (uncovered) arenas, their parallel composition additionally has polarities

$$\mathrm{pol}_{A_1 \parallel A_2}((i, a)) = \mathrm{pol}_{A_i}(a).$$

Note that configurations of a parallel composition $\mathsf{q}_1 \parallel \mathsf{q}_2$ have the form $\{1\} \times x_1 \cup \{2\} \times x_2$, which we will write $x_1 \parallel x_2 \in \mathscr{C}(\mathsf{q}_1 \parallel \mathsf{q}_2)$. With this notation, configurations of $\mathsf{q}_1 \parallel \mathsf{q}_2$ are exactly those of the form $x_1 \parallel x_2$, with $x_1 \in \mathscr{C}(\mathsf{q}_1)$ and $x_2 \in \mathscr{C}(\mathsf{q}_2)$. We prove in passing the following lemma, stating compatibility between the meet and parallel composition.

▶ **Lemma 59.** *Let* $\mathsf{q}_1, \mathsf{q}_2, \mathsf{p}_1, \mathsf{p}_2$ *be ees. Then, we have:*

$$(\mathsf{q}_1 \parallel \mathsf{q}_2) \wedge (\mathsf{p}_1 \parallel \mathsf{p}_2) = (\mathsf{q}_1 \wedge \mathsf{p}_1) \parallel (\mathsf{q}_2 \wedge \mathsf{p}_2)$$

**Proof.** We show that $(\mathsf{q}_1 \wedge \mathsf{p}_1) \parallel (\mathsf{q}_2 \wedge \mathsf{p}_2)$ is a greatest lower bound of $\mathsf{q}_1 \parallel \mathsf{q}_2$ and $\mathsf{p}_1 \parallel \mathsf{p}_2$. First, it is a lower bound: indeed, if $x \parallel y \in \mathscr{C}((\mathsf{q}_1 \wedge \mathsf{p}_1) \parallel (\mathsf{q}_2 \wedge \mathsf{p}_2))$, we have $x \in \mathscr{C}(\mathsf{q}_1 \wedge \mathsf{p}_1)$ and $y \in \mathscr{C}(\mathsf{q}_2 \wedge \mathsf{p}_2)$. In particular, $x \in \mathscr{C}(\mathsf{q}_1) \cap \mathscr{C}(\mathsf{p}_1)$ and $y \in \mathscr{C}(\mathsf{q}_2) \cap \mathscr{C}(\mathsf{p}_2)$. It follows that $x \parallel y \in \mathscr{C}(\mathsf{q}_1 \parallel \mathsf{q}_2)$ and $x \parallel y \in \mathscr{C}(\mathsf{p}_1 \parallel \mathsf{p}_2)$ as required.

Finally, it is the greatest lower bound. Let $\mathsf{q}' \preccurlyeq \mathsf{q}_1 \parallel \mathsf{q}_2, \mathsf{p}_1 \parallel \mathsf{p}_2$. In particular, $\mathscr{C}(\mathsf{q}') \subseteq \mathscr{C}(\mathsf{q}_1 \parallel \mathsf{q}_2)$, so they have the form $x \parallel y$ such that $x \in \mathscr{C}(\mathsf{q}_1) \cap \mathscr{C}(\mathsf{p}_1)$ and $y \in \mathscr{C}(\mathsf{q}_2) \cap \mathscr{C}(\mathsf{p}_2)$. But the same reasoning holds for any covering chain of $x \parallel y$ in $\mathscr{C}(\mathsf{q}')$, yielding by projection covering chains for $x$ in $\mathscr{C}(\mathsf{q}_1) \cap \mathscr{C}(\mathsf{p}_1)$ and for $y$ in $\mathscr{C}(\mathsf{q}_2) \cap \mathscr{C}(\mathsf{p}_2)$. Hence, by Lemma 53, $x \in \mathscr{C}(\mathsf{q}_1 \wedge \mathsf{p}_1)$ and $y \in \mathscr{C}(\mathsf{q}_2 \wedge \mathsf{p}_2)$. ◀

As usual, (uncovered) strategies **from** $A$ **to** $B$ will be (uncovered) strategies on the compound game $A^\perp \parallel B$; so we will also write $\sigma : A \rightarrow\!\!\!\!\!+ B$ for $\sigma : A^\perp \parallel B$.

### A.3.3 Strategies as families of configurations

To build strategies and reason on them, it will be convenient to characterise them in terms of the configurations they can reach, rather than in terms of their causal structure.

▶ Proposition 60. Consider $(|E|, \mathcal{E})$ where $|E|$ is a set of *events*, and $\mathcal{E}$ is a set of finite subsets of $|E|$ which is:

- *Covering:* for all $e \in E$, there is $x \in \mathcal{E}$ such that $e \in x$.
- *Consistent:* for all $x, y \in \mathcal{E}$, $x \cup y \in \mathcal{E}$.
- *Stable:* for all $x, y \in \mathcal{E}$, $x \cap y \in \mathcal{E}$.
- *Coincidence-free:* for all $x \in \mathcal{E}$ and $e_1, e_2 \in x$ distinct, there is $y \in \mathcal{E}$ such that

$$e_1 \in y \Leftrightarrow e_2 \notin y$$

Let us call a **configuration-ees** a pair $(|E|, \mathcal{E})$ that satisfies these axioms. First, for any $\mathsf{q}$ an ees, $(|\mathsf{q}|, \mathscr{C}(\mathsf{q}))$ is a configuration-ees.

Second, for any configuration-ees $(|E|, \mathcal{E})$, there exists a unique causal order $\leq_E$ making $(|E|, \leq_E)$ an ees such that $\mathcal{E} = \mathscr{C}(E)$.

**Proof.** If $\mathsf{q}$ is an ees, that $(|\mathsf{q}|, \mathscr{C}(\mathsf{q}))$ is a configuration-ees is direct to check.

Now, consider $(|E|, \mathcal{E})$ a configuration-ees. We set $e_1 \leq_E e_2$ when for all $x \in \mathcal{E}$, if $e_2 \in x$ then $e_1 \in x$. First, it is a partial order. Reflexivity and transitivity are clear. For antisymmetry, assume $e_1 \leq_E e_2$ and $e_2 \leq_E e_1$. Assume that $e_1 \neq e_2$. Since $\mathcal{E}$ is covering, there is $x \in \mathcal{E}$ such that $e_2 \in x$. But then by coincidence-freeness, there is $y \in \mathcal{E}$ such that $e_1 \in y \Leftrightarrow e_2 \notin y$. *W.l.o.g.*, since $e_1 \leq_E e_2$, necessarily $e_1 \in y$ and $e_2 \notin y$. But since $e_2 \leq_E e_1$, we get a contradiction.

Now, to establish the remaining properties, we will make use of the following fact: for all $e \in |E|$, we have (the two inclusions are direct observations from the definitions):

$$[e]_E = \cap\{x \in \mathcal{E} \mid e \in x\}$$

That implies first that as needed, $[e]_E$ is always finite, concluding that $(|E|, \leq_E)$ is an ees. Now let us prove that its $\mathcal{E} = \mathscr{C}(E)$. Take $x \in \mathcal{E}$. Then it is down-closed for $\leq_E$: indeed if $e \in x$ and $e' \leq_E e$, then by definition $e' \in x$ as well. Reciprocally, assume that $x \in \mathscr{C}(E)$. But clearly we have (again, the two inclusions are obvious)

$$x = \cup\{[e]_E \mid e \in x\}$$

Since $[e]_E = \cap\{x \in \mathcal{E} \mid e \in x\}$, by stability $[e]_E \in \mathcal{E}$. Since $x = \cup\{[e]_E \mid e \in X\}$, we conclude by completeness that $x \in \mathcal{E}$ as required.

Finally, uniqueness comes from the observation that in any ees $(|\mathsf{q}|, \leq_\mathsf{q})$, for all $e_1, e_2 \in |\mathsf{q}|$, we have $e_1 \leq_\mathsf{q} e_2$ iff for all $x \in \mathscr{C}(\mathsf{q})$, if $e_2 \in x$ then $e_1 \in x$. The two directions are obvious. ◀

It will be convenient sometimes to use this proposition and define ees via their configurations, rather than directly via their causal order. In order to extend this to strategies we prove the following proposition.

▶ Proposition 61. Let $A$ be an (uncovered) arena. (Uncovered) strategies on $A$ are in one-to-one correspondance (through the constructions of Proposition 60) with configuration-ees $(|\sigma|, \boldsymbol{\sigma})$ with $|\sigma| \subseteq |A|$ and $\boldsymbol{\sigma} \subseteq \mathscr{C}(A)$, which additionally satisfy:

- *Receptivity:* for any $x \in \boldsymbol{\sigma}$, if $x \subseteq^- y$ then $y \in \boldsymbol{\sigma}$ as well.
- *Courtesy:* if $x \overset{a_1^+}{\longrightarrow\!\subset} \overset{a_2}{\longrightarrow\!\subset}$ in $\boldsymbol{\sigma}$ and $x \overset{a_2}{\longrightarrow\!\subset}$ in $\mathscr{C}(A)$, then $x \overset{a_2}{\longrightarrow\!\subset}$ in $\boldsymbol{\sigma}$ as well.

where $x \subseteq^- y$ (resp. $x \subseteq^+ y$) means that $x \subseteq y$ and $\mathrm{pol}_A(y \setminus x) \subseteq \{-\}$ (resp. $\mathrm{pol}_A(y \setminus x) \subseteq \{+\}$).

We say that such a configuration-ees is a **configuration-strategy**.

**Proof.** Firstly, it is trivial that if $\sigma : A$, then $\mathscr{C}(\sigma)$ satisfies receptivity above. For courtesy, assume that $x \overset{a_1^+}{\longrightarrow\!\subset} \overset{a_2}{\longrightarrow\!\subset}$ in $\mathscr{C}(\sigma)$ and $x \overset{a_2}{\longrightarrow\!\subset}$ in $\mathscr{C}(A)$. If $a_1 \leq_\sigma a_2$, then necessarily $a_1 \rightarrowtriangle_\sigma a_2$. But then by standard courtesy we have $a_1 \rightarrowtriangle_A a_2$ as well, contradicting $x \overset{a_2}{\longrightarrow\!\subset}$ in $\mathscr{C}(A)$.

Now, we need to check that if $(|\sigma|, \leq_\sigma)$ is an ees with $|\sigma| \subseteq |A|, \mathscr{C}(\sigma) \subseteq \mathscr{C}(A)$ satisfying the two additional conditions above, then it is a strategy. First, it satisfies receptivity by Lemma 56. For courtesy, assume that $a_1 \rightarrowtriangle_\sigma a_2$ with $\mathrm{pol}_A(a_1) = +$ or $\mathrm{pol}_A(a_2) = -$. If $a_1 \rightarrowtriangle_A a_2$ we are done, otherwise $a_1$ and $a_2$ are incomparable in $A$. Take $x = [a_2]_\sigma \setminus \{a_1, a_2\} \in \mathscr{C}(\sigma)$, then we have $x \stackrel{a_2}{\longrightarrow\!\!\!\subset}$ in $A$. If $\mathrm{pol}_A(a_2) = -$, then by receptivity we have $x \stackrel{a_2}{\longrightarrow\!\!\!\subset}$ in $\mathscr{C}(\sigma)$, contradicting $a_1 \twoheadrightarrow_\sigma a_2$. Otherwise, we have $\mathrm{pol}_A(a_2) = +$. Then, by courtesy we get that $x \stackrel{a_2}{\longrightarrow\!\!\!\subset}$ in $\mathscr{C}(\sigma)$ again, which is a contradiction. So, $a_1 \twoheadrightarrow_A a_2$, and we have proved that $\sigma$ is an (uncovered) strategy. ◀

Finally, we give a final characterisations of configuration-ees that correspond to strategies, which will be useful in the coming proofs. For that, we recall first the "Scott order" between configurations of an arena.

▶ **Lemma 62.** *Let $A$ be an arena. The **Scott order** on $\mathscr{C}(A)$ is the partial order defined by, for $x, y \in \mathscr{C}(A)$:*

$$x \sqsubseteq_A y \Leftrightarrow x \supseteq_A^- z \subseteq_A^+ y$$

*for some $z \in \mathscr{C}(A)$.*

**Proof.** First of all, we observe that in the definition, $z$ is necessarily $x \cap y$. Indeed clearly $z \subseteq x \cap y$, and to show that $x \cap y \subseteq z$, notice that for $a \in x \cap y$ either $\mathrm{pol}_A(a) = -$ and then $a \in z$ follows from $z \subseteq^+ y$, or $\mathrm{pol}_A(a) \neq -$ and $a \in z$ follows from $z \subseteq^- x$.

We need to check that it is a partial order. Reflexivity is trivial. For transitivity, assume that $x \sqsubseteq_A y$ and $y \sqsubseteq_A z$. The situation is summarized by the following diagram:

$$x \supseteq^- x \cap y \subseteq^+ y \supseteq^- y \cap z \subseteq^+ z$$

But then, necessarily $x \cap y \cap z \in \mathscr{C}(A)$ – it is down-closed as the intersection of down-closed sets. But then it is easy to see that $x \cap y \cap z \subseteq^- x \cap y$, and $x \cap y \cap z \subseteq^+ y \cap z$. Finally, we have that $x \cap y \cap z = x \cap z$: indeed, if $a \in x \cap z$, then if $\mathrm{pol}_A(a) = -$ we deduce from $y \cap z \subseteq^+ z$ that $a \in y \cap z \subseteq y$, and symmetrically if $\mathrm{pol}_A(a) \neq -$. Transitivity follows. Finally, for antisymmetry, assume we have the following situation:

$$x \supseteq^- x \cap y \subseteq^+ y \supseteq^- y \cap x \subseteq^+ x$$

It is immediate that $y = x \cap y$, so $y \subseteq x$. Symmetrically, $x \subseteq y$ hence $x = y$. ◀

We now give our final characterization of configuration-strategies, given via a discrete fibration like property (as in [4]).

▶ Proposition 63. Let $(|\sigma|, \boldsymbol{\sigma})$ be a configuration-ees with $|\sigma| \subseteq |A|$ and $\boldsymbol{\sigma} \subseteq \mathscr{C}(A)$. Then it is a configuration-strategy iff for all $x \in \boldsymbol{\sigma}$, for all $y \sqsubseteq_A x$, then $y \in \boldsymbol{\sigma}$.

**Proof.** *If.* Receptivity is clear, since if $x \subseteq_A^- y$, then $y \sqsubseteq_A x$ by definition. For courtesy, assume $x \in \boldsymbol{\sigma}$ with $x \stackrel{a_1^+}{\longrightarrow\!\!\!\subset} \stackrel{a_2}{\longrightarrow\!\!\!\subset}$ in $\boldsymbol{\sigma}$ and $x \stackrel{a_2}{\longrightarrow\!\!\!\subset}$ in $\mathscr{C}(A)$. In particular, we have $x \cup \{a_2\} \subseteq_A^+ x \cup \{a_1, a_2\}$ with the latter in $\boldsymbol{\sigma}$. By hypothesis, it follows that $x \cup \{a_2\} \in \boldsymbol{\sigma}$ as required.

*Only if.* Assume that $y \in \boldsymbol{\sigma}$, with $x \sqsubseteq_A y$. As observed earlier, we necessarily have $x \supseteq_A^- x \cap y \subseteq_A^+ y$. If we can show that $x \cap y \in \boldsymbol{\sigma}$, then we are done by receptivity. So we only have to prove that for all $x \subseteq_A^+ y$, if $y \in \boldsymbol{\sigma}$, then $x \in \boldsymbol{\sigma}$ as well. It is in fact sufficient to prove the above if $x = y \cup \{a^+\}$, the many-step version will then follow by induction.

Hence, take $y \in \boldsymbol{\sigma}$ with $x \overset{a^+}{\longrightarrow\!\!\subset} y$. From the correspondence of Proposition 60 we know that $\boldsymbol{\sigma}$ is the set of configurations of a strategy $(|\sigma|, \leq_\sigma)$ which is courteous, hence if we had $a \rightarrowtriangle_\sigma a'$ for $a' \in y$ we would have $a \rightarrowtriangle_A a'$ as well, contradicting that $x \in \mathscr{C}(A)$. Hence no event depends on $a$ in $y$ for $\leq_\sigma$, so $x$ is down-closed, *i.e.* $x \in \boldsymbol{\sigma}$. ◀

### A.3.4 Copycat strategies

As a first example of a strategy, which will also play a major role in the rest of the development, we introduce the copycat strategy. In fact we define with slightly more generality a notion of copycat-like strategies; those will later give the structural morphisms corresponding to the symmetric monoidal structure on the category of strategies.

First, we give a notion of map between arenas – note that this is an instance of the usual notion of map between event structures.

▶ **Definition 64.** Let $A, B$ be arenas. A **map of arenas** from $A$ to $B$ is a function on events $f : |A| \to |B|$ preserving polarity, such that for all $x \in \mathscr{C}(A)$, its direct image $f\,x \in \mathscr{C}(B)$ is a configuration as well; and which is *locally injective*, in the sense that for all $e_1, e_2 \in x \in \mathscr{C}(A)$, if $f\,e_1 = f\,e_2$ then $e_1 = e_2$.

An **isomorphism of arenas** is simply an iso in the category of arenas and maps between them. We write $f : A \cong B$ to denote that $f$ is an iso between $A$ and $B$.

Isomorphisms of arenas will be simply lifted to copycat strategies, that will eventually also be isos in the category of arenas and strategies between them that we aim to form. From $f : A \cong B$, we define a strategy $\mathit{c\!c}_f : A^\perp \parallel B$ (a *copycat strategy*) through its configurations, using the Scott order.

▶ Proposition 65. Let $f : A \cong B$. We define a family $\mathbb{C}_f$ (on events/polarity $A^\perp \parallel B$) comprising all $x_A \parallel x_B \in \mathscr{C}(A^\perp \parallel B)$ such that:

$$x_B \sqsubseteq_B f(x_A)$$

So defined this is a configuration-strategy, corresponding via Proposition 61 to a strategy $\mathit{c\!c}_f : A^\perp \parallel B$.

**Proof.** We check first the axioms of a configuration-ees.

*Covering.* Let $c \in A^\perp \parallel B$. Without loss of generality, assume that $c = (1, a)$ (the other case is symmetric). We have $[a]_A \in \mathscr{C}(A)$, by construction. Then, we set $x = [a] \parallel f\,[a]$. It is immediate by definition that $f\,[a] \sqsubseteq_B [a]$, so $x \in \mathbb{C}_f$ by definition; and $c \in x$ by construction.

*Consistent.* Let $x_A \parallel x_B \in \mathbb{C}_f$, and $y_A \parallel y_B \in \mathbb{C}_f$. Then, by consistency on $A$ and $B$, $x_A \cup y_A \in \mathscr{C}(A)$ and $x_B \cup y_B \in \mathscr{C}(B)$. Moreover, we have:

$$x_B \cup y_B \sqsubseteq_B f\,x_A \cup f\,y_A$$

as follows immediately from $x_B \sqsubseteq_B f\,x_A$ and $y_B \sqsubseteq_B f\,y_A$. Hence, we have as required $x_A \cup y_A \parallel x_B \parallel y_B \in \mathbb{C}_f$.

*Stable.* Same reasoning.

*Coincidence-free.* Let $x_A \parallel x_B \in \mathbb{C}_f$, and $c, c' \in x_A \parallel x_B$. If both $c, c'$ are on the same side, say wlog $c = (1, a)$ and $c' = (1, a')$ with $a, a' \in x_A$, then there is $x'_A \in \mathscr{C}(A)$ with $x'_A \subseteq x_A$ and $a \in x'_A \Leftrightarrow a \notin x'_A$. But then we have $x'_A \parallel f\,x'_A \cap x_B \in \mathbb{C}_f$, and it separates $c$ and $c'$. From now on, we can therefore assume wlog that $c = (1, a)$ and $c' = (2, b)$ with $a \in x_A$ and $b \in x_B$. If $f\,a \notin f\,x_A \cap x_B$ or $b \notin f\,x_A \cap x_B$, assume wlog it is the second. Then $x_A \parallel f\,x_A \cap x_B \in \mathbb{C}_f$ and contains $(1, a)$ but not $(2, b)$. So, the only case left is when

$f\,a, b \in f\,x_A \cap x_B$. But then there is $y \in \mathscr{C}(B)$ such that $y \subseteq f\,x_A \cap x_B$ and $f\,a \in y \Leftrightarrow b \notin y$. Then, $f^{-1}\,y \parallel y \in \mathbb{C}_f$ and separates $(1, a)$ and $(2, b)$.

Now, we check that it is a configuration-strategy.

*Receptivity.* Take $x_A \parallel x_B \in \mathbb{C}_f$ such that $x_A \parallel x_B \subseteq^{-}_{A^\perp \parallel B} x'_A \parallel x'_B$. Then it is clear by construction that $x'_A \parallel x'_B \in \mathbb{C}_f$ as well.

*Courtesy.* Take $x_A \parallel x_B \in \mathbb{C}_f$, and assume that

$$x_A \parallel x_B \overset{c_1^+}{\longrightarrow\!\subset} \overset{c_2^+}{\longrightarrow\!\subset}$$

Where $(x_A \parallel x_B) \cup \{c_2\} \in \mathscr{C}(A^\perp \parallel B)$. Assume wlog that $c_2 = (2, b)$; so that $x_B \longrightarrow\!\subset b^+$. But necessarily $b = f\,a$ for some $a \in x_A$ (by definition of $\mathbb{C}_f$ that is the case for $(x_A \parallel x_B) \cup \{c_1, c_2\}$, but $a$ cannot match $c_1$ for polarity reasons. Hence, $x_A \parallel (x_B \cup \{b\}) \in \mathbb{C}_f$ as well, concluding the proof. ◀

Though defining copycat strategies as above via their configurations rather than concretely as an ees makes for smoother proofs, it will be also convenient to have a concrete understanding of immediate causality in $\mathfrak{c}_f$.

▶ Proposition 66. Let $f : A \cong B$ be an iso. Then, the immediate causal links of $\mathfrak{c}_f$ are exactly those of the form:

$$\begin{array}{rcll} (1, a) & \rightarrowtriangle_{\mathfrak{c}_f} & (2, f\,a) & (\mathrm{pol}_A(a) = +) \\ (2, f\,a) & \rightarrowtriangle_{\mathfrak{c}_f} & (1, a) & (\mathrm{pol}_A(a) = -) \\ (1, a_1) & \rightarrowtriangle_{\mathfrak{c}_f} & (1, a_2) & (a_1^- \rightarrowtriangle_A a_2^+) \\ (2, b_1) & \rightarrowtriangle_{\mathfrak{c}_f} & (2, b_2) & (b_1^+ \rightarrowtriangle_B b_2^-) \end{array}$$

**Proof.** Recall from Proposition 60 that $\leq_{\mathfrak{c}_f}$ is defined as

$$c_1 \leq_{\mathfrak{c}_f} c_2 \quad \Leftrightarrow \quad \forall x \in \mathbb{C}_f,\ c_2 \in x \implies c_1 \in x$$

It is a direct verification from this definition that the immediate causal links proposed are strict inequalities. We detail the first case: take $x_A \parallel x_B \in \mathbb{C}_f$, and assume that $f\,a \in x_B$ with $\mathrm{pol}_A(a) = +$. By definition we have

$$x_B \sqsubseteq_B f\,x_A.$$

Since $\mathrm{pol}_B(f\,a) = +$, by definition of the Scott order we have $f\,a \in f\,x_A$ as well, hence $a \in x_A$. The second case is symmetric, and the two other cases are obvious as the pairs proposed are already in strict inequality in the game.

Now, we need to also check that those are immediate causal links, *i.e.* that there are no events in between. Given a configuration-ees $\mathcal{A}$, given $a_1 <_A a_2$, so as to establish $a_1 \rightarrowtriangle_A a_2$ it suffices to establish that there is $x \in \mathcal{A}$ such that $x \overset{a_1}{\longrightarrow\!\subset} \overset{a_2}{\longrightarrow\!\subset}$. For the first case, we have

$$[a)_A \parallel f\,[a)_A \overset{(1,a)}{\longrightarrow\!\subset} [a]_A \parallel f\,[a)_A \overset{(2,f\,a)}{\longrightarrow\!\subset} [a]_A \parallel f\,[a]_A$$

where it is easy to check that all those are in $\mathbb{C}_f$. The second case is symmetric. For the third case we first note that $x = [a_2)_A \setminus \{a_1\} \in \mathscr{C}(A)$. Indeed if it was not down-closed, there would be $a \in [a_2)_A$ such that $a_1 <_A a$, with then $a_1 <_A a <_A a_2$, contradicting $a_1 \rightarrowtriangle_A a_2$. Hence we have the chain:

$$x \parallel f\,[a_2)_A \overset{(1,a_1)}{\longrightarrow\!\subset} [a_2)_A \parallel f\,[a_2)_A \overset{(1,a_2)}{\longrightarrow\!\subset} [a_2]_A \parallel f\,[a_2)_A$$

where it is immediate from the definition that each of those are in $\mathbb{C}_f$. ◀

This way we recover the standard definition of copycat, which is usually via the transitive closure of the causality relation above.

## A.4 A category of strategies

Now, we give the definition of the composition of strategies, and we prove that this gives a category.

The recipe for composition – *parallel interaction* followed by *hiding* – is usual in game semantics: given $\sigma : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$, first embed them both in $A \parallel B \parallel C$ (ignoring polarities), let them interact by constructing the "least causal agreement" between the two strategies, and then hide away the synchronized events (*i.e.* those in $B$). We will first detail the composition, and then the hiding.

### A.4.1 Interaction

First, we define the *interaction* of two strategies.

▶ **Definition 67.** Let $\sigma : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$ be uncovered strategies, where $B$ is an arena (*i.e.* no neutral events). Then, the **interaction** $\tau \circledast \sigma$ is defined as the meet:

$$\tau \circledast \sigma = (\sigma \parallel C) \wedge (A \parallel \tau)$$

Note that for now, this operation is only well-defined as yielding an ees: $\sigma, C, A, \tau$ are all ees, parallel composition is defined on ees, and the meet yields an ees.

There is a slight abuse of notation here: there is an implicit renaming (to a ternary parallel composition) going on to ensure that $\sigma \parallel C$ and $A \parallel \tau$ both have events a subset of those of $A \parallel B \parallel C$. We keep this renaming implicit (as is often done in game semantics) because having it explicit overloads notations with no gain in clarity, but one must be aware that it is there.

Interaction between uncovered strategies always yields an uncovered strategy:

▶ Proposition 68. If $\sigma : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$ are as in Definition 67 above, then $\tau \circledast \sigma : A^\perp \parallel B^0 \parallel C$ is an uncovered strategy.

**Proof.** By definition of the meet, $|\tau \circledast \sigma| \subseteq |A^\perp \parallel B^0 \parallel C|$. By Lemma 53, its configurations are those sets of events $x_A \parallel x_B \parallel x_C$ of $A \parallel B \parallel C$ such that there is a chain:

$$\emptyset = x_A^0 \parallel x_B^0 \parallel x_C^0 \quad -\!\!\subset \quad x_A^1 \parallel x_B^1 \parallel x_C^1 \quad -\!\!\subset \quad \dots \quad -\!\!\subset \quad x_A^n \parallel x_B^n \parallel x_C^n$$

such that $x_A^n \parallel x_B^n \parallel x_C^n = x_A \parallel x_B \parallel x_C$, for all $0 \leq i \leq n$, $x_A^i \parallel x_B^i \in \mathscr{C}(\sigma)$ and $x_B^i \parallel x_C^i \in \mathscr{C}(\tau)$. By construction, we know it is an ees. To show that it is an uncovered strategy, we use the characterizations of receptivity and courtesy of Proposition 61.

*Receptivity.* Consider $x_A \parallel x_B \parallel x_C \in \mathscr{C}(\tau \circledast \sigma)$ with a covering chain as above, with $x_A \parallel x_B \parallel x_C \overset{e^-}{-\!\!\subset}$ in $A^\perp \parallel B^0 \parallel C$. Necessarily, $e$ is in $A$ or $C$. Wlog, assume it is in $C$. Hence $x_B \parallel x_C \in \mathscr{C}(\tau)$ and $x_C \overset{c^-}{-\!\!\subset} x'_C$ in $C$. By receptivity, $x_B \parallel x'_C \in \mathscr{C}(\tau)$ as well. Appending this to the covering chain, we get a witness that $x_A \parallel x_B \parallel x'_C \in \mathscr{C}(\tau \circledast \sigma)$.

*Courtesy.* Consider $x_A \parallel x_B \parallel x_C$ with a covering chain as above, and

$$x_A \parallel x_B \parallel x_C \quad \overset{e^+}{-\!\!\subset} \quad x'_A \parallel x'_B \parallel x'_C \quad \overset{e'}{-\!\!\subset} \quad x''_A \parallel x''_B \parallel x''_C$$

where the polarity of $e$ is taken in $A^\perp \parallel B^0 \parallel C$. Necessarily $e$ is in $A$ or $C$, wlog let us say it is in $C$. If $e'$ is in $B$ or $C$, then the (end of the) covering chain above can be rewritten as

$$x_A \parallel x_B \parallel x_C \quad \overset{e^+}{-\!\!\subset} \quad x_A \parallel x_B \parallel x'_C \quad \overset{e'}{-\!\!\subset} \quad x_A \parallel x'_B \parallel x''_C.$$

But then, by courtesy for $\tau$, it follows that $x_B \parallel x_C \overset{e'}{\longrightarrow\!\!\!\!\subset} x'_B \parallel y_C \overset{e}{\longrightarrow\!\!\!\!\subset} x'_B \parallel x''_C$ in $\mathscr{C}(\tau)$ as well, meaning that

$$x_A \parallel x_B \parallel x_C \quad \overset{e'}{\longrightarrow\!\!\!\!\subset} \quad x_A \parallel x'_B \parallel y_C$$

completes the covering chain of $x_A \parallel x_B \parallel x_C$ so as to justify that $x_A \parallel x'_B \parallel x_C$ extends via $e'$ in $\mathscr{C}(\tau \circledast \sigma)$, as required. If $e'$ is in $A$, then similarly $x_A \parallel x_B \parallel x_C \overset{e'}{\longrightarrow\!\!\!\!\subset}$ (with no need of courtesy from either $\sigma$ or $\tau$). ◀

Of course this operation is not the definition notion of composition of strategies: it lacks hiding. Without hiding, we would not get a category of games, as in particular copycat would not be idempotent. However, it will be important for the sequel to note that already, without hiding, we have an associative notion of composition.

To make this precise, we need however to precise something. By an uncovered strategy **from** $A$ **to** $B$, where $A$ and $B$ are covered arenas, we mean an uncovered strategy $\sigma : A^\perp \parallel N^0 \parallel B$ for some entirely neutral arena $N^0$.

▶ Proposition 69. Interaction of (uncovered) strategies is associative.

**Proof.** Let $\sigma : A^\perp \parallel N^0 \parallel B$, $\tau : B^\perp \parallel M^0 \parallel C$ and $\delta : C^\perp \parallel P^0 \parallel D$.

The uncovered strategies $\sigma$ and $\tau$ interact on $B$, so their interaction is defined as the meet $(\sigma \parallel M \parallel C) \wedge (A \parallel N \parallel \tau)$. By Lemma 59, we have:

$$(\tau \circledast \sigma) \parallel P \parallel D = (\sigma \parallel M \parallel C \parallel P \parallel D) \wedge (A \parallel N \parallel \tau \parallel P \parallel D).$$

Now, we note that

$$\begin{array}{rcl} \delta \circledast (\tau \circledast \sigma) & = & (A \parallel N \parallel B \parallel M \parallel \delta) \wedge ((\tau \circledast \sigma) \parallel P \parallel D) \\ & = & (A \parallel N \parallel B \parallel M \parallel \delta) \wedge ((\sigma \parallel M \parallel C \parallel P \parallel D) \\ & & \wedge (A \parallel N \parallel \tau \parallel P \parallel D)) \end{array}$$

the meet is a least upper bound (Proposition 52) so is associative, hence the dual reasoning implies equality with $(\delta \circledast \tau) \circledast \sigma$. ◀

Since $\delta \circledast (\tau \circledast \sigma) = (\delta \circledast \tau) \circledast \sigma$, we can – and will – write unambiguously $\delta \circledast \tau \circledast \sigma$ for either of them.

## A.4.2 Hiding

Now, we introduce the second phase of composition: *hiding*. It will be presented as an operation which takes an uncovered strategy, and produces a covered strategy by removing neutral events. First, we define the hiding of an uncovered game.

▶ **Definition 70.** Let $A$ be an uncovered game. Its **hiding**, written $A_\downarrow$, has components:

$$\begin{array}{rcl} |A_\downarrow| & = & |A| \cap \{a \in A \mid \mathrm{pol}_A(a) \neq 0\} \\ \leq_{A_\downarrow} & = & \leq_A \cap |A_\downarrow|^2 \end{array}$$

It is direct to prove that this still defines a game. Similarly, we define the hiding of an uncovered strategy.

▶ **Definition 71.** Let $\sigma : A$ be an uncovered strategy on an uncovered game $A$. Then we define $\sigma_\downarrow$ as having components:

$$\begin{array}{rcl} |\sigma_\downarrow| & = & |\sigma| \cap |A_\downarrow| \\ \leq_{\sigma_\downarrow} & = & \leq_\sigma \cap |A_\downarrow|^2 \end{array}$$

which is, as is simply to check, an ees.

So we simply ignore the neutral events, which are thought of still occuring silently in the background. For now $\sigma_\downarrow$ is only defined as an ees – but in the sequel we will see that it is indeed a (covered) strategy on $A_\downarrow$. First, we describe its configurations.

▶ **Lemma 72.** *Let $\sigma : A$ be an uncovered strategy. Then, we have:*

$$\mathscr{C}(\sigma_\downarrow) = \{x \cap |A_\downarrow| \mid x \in \mathscr{C}(A)\}$$

*Moreover, for any $x \in \mathscr{C}(\sigma_\downarrow)$, there is a unique minimal* $\mathrm{wit}(x) \in \mathscr{C}(\sigma)$*, called the* ***witness*** *of $x$, s.t.* $\mathrm{wit}_\sigma(x) \cap |A_\downarrow| = x$.

**Proof.** First, we prove the characterization of $\mathscr{C}(\sigma_\downarrow)$. If $x \in \mathscr{C}(\sigma_\downarrow)$, then in particular $x \subseteq |A|$. It follows that $[x]_A \in \mathscr{C}(A)$. Clearly, $x \subseteq [x]_A \cap |\sigma_\downarrow|$. In the other direction, take $a' \in [x]_A \cap |\sigma_\downarrow|$. By definition, there is $a \in x$ such that $a' \leq_\sigma a$. But $a' \in |\sigma_\downarrow|$, so by definition $a' \leq_{\sigma_\downarrow} a$ as well, so $a' \in x$. Likewise, for $x \in \mathscr{C}(\sigma)$, we have $x \cap |\sigma_\downarrow| \in \mathscr{C}(\sigma_\downarrow)$: if $a \in x \cap |\sigma_\downarrow|$ with $a' \leq_{\sigma_\downarrow} a$, then $a' \leq_\sigma a$ as well, so $a' \in x$.

For $x \in \mathscr{C}(\sigma_\downarrow)$, we define $\mathrm{wit}_\sigma(x) = [x]_\sigma$. As observed above it is indeed a witness. For minimality, observe that if $y \in \mathscr{C}(\sigma)$ such that $y \cap |\sigma_\downarrow| = x$, then obviously $x \subseteq y$, hence since $y \in \mathscr{C}(\sigma)$ is down-closed we have $[x]_\sigma \subseteq y$ as well. ◀

We now need to check that so defined, the hiding of an uncovered strategy is, as expected, a strategy.

▶ Proposition 73. If $\sigma : A$ is an uncovered strategy on an uncovered game $A$, then $\sigma_\downarrow : A_\downarrow$ is a covered strategy.

**Proof.** We already know that $\sigma_\downarrow$ is an ees, it remains to prove that it is receptive and courteous. For that, we are going to use the characterization of those in Proposition 61 and that of configurations of $\sigma_\downarrow$ in Lemma 72.

*Receptivity.* Assume $x \in \mathscr{C}(\sigma_\downarrow)$, and $x \overset{a^-}{\longrightarrow\!\!\subset}$ in $\mathscr{C}(A_\downarrow)$. But then $[x]_\sigma \in \mathscr{C}(\sigma)$, with also $[x]_\sigma \in \mathscr{C}(A)$ since $\sigma : A$. Moreover, we still have $[x]_\sigma \overset{a^-}{\longrightarrow\!\!\subset}$ in $\mathscr{C}(A)$: indeed, by definition of arenas $a^-$ only depends on visible events, which are all in $x$, hence in $[x]_\sigma$. By receptivity of $\sigma$, we have $[x]_\sigma \cup \{a\} \in \mathscr{C}(\sigma)$ as well, hence $x \cup \{a\} \in \mathscr{C}(\sigma_\downarrow)$ as required.

*Courtesy.* Assume that $x \overset{a_1^+}{\longrightarrow\!\!\subset} \overset{a_2}{\longrightarrow\!\!\subset}$ in $\mathscr{C}(\sigma_\downarrow)$, with $x \overset{a_2}{\longrightarrow\!\!\subset}$ in $\mathscr{C}(A_\downarrow)$. Then, we have in $\sigma$ a covering chain

$$[x]_\sigma \quad \overset{e_1^0}{\longrightarrow\!\!\subset} \quad \dots \quad \overset{e_n^0}{\longrightarrow\!\!\subset} \quad \overset{a_1^+}{\longrightarrow\!\!\subset} \quad \overset{e_{n+1}^0}{\longrightarrow\!\!\subset} \quad \dots \quad \overset{e_{n+p}^0}{\longrightarrow\!\!\subset} \quad \overset{a_2}{\longrightarrow\!\!\subset}$$

obtained by adding to $[x]_\sigma$ (a linearization of) the dependencies of $a_1^+$ not in $[x]_\sigma$ (which are necessarily neutral), and then adding (a linearization of) the dependencies of $a_2$ not in $[x]_\sigma \cup [a_1]_\sigma$ (again, necessarily neutral). By $p+1$ uses of the characterization of courtesy in Proposition 61 for $\sigma$, it follows that the following is also a covering chain in $\sigma$

$$[x]_\sigma \quad \overset{e_1^0}{\longrightarrow\!\!\subset} \quad \dots \quad \overset{e_n^0}{\longrightarrow\!\!\subset} \quad \overset{e_{n+1}^0}{\longrightarrow\!\!\subset} \quad \dots \quad \overset{e_{n+p}^0}{\longrightarrow\!\!\subset} \quad \overset{a_2}{\longrightarrow\!\!\subset} \quad \overset{a_1^+}{\longrightarrow\!\!\subset},$$

hence $\sigma_\downarrow \overset{a_2}{\longrightarrow\!\!\subset}$ as needed. ◀

We are finally in position to define the composition of covered strategies.

▶ **Definition 74.** Let $\sigma : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$ be two covered strategies. Then we define:

$$\tau \odot \sigma = (\tau \circledast \sigma)_\downarrow$$

which is a covered strategies by Propositions 68 and 73.

Before going on to prove associativity, we establish the following lemma on hiding, which is going to be useful later on.

▶ **Lemma 75.** *Let $\sigma : A$ be an uncovered strategy, and take $x \longrightarrow\!\!\!\!\subset x'$ in $\mathscr{C}(\sigma_\downarrow)$. Then, there exists a chain*

$$\mathrm{wit}_\sigma(x) = y_0 \;\; \longrightarrow\!\!\!\!\subset \;\; \dots \;\; \longrightarrow\!\!\!\!\subset \;\; y_n = \mathrm{wit}_\sigma(x')$$

*in $\mathscr{C}(\sigma)$.*

**Proof.** First of all, remark that $\mathrm{wit}_\sigma(x) \subseteq \mathrm{wit}_\sigma(x')$. Indeed, otherwise $\mathrm{wit}_\sigma(x) \cap \mathrm{wit}_\sigma(x') \in \mathscr{C}(\sigma)$ (as configurations are stable under intersections) is strictly included in $\mathrm{wit}_\sigma(x)$, but its visible events are those of $x$, contradicting minimality of $\mathrm{wit}_\sigma(x)$ as a witness for $x$.

Since $\mathrm{wit}_\sigma(x') \in \mathscr{C}(\sigma)$, it has a covering chain

$$\emptyset = y_0 \;\; \longrightarrow\!\!\!\!\subset \;\; \dots \;\; \longrightarrow\!\!\!\!\subset \;\; y_p = \mathrm{wit}_\sigma(x')$$

where for all $0 \leq i \leq p$, $y_i \in \mathscr{C}(\sigma)$. But configurations are stable under unions (consistency), therefore taking the union of all elements of the chain with $\mathrm{wit}_\sigma(x) \subseteq \mathrm{wit}_\sigma(x')$, we get a chain

$$\mathrm{wit}_\sigma(x) = y'_0 \;\; \longrightarrow\!\!\!\!\subset^\bullet \;\; \dots \;\; \longrightarrow\!\!\!\!\subset^\bullet \;\; y'_p = \mathrm{wit}_\sigma(x')$$

in $\mathscr{C}(\sigma)$ where $x \longrightarrow\!\!\!\!\subset^\bullet y$ means that $x \longrightarrow\!\!\!\!\subset y$ or $x = y$. Simplifying this chain taking only the progressing steps, we get

$$\mathrm{wit}_\sigma(x) = x_0 \;\; \longrightarrow\!\!\!\!\subset \;\; \dots \;\; \longrightarrow\!\!\!\!\subset \;\; x_n = \mathrm{wit}_\sigma(x')$$

as required. ◀

### A.4.3 Associativity

Toward establishing the categorical structure of strategies, we show that the notion of composition of covered strategies we just introduced is associative. This relies on the lemma below.

▶ **Lemma 76.** *Let $\sigma : A^\perp \parallel M^0 \parallel B$ and $\tau : B^\perp \parallel N^0 \parallel C$ be two uncovered strategies (with $B$ a covered arena). Then,*

$$(\tau \circledast \sigma)_\downarrow = (\tau_\downarrow \circledast \sigma_\downarrow)_\downarrow : A^\perp \parallel C$$

**Proof.** We rely on Proposition 61 and show that these two covered strategies have the same configurations, so the proof boils down to the two inclusions.

$\subseteq$. Let $x_A \parallel x_C \in (\tau \circledast \sigma)_\downarrow$. By Lemma 72 there is a unique minimal

$$\mathrm{wit}_{\tau\circledast\sigma}(x_A \parallel x_C) = x_A \parallel x_M \parallel x_B \parallel x_N \parallel x_C \in \mathscr{C}(\tau \circledast \sigma)$$

We show that then, $x_A \parallel x_B \parallel x_C \in \mathscr{C}(\tau_\downarrow \circledast \sigma_\downarrow)$. Take a covering chain for $x = x_A \parallel x_M \parallel x_B \parallel x_N \parallel x_C \in \mathscr{C}(\tau \circledast \sigma)$, *i.e.* a chain

$$(x^i)_{0 \leq i \leq n} = (x^i_A \parallel x^i_M \parallel x^i_B \parallel x^i_N \parallel x^i_C)_{0 \leq i \leq n}$$

with $x_i \longrightarrow\!\!\!\!\subset x_{i+1}$ (for $0 \leq i \leq n-1$), $x^0 = \emptyset$, $x_n = x$, and for all $0 \leq i \leq n$, $x^i_A \parallel x^i_M \parallel x^i_B \in \mathscr{C}(\sigma)$ and $x^i_B \parallel x^i_N \parallel x^i_C \in \mathscr{C}(\tau)$. The existence of this chain is garanteed by Lemma 53. Projecting this chain to $A, B, C$ we get:

$$y^i = (x^i_A \parallel x^i_B \parallel x^i_C)_{0 \leq i \leq n}$$

where for all $0 \leq i \leq n$, $x^i_A \parallel x^i_B \in \mathscr{C}(\sigma)$ and $x^i_B \parallel x^i_C \in \mathscr{C}(\tau)$, and for all $0 \leq i \leq n-1$, $y^i \longrightarrow\!\!\!\!\subset^{\bullet} y^{i+1}$, meaning either $y^i = y^{i+1}$ or $y^i \longrightarrow\!\!\!\!\subset y^{i+1}$. Removing duplicates, we obtain a covering chain for $x_A \parallel x_B \parallel x_C$ satisfying the required conditions to testify (following Lemma 53) that $x_A \parallel x_B \parallel x_C \in \mathscr{C}(\tau_\downarrow \circledast \sigma_\downarrow)$. Therefore, $x_A \parallel x_C \in \mathscr{C}((\tau_\downarrow \circledast \sigma_\downarrow)_\downarrow)$.

$\supseteq$. Let $x_A \parallel x_C \in \mathscr{C}((\tau_\downarrow \circledast \sigma_\downarrow)_\downarrow)$. By Lemma 72 there is a unique minimal

$$\mathrm{wit}_{\tau_\downarrow \circledast \sigma_\downarrow}(x_A \parallel x_C) = x_A \parallel x_B \parallel x_C \in \mathscr{C}(\tau_\downarrow \circledast \sigma_\downarrow)$$

By Lemma 53, that means that there is a chain

$$(x^i)_{0 \leq i \leq n} = (x^i_A \parallel x^i_B \parallel x^i_C)_{0 \leq i \leq n}$$

such that $x^0 = \emptyset$, $x^n = x_A \parallel x_B \parallel x_C$, for all $0 \leq i \leq n-1$, $x^i \longrightarrow\!\!\!\!\subset x^{i+1}$ and for all $0 \leq i \leq n$, $x^i_A \parallel x^i_B \in \mathscr{C}(\sigma_\downarrow)$ and $x^i_B \parallel x^i_C \in \mathscr{C}(\tau_\downarrow)$. For each $0 \leq i \leq n$, we observe

$$x^i_A \parallel x^i_M \parallel x^i_B \parallel x^i_N \parallel x^i_C \in \mathscr{C}(\sigma \parallel N \parallel C) \cap \mathscr{C}(A \parallel M \parallel \tau)$$

where $x^i_A \parallel x^i_M \parallel x^i_B = \mathrm{wit}_\sigma(x^i_A \parallel x^i_B)$ and $x^i_B \parallel x^i_N \parallel x^i_C = \mathrm{wit}_\tau(x^i_B \parallel x^i_C)$. It remains to complete this into a covering chain, *i.e.*, for each $0 \leq i \leq n$, provide:

$$x^i_A \parallel x^i_M \parallel x^i_B \parallel x^i_N \parallel x^i_C \;\longrightarrow\!\!\!\!\subset\; \dots \;\longrightarrow\!\!\!\!\subset\; x^{i+1}_A \parallel x^{i+1}_M \parallel x^{i+1}_B \parallel x^{i+1}_N \parallel x^{i+1}_C$$

in $\mathscr{C}(\sigma \parallel N \parallel C) \cap \mathscr{C}(A \parallel M \parallel \tau)$. There are several cases, depending on the location of the extension $x^i \longrightarrow\!\!\!\!\subset x^{i+1}$. If it is in $A$ or $C$, say *w.l.o.g.* that it is in $C$. Then by Lemma 75, there is

$$x^i_B \parallel x^i_N \parallel x^i_C \;\longrightarrow\!\!\!\!\subset\; \dots \;\longrightarrow\!\!\!\!\subset\; x^{i+1}_B \parallel x^{i+1}_N \parallel x^{i+1}_C$$

where, by necessity, $x^i_B = x^{i+1}_B$. Taking the componentwise parallel composition with $x^i_A \parallel x^i_M$, we get the required chain segment. Suppose now that the extension $x^i \longrightarrow\!\!\!\!\subset x^{i+1}$ is in $B$; say *w.l.o.g.* that it is positive for $\tau$, *i.e.* has negative polarity in $B$. As just above, by Lemma 75 we get a chain segment

$$x^i_B \parallel x^i_N \parallel x^i_C \;\longrightarrow\!\!\!\!\subset\; \dots \;\longrightarrow\!\!\!\!\subset\; x^{i+1}_B \parallel x^{i+1}_N \parallel x^{i+1}_C$$

where, this time, $x^i_C = x^{i+1}_C$. Taking the componentwise parallel composition with $x^i_A \parallel x^i_M$, we get again the required chain segment – exploiting that $x^i_A \parallel x^i_M \parallel x^{i+1}_B \in \mathscr{C}(\sigma)$ by receptivity. Appending all these segments, we get a covering chain witnessing that $x_A \parallel x_M \parallel x_B \parallel x_N \parallel x_C \in \mathscr{C}(\tau \circledast \sigma)$, hence $x_A \parallel x_C \in \mathscr{C}((\tau \circledast \sigma)_\downarrow)$. ◀

From that, we can immediately deduce:

▶ Proposition 77. Composition of covered strategies is associative.

**Proof.** Consider (covered) strategies $\sigma : A^\perp \parallel B, \tau : B^\perp \parallel C$, and $\delta : C^\perp \parallel D$. We use the following equational reasoning:

$$\begin{aligned}
\delta \odot (\tau \odot \sigma) &=_1 (\delta \circledast (\tau \odot \sigma))_\downarrow \\
&=_2 (\delta_\downarrow \circledast (\tau \circledast \sigma)_\downarrow)_\downarrow \\
&=_3 (\delta \circledast (\tau \circledast \sigma))_\downarrow \\
&=_4 ((\delta \circledast \tau) \circledast \sigma)_\downarrow \\
&=_5 ((\delta \circledast \tau)_\downarrow \circledast \sigma_\downarrow)_\downarrow \\
&=_6 ((\delta \odot \tau) \circledast \sigma)_\downarrow \\
&=_7 ((\delta \odot \tau) \odot \sigma
\end{aligned}$$

where lines $1, 2, 6, 7$ are by definition of composition and the observation that for $\sigma$ covered, $\sigma_\downarrow = \sigma$; 3 and 5 are by Lemma 76; and 4 is by Proposition 69. ◀

Before we finally obtain a category, it remains to show that copycat is neutral for composition. In fact we will directly show a bit more: we will analyse in general the result of composing a strategy with a copycat strategy. This additional information will be useful later when constructing the compact closed structure of the category.

### A.4.4 Global and local renaming

It is convenient to introduce some tools for transporting ees and strategies along arena isomorphisms.

#### A.4.4.1 Global renaming.

First we introduced a global notion of renaming, defined at the level of ees.

▶ **Definition 78.** If $A$ is an arena, we say that an ees $\mathsf{q}$ is an **ees on** $A$ if $\mathscr{C}(\mathsf{q}) \subseteq \mathscr{C}(A)$ (however, $\mathsf{q}$ may fail receptivity or courtesy).

In constructing the compact closed structure of $\mathsf{Det}$, we will use the following *global renaming* operation on such a $\mathsf{q}$:

▶ **Definition 79.** Let $\mathsf{q}$ be an ees on arena $A$ and $f : A \cong A'$ be an arena isomorphism. Then, we define $f * \mathsf{q}$ as having events

$$|f * \mathsf{q}| = \{f\,e \mid e \in |\mathsf{q}|\}$$

and causality transported by $f$, *i.e.*

$$f\,e \leq_{f*\mathsf{q}} f\,e' \quad \Leftrightarrow \quad e \leq_{\mathsf{q}} e'$$

It is the **global renaming of** $\mathsf{q}$ **following** $f$, clearly an ees on $A'$.

From the definition, we immediately have:

▶ **Lemma 80.** *Let* $\mathsf{q}$ *be an ees on arena* $A$ *and* $f : A \cong A'$ *be an arena isomorphism. Then,*

$$\mathscr{C}(f * \mathsf{q}) = \{f\,x \mid x \in \mathscr{C}(\mathsf{q})\}$$

*where* $f\,x$ *is the direct image of* $x \in \mathscr{C}(\mathsf{q})$ *by* $f$.

**Proof.** Obvious. ◀

A direct observation is that the meet commutes with renaming.

▶ **Lemma 81.** *Let* $\mathsf{q}, \mathsf{q}'$ *be two ees on arena* $A$ *and* $f : A \cong A'$ *be an arena isomorphism. Then,*

$$(f * \mathsf{q}) \wedge (f * \mathsf{q}') = f * (\mathsf{q} \wedge \mathsf{q}')$$

**Proof.** Obvious by Lemma 53 – a securing chain can be transported either way by Lemma 80. ◀

#### A.4.4.2 Local renaming.

Now, we introduce a particular case of global renaming called *local renaming*, defined at the level of strategies from one arena into another.

▶ **Definition 82.** Let $\sigma : A^\perp \parallel B$ be a covered strategy. Let $f : A' \cong A$ and $g : B \cong B'$ be isomorphisms. Then, we define $g \cdot \sigma \cdot f : A'^\perp \parallel B'$ as the ees $(f^{-1} \parallel g) * \sigma$. It is obvious that this is indeed a strategy as $f$ and $g$ are order-isomorphisms and preserve polarities.

For $\sigma : A^\perp \parallel B$, we will also write $g \cdot \sigma$ for $g \cdot \sigma \cdot \mathrm{id}_A$ and $\sigma \cdot f$ for $\mathrm{id}_B \cdot \sigma \cdot f$. Clearly, $\mathrm{id}_B \cdot \sigma \cdot \mathrm{id}_A = \sigma$.

It is convenient to characterise the effect of renaming on configurations.

▶ **Lemma 83.** *Let* $\sigma : B^\perp \parallel C$ *be a strategy, and* $f : A \cong B, g : C \cong D$ *be isomorphisms of arenas. Then,*

$$\mathscr{C}(g \cdot \sigma \cdot f) = \{(f^{-1}\, x_B \parallel g\, x_C) \mid (x_B \parallel x_C) \in \mathscr{C}(\sigma)\}$$

**Proof.** Follows from Lemma 80. ◀

Renaming gives a sort of action of isomorphisms of arenas on strategies, as expressed and proved below.

▶ **Lemma 84.** *For* $\sigma : A_1^\perp \parallel B_1$ *a strategy and* $f_1 : A_1 \cong A_2, f_2 : A_2 \cong A_3, g_1 : B_1 \cong B_2, g_2 : B_2 \cong B_3$ *isomorphisms of arenas, we have:*

$$\begin{array}{rcl} \mathrm{id}_{B_1} \cdot \sigma \cdot \mathrm{id}_{A_1} & = & \sigma \\ g_2 \cdot (g_1 \cdot \sigma) & = & (g_2 \circ g_1) \cdot \sigma \\ (\sigma \cdot f_1) \cdot f_2 & = & \sigma \cdot (f_2 \circ f_1) \end{array}$$

**Proof.** Direct using Proposition 61 (in particular that strategies with the same configurations are equal) and Lemma 83. ◀

### A.4.5 Composition with copycat strategies

With that, we state and prove the following key lemma, expressing the result of composing a strategy with a copycat strategy.

▶ **Lemma 85.** *Let* $\sigma : A^\perp \parallel B$ *be a covered strategy, and* $g : B \cong B'$, $f : A \cong A'$ *be isomorphisms. Then,*

$$\begin{array}{rcl} \text{\ae}_g \odot \sigma & = & g \cdot \sigma \\ \sigma \odot \text{\ae}_f & = & \sigma \cdot f \end{array}$$

**Proof.** We prove that for $\sigma : A$ and $f : A \cong B$, $\text{æ}_f \odot \sigma = f \cdot \sigma$. The statement of the lemma follows from the same reasoning, with some slight purely notational complications. To prove that the two strategies are equal we show that they have the same configurations, exploiting Proposition 61.

Let $x \in \mathscr{C}(\text{æ}_f \odot \sigma)$. By definition of composition it has a witness $y \parallel x \in \mathscr{C}(\text{æ}_f \circledast \sigma)$, with $y \in \mathscr{C}(\sigma)$ and $y \parallel x \in \mathscr{C}(\text{æ}_f)$, *i.e.* by Proposition 65,

$$x \sqsubseteq_B f\, y$$

which, since $f$ is an order-isomorphism and preserves polarities, amounts to $f^{-1}\, x \sqsubseteq_A y$ with $y \in \mathscr{C}(\sigma)$. Now, by the characterisation of strategies in Proposition 63, it follows that $f^{-1}\, x \in \mathscr{C}(\sigma)$, so $x \in \mathscr{C}(f \cdot \sigma)$ by Lemma 83.

For the other inclusion, let $x \in \mathscr{C}(f \cdot \sigma)$. Then, we have

$$f^{-1}\, x \parallel x \in \mathscr{C}(\text{æ}_f \circledast \sigma).$$

Indeed $f^{-1}\, x \in \mathscr{C}(\sigma)$ (by Lemma 83) and $f^{-1}\, x \parallel x \in \mathscr{C}(\text{æ}_f)$, but we also need to construct a covering chain (following Lemma 53). One can easily be constructed by taking any covering chain $\emptyset = x_0 {-\!\!\subset} \dots {-\!\!\subset} x_n = x$ for $x$ in $\sigma$: for each $0 \leq i \leq n$, we have that $f^{-1}\, x_i \in \mathscr{C}(\sigma)$ and $f^{-1}\, x_i \parallel x_i \in \mathscr{C}(\text{æ}_f)$. If $x_{i+1} = x_i \cup \{a\}$ with $\mathrm{pol}_A(a) = -$, then

$$f^{-1}\, x_i \parallel x_i \;\; {-\!\!\subset} \;\; f^{-1}\, x_i \parallel x_{i+1} \;\; {-\!\!\subset} \;\; f^{-1}\, x_{i+1} \parallel x_{i+1}$$

is such that $f^{-1}\, x_i \in \mathscr{C}(\sigma)$ and $f^{-1}\, x_i \parallel x_{i+1} \in \mathscr{C}(\text{æ}_f)$ as well. Symmetrically if $\mathrm{pol}_A(a) = +$, then $f^{-1}\, x_{i+1} \parallel x_i$ provides an intermediate step. Concatenating all those together, we get a covering chain for $f^{-1}\, x \parallel x \in \mathscr{C}(\text{æ}_f \circledast \sigma)$. Hence, $x \in \mathscr{C}(\text{æ}_f \odot \sigma)$. ◀

For $A$ a covered arena, write $\text{æ}_A : A^\perp \parallel A$ for the copycat strategy obtained by lifting the identity isomorphism, *i.e.* $\text{æ}_A = \text{æ}_{\mathrm{id}_A}$. From all of the above, we immediately deduce:

▶ **Corollary 86.** *There is a category* Det *with covered arenas as objects, strategies $\sigma : A^\perp \parallel B$ as morphisms from $A$ to $B$, and $\text{æ}_A : A^\perp \parallel A$ as identities.*

**Proof.** We know that composition is associative by Proposition 77. Finally, for $\sigma : A^\perp \parallel B$, we have $\text{æ}_B \odot \sigma = \mathrm{id}_B \cdot \sigma = \sigma$ where the first equality follows from Lemma 85, and the second by definition. Likewise $\sigma \odot \text{æ}_A = \sigma$, and we have a category. ◀

## A.5 Compact closed structure

We have constructed above the category Det of games and strategies. Before going on to equip Det with term annotations, we will first investigate here its further structure – we will show in praticular that it is compact closed.

### A.5.1 Symmetric monoidal structure.

#### A.5.1.1 Tensor.

First of all, we construct a bifunctor

$$- \otimes - \;: \mathsf{Det} \times \mathsf{Det} \to \mathsf{Det}$$

▶ **Definition 87** (Tensor of arenas). Let $A, B$ be two arenas. Their **tensor** $A \otimes B$ is simply defined as $A \parallel B$.

Even though $A \otimes B$ is simply defined as $A \parallel B$, we find it useful to have introduce the distinct notation $\otimes$ for it in order to emphasize the role it plays in the category, and to disambiguate the functorial action of the tensor with the parallel composition of partial orders.

We now introduce the bifunctorial action of the tensor.

▶ Proposition 88. Let $\sigma_1 : A_1^\perp \parallel B_1$ and $\sigma_2 : A_2^\perp \parallel B_2$ be strategies. We define:

$$\sigma_1 \otimes \sigma_2 = \gamma * (\sigma_1 \parallel \sigma_2) : (A_1 \otimes A_2)^\perp \parallel (B_1 \otimes B_2)$$

with $\gamma : (A_1^\perp \parallel B_1) \parallel (A_2^\perp \parallel B_2) \to (A_1^\perp \parallel A_2^\perp) \parallel (B_1 \parallel B_2)$ the obvious isomorphism. This is a strategy, and its configurations are exactly those

$$(x_{A_1} \parallel x_{A_2}) \parallel (x_{B_1} \parallel x_{B_2}) \in \mathscr{C}((A_1 \parallel A_2)^\perp \parallel (B_1 \parallel B_2))$$

such that $x_{A_1} \parallel x_{B_1} \in \mathscr{C}(\sigma_1)$ and $x_{A_2} \parallel x_{B_2} \in \mathscr{C}(\sigma_2)$.

**Proof.** Obvious by construction and Lemma 80. ◀

Towards functoriality, we show first that the tensor preserves copycat.

▶ **Lemma 89.** *Let $A, B$ be arenas. Then, $\text{æ}_{A\otimes B} = \text{æ}_A \otimes \text{æ}_B$.*

**Proof.** We use Proposition 61 and show that the two strategies have the same configurations. By Proposition 65, configurations of $\text{æ}_{A\otimes B}$ are those

$$(x_A^l \parallel x_B^l) \parallel (x_A^r \parallel x_B^r) \in \mathscr{C}((A \parallel B)^\perp \parallel (A \parallel B))$$

such that $x_A^r \parallel x_B^r \sqsubseteq_{A\parallel B} x_A^l \parallel x_B^l$. But by definition of the Scott order, this is equivalent to $x_A^r \sqsubseteq_A x_A^l$ and $x_B^r \sqsubseteq_B x_B^l$, which by Proposition 88 corresponds to configurations of $\text{æ}_A \otimes \text{æ}_B$. ◀

Finally, we prove bifunctoriality.

▶ Proposition 90. Let $\sigma_1 : A_1^\perp \parallel B_1, \sigma_2 : A_2^\perp \parallel B_2, \tau_1 : B_1^\perp \parallel C_1$ and $\tau_2 : B_2^\perp \parallel C_2$. Then,

$$(\tau_1 \odot \sigma_1) \otimes (\tau_2 \odot \sigma_2) = (\tau_1 \otimes \tau_2) \odot (\sigma_1 \otimes \sigma_2)$$

**Proof.** We give a direct equational proof at the level of interactions. In the proof below we overload the symbol $\gamma$ to denote each time the canonical isomorphisms of arenas whose precise identity can be uniquely recovered from the context.

$$\begin{aligned}
& (\tau_1 \otimes \tau_2) \circledast (\sigma_1 \otimes \sigma_2) \\
= \quad & (\gamma * (\sigma_1 \parallel \sigma_2) \parallel (C_1 \parallel C_2)) \wedge ((A_1 \parallel A_2) \parallel \gamma * (\tau_1 \parallel \tau_2)) \\
= \quad & (\gamma * ((\sigma_1 \parallel C_1) \parallel (\sigma_2 \parallel C_2))) \wedge (\gamma * ((A_1 \parallel \tau_1) \parallel (A_2 \parallel \tau_2))) \\
= \quad & \gamma * (((\sigma_1 \parallel C_1) \parallel (\sigma_2 \parallel C_2)) \wedge ((A_1 \parallel \tau_1) \parallel (A_2 \parallel \tau_2)) \\
= \quad & \gamma * (((\sigma_1 \parallel C_1) \wedge (A_1 \parallel \tau_1)) \parallel ((\sigma_2 \parallel C_2) \wedge (A_2 \parallel \tau_2))) \\
= \quad & \gamma * ((\tau_1 \circledast \sigma_1) \parallel (\tau_2 \circledast \sigma_2))
\end{aligned}$$

where equalities follow in order by unfolding definitions, by definition of renaming (note that the symbol $\gamma$ does not denote the same isomorphism), by Lemma 81, by Lemma 59, and finally by definition of interactions. From the equality we established, the one we seek follows by hiding. ◀

#### A.5.1.2 Symmetric monoidal structure.

To construct a symmetric monoidal category, we additionally need to describe the structural morphisms, show the necessary naturality and coherence conditions.

First of all, observe that the category IsoAr having arenas as objects and isomorphisms between them as morphisms already has a symmetric monoidal structure. The tensor operation on arenas is defined as above, and it extends to maps in the obvious way. There is an empty arena 1. Furthermore, for all arenas $A, B$ and $C$ we have isomorphisms of arenas as below

$$
\begin{array}{rcrcl}
\rho_A & : & A \otimes 1 & \cong & A \\
\lambda_A & : & 1 \otimes A & \cong & A \\
s_{A,B} & : & A \otimes B & \cong & B \otimes A \\
\alpha_{A,B,C} & : & (A \otimes B) \otimes C & \cong & A \otimes (B \otimes C)
\end{array}
$$

which are natural in $A, B$ and $C$; and satisfy the coherence conditions required of the structural maps of a symmetric monoidal category. These isomorphisms can be easily lifted to copycat strategies:

$$
\begin{array}{rcrcl}
\text{æ}_{\rho_A} & : & (A \otimes 1)^\perp & \parallel & A \\
\text{æ}_{\lambda_A} & : & (1 \otimes A)^\perp & \parallel & A \\
\text{æ}_{s_{A,B}} & : & (A \otimes B)^\perp & \parallel & B \otimes A \\
\text{æ}_{\alpha_{A,B,C}} & : & ((A \otimes B) \otimes C)^\perp & \parallel & A \otimes (B \otimes C)
\end{array}
$$

From now on, to alleviate notations we will write simply $\rho_A$ for $\text{æ}_{\rho_A}$, $\lambda_A$ for $\text{æ}_{\lambda_A}$, and so on; as long as it is unambiguous. It remains to prove that these *structural strategies* satisfy the necessary coherence and naturality conditions. Coherence (*e.g.* Mac Lane's pentagon along with the triangle identities and that $s_{A,B}$ and $s_{B,A}$ are inverses) will simply follow from coherence in the category of arenas and isomorphisms using this lemma.

▶ **Lemma 91.** *The operation which to an arena $A$ associates $A$ and to an isomorphism $f : A \cong B$ associates $\text{æ}_f$ extends to a functor:*

$$\text{æ}_{\_} : \mathsf{IsoAr} \to \mathsf{Det}$$

**Proof.** Preservation of identities is by definition. For preservation of composition, take $f : A \cong B$ and $g : B \cong C$. We calculate:

$$
\begin{array}{rcl}
\text{æ}_g \odot \text{æ}_f & = & g \cdot \text{æ}_f \\
& = & g \cdot (\text{æ}_f \odot \text{æ}_A) \\
& = & g \cdot (f \cdot \text{æ}_A) \\
& = & (g \circ f) \cdot \text{æ}_A \\
& = & \text{æ}_{g \circ f} \odot \text{æ}_A \\
& = & \text{æ}_{g \circ f}
\end{array}
$$

where the first equality is by Lemma 85, the second by Corollary 86, the third by Lemma 85 again, the fourth by Lemma 84, and we conclude by Lemma 85 and Corollary 86 again. ◀

Hence, all coherence laws follow immediately from those in IsoAr. Finally, we have to prove that the structural strategies are natural.

▶ **Lemma 92.** *The families of strategies $\lambda_A, \rho_A, s_{A,B}$ and $\alpha_{A,B,C}$ are natural in $A, B, C$.*

**Proof.** For all those families, naturality follows directly from Lemma 85 and definition of the structural strategies. Below we only show naturality for $s_{A,B}$, the other cases following in the same way.

Let $\sigma : A_1^\perp \parallel A_2$ and $\tau : B_1^\perp \parallel B_2$ be two strategies. We first note that:

$$s_{A_2,B_2} \cdot (\sigma \otimes \tau) = (\tau \otimes \sigma) \cdot s_{A_1,B_1}$$

as can be established directly via Lemma 83 and Proposition 88. But then the required naturality square follows by Lemma 85. ◀

We have finished the proof of:

▶ Proposition 93. The tuple $(\mathsf{Det}, \otimes, 1)$ is a symmetric monoidal category.

## A.5.2 Compact closure.

Finally, we show that Det is compact closed. First we define the unit and co-unit.

▶ **Definition 94.** Let $A$ be an arena. Then we have two strategies:

$$\eta_A : 1^\perp \parallel (A^\perp \otimes A) \qquad \epsilon_A : (A \otimes A^\perp)^\perp \parallel 1$$

defined as renamings of $\text{æ}_A : A^\perp \parallel A$.

We can finally state and prove the main result of this section.

▶ Proposition 95. The category Det is compact closed.

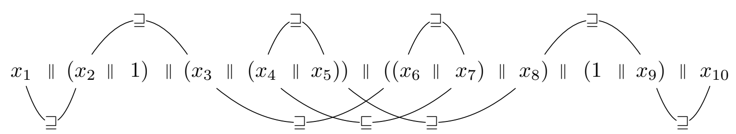
**Proof.** We only have to check the two equations for duals in a compact closed category, *i.e.* for any arena $A$, we have:

$$\begin{aligned} \text{æ}_A &= \lambda_A \odot (\epsilon_A \otimes \text{æ}_A) \odot \alpha^{-1}_{A,A^\perp,A} \odot (\text{æ}_A \otimes \eta_A) \odot \rho_A^{-1} \\ \text{æ}_{A^\perp} &= \rho_{A^\perp} \odot (\text{æ}_{A^\perp} \otimes \epsilon_A) \odot \alpha_{A^\perp,A,A^\perp} \odot (\eta_A \otimes \text{æ}_{A^\perp}) \odot \lambda^{-1}_{A^\perp} \end{aligned}$$

We focus on the first, as the two equations are similar. By repeated applications of Lemma 76, the right hand side of the equation is equal to

$$(\lambda_A \circledast (\epsilon_A \otimes \text{æ}_A) \circledast \alpha^{-1}_{A,A^\perp,A} \circledast (\text{æ}_A \otimes \eta_A) \circledast \rho_A^{-1})_\downarrow$$

Those are all copycat strategies – unfolding the interaction, and using the caracterisation of configurations of copycat strategies in Proposition 65, we get that the configurations of $\lambda_A \circledast (\epsilon_A \otimes \text{æ}_A) \circledast \alpha^{-1}_{A,A^\perp,A} \circledast (\text{æ}_A \otimes \eta_A) \circledast \rho_A^{-1}$ are exactly those of the form below:



where for each $i$, $x_i \in \mathscr{C}(A)$, related by the constraints pictured. But by Lemma 62, the Scott order is a partial order, therefore its projection $x_{10} \sqsubseteq x_1$ and $x_1 \parallel x_{10} \in \mathscr{C}(\text{æ}_A)$. Reciprocally, for any $x_1 \parallel x_{10} \in \mathscr{C}(\text{æ}_A)$, it is easy to construct a witness as above, establishing the equation. ◀

This concludes the construction of the compact closed category Det, which will serve as carrying the causal backbone (the acyclicity witnesses) for our formulation as strategies of expansion trees. In the next section, we enrich this category with the term assignments used to compute the Herbrand witnesses.

## A.6 Further structure of Det

In this subsection, we introduce some further structure of Det that will be helpful in giving the interpretation of the sequent calculus and proving some of its properties.

### A.6.1 Order-enrichment

Here, we prove – though it is really more of an observation – that Det is actually an order-enriched category: each homset is partially ordered by $\preccurlyeq$ and all our operations on strategies (composition and tensor) are compatible with it.

▶ Proposition 96. Det is an order-enriched category, where for any two arenas $A$ and $B$, $\mathsf{Det}(A, B)$ is the partial order of strategies from $A$ to $B$, ordered by $\preccurlyeq$.

**Proof.** We know that $\preccurlyeq$ is a partial order on ees, so it specializes into a partial order on strategies.

First, we prove that composition of strategies is compatible with $\preccurlyeq$. Take $A, B$ and $C$ arenas, and strategies $\sigma_1 \preccurlyeq \sigma_2 : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$ strategies; and $x_A \parallel x_C \in \mathscr{C}(\tau \odot \sigma_3)$. Consider its witness

$$x_A \parallel x_B \parallel x_C \in \mathscr{C}(\tau \circledast \sigma_1).$$

By Lemma 53 and the fact that $\mathscr{C}(\sigma_1) \subseteq \mathscr{C}(\sigma_2)$, it is then direct that $x_A \parallel x_B \parallel x_C \in \mathscr{C}(\tau \circledast \sigma_2)$ as well, hence $x_A \parallel x_C \in \mathscr{C}(\tau \odot \sigma_2)$. The argument is clearly symmetric, so composition preserves the order in its two components.

If $\sigma_1 \preccurlyeq \sigma_2 : A$ and $\tau : B$, then it is immediate from the definition that $\sigma_1 \parallel \tau \preccurlyeq \sigma_2 \parallel \tau$. From that and the definition of tensor, it follows directly that the tensor of strategies preserves the order as well. ◀

Intuitively, $\sigma_1 \preccurlyeq \sigma_2$ means that $\sigma_1$ plays fewer moves than $\sigma_2$, with more constrained causality. This order enrichment will play a minor role in our interpretation – namely, we will use it to prove that the interpretation of first-order MLL, though it does not preserve cut elimination in general, does preserve it in a lax sense, following an enrichement with terms of the order above.

### A.6.2 Functorial shifts

Our course, a key component of our interpretation will be to show how formula constructors map to operations on arenas. In particular, since moves played by strategies are meant to correspond to quantifiers, we need already at this level to study the properties of the addition of a single move as a prefix to an arena. Following the literature, we refer to this operation as a *shift*.

#### A.6.2.1 Shifts on arenas.

First, we define the shift operations on arenas.

▶ **Definition 97.** Let $A$ be an arena. We define its **up-shift** $\uparrow A$ as the tuple $(|\uparrow A|, \leq_{\uparrow A}, \mathrm{pol}_{\uparrow A})$, where $|\uparrow A| = |A| \uplus \{\bullet\}$ (where, *w.l.o.g.* we assume that $\bullet \notin |A|$); $a_1 \leq_{\uparrow A} a_2$ iff $a_1 = \bullet$ or $a_1 \leq_A a_2$; and the polarity function is obtained by extending $\mathrm{pol}_A$ with $\mathrm{pol}_{\uparrow A}(\bullet) = -$.

Likewise, the **down-shift**, $\downarrow A$, is as $\uparrow A$ with the exception of $\mathrm{pol}_{\downarrow A}(\bullet) = +$.

Later on, the up-shift will be used to model the addition of a universal quantifier, whereas the down-shift will be used for existential quantifiers.

#### A.6.2.2 Shifts on strategies.

We now extend these two operations to two functors $\uparrow, \downarrow : \mathsf{Det} \to \mathsf{Det}$.

▶ **Lemma 98.** *Let $\sigma : A^\perp \parallel B$ be a strategy. Then, the set*

$$\mathscr{C}(\uparrow \sigma) = \left\{ x_{\uparrow A} \parallel x_{\uparrow B} \quad | \quad \begin{array}{l} x_A \parallel x_B \in \mathscr{C}(\sigma) \\ \bullet \in x_{\uparrow A} \implies \bullet \in x_{\downarrow B} \end{array} \right\}$$

*where $x_A = x_{\uparrow A} \cap |A|$ (and likewise for $x_B$) is a configuration-strategy. It corresponds to (via Proposition 61) a strategy:*

$$\uparrow \sigma : (\uparrow A)^\perp \parallel (\uparrow B)$$

*Symmetrically, we have $\downarrow \sigma : (\downarrow A)^\perp \parallel (\downarrow B)$.*

**Proof.** Direct verification. ◀

▶ Proposition 99. The construction above yields order-enriched functors:

$$\uparrow, \downarrow : \mathsf{Det} \to \mathsf{Det}$$

**Proof.** We prove it for $\uparrow$, the other case is symmetric. It is a direct verification that $\uparrow$ preserves copycat, and that it preserves the order. Let us show that it also preserves composition.

Let $x_{\uparrow A} \parallel x_{\uparrow C} \in \mathscr{C}(\uparrow(\tau \odot \sigma))$. Consider its projection $x_A \parallel x_C \in \mathscr{C}(\tau \odot \sigma)$, which has a witness $x_A \parallel x_B \parallel x_C \in \mathscr{C}(\tau \circledast \sigma)$ along with a covering chain:

$$x_A^0 \parallel x_B^0 \parallel x_C^0 {-\!\!\subset} \dots {-\!\!\subset} x_A^n \parallel x_B^n \parallel x_C^n$$

where $x_K^0 = \emptyset$ and $x_K^n = x_K$. This covering chain exists by Lemma 53. Assume first that $\bullet \in x_{\uparrow A}$. Then, it is immediate that

$$\begin{array}{rcl} \emptyset \parallel \emptyset \parallel \emptyset & {-\!\!\subset} & \emptyset \parallel \emptyset \parallel \{\bullet\} \\ & {-\!\!\subset} & \emptyset \parallel \{\bullet\} \parallel \{\bullet\} \\ & {-\!\!\subset} & \{\bullet\} \parallel \{\bullet\} \parallel \{\bullet\} \\ & {-\!\!\subset} & \{\bullet\} \cup x_A^1 \parallel \{\bullet\} \cup x_B^1 \parallel \{\bullet\} \cup x_C^1 \\ & {-\!\!\subset} & \dots \\ & {-\!\!\subset} & \{\bullet\} \cup x_A^n \parallel \{\bullet\} \cup x_B^n \parallel \{\bullet\} \cup x_C^n \end{array}$$

is a covering chain in $\uparrow \tau \circledast \uparrow \sigma$. But as $\bullet \in x_{\uparrow A}$, we actually have $x_{\uparrow A} = \{\bullet\} \cup x_A$. Thus $\bullet \in x_{\uparrow C}$ as well and necessarily $x_{\uparrow C} = \{\bullet\} \cup x_C$. So this establishes that $x_{\uparrow A} \parallel x_{\uparrow C} \in \uparrow \tau \odot \uparrow \sigma$. We can easily adapt the reasoning if $\bullet \notin x_{\uparrow A}$ (in which case $x_A$ is empty and the covering chain in the interaction need only reach $B$ and $C$).

Reciprocally, take $x = x_{\uparrow A} \parallel x_{\uparrow C} \in \uparrow \tau \odot \uparrow \sigma$, and its projection $x_A \parallel x_C$ to $A^\perp \parallel C$. By definition, there is $x_{\uparrow A} \parallel x_{\uparrow B} \parallel x_{\uparrow C} \in \uparrow \tau \circledast \uparrow \sigma$, along with a covering chain $\rho$. Projecting this configuration of the interaction to $A \parallel B \parallel C$ yields a covering chain for $x_A \parallel x_B \parallel x_C \in \tau \circledast \sigma$, so $x_A \parallel x_C \in \tau \odot \sigma$. Finally if $\bullet \in x_{\uparrow A}$ then necessarily $\bullet \in x_{\uparrow B}$ as well by definition of $\uparrow \sigma$, and likewise $\bullet \in x_{\uparrow C}$ by definition of $\uparrow \tau$, which concludes the proof that $x \in \uparrow(\tau \odot \sigma)$. ◀

### A.6.3 Countable tensor power

In this subsection, we introduce the construction which to any arena $A$ associates its countable tensor power $\otimes^\omega A$ – which has countably many copies of $A$ in parallel – along with its functorial action. This will serve as the basis for our two dual constructions on games $?A$ and $!A$ to come, used in defining the interpretation.

#### A.6.3.1 Action on arenas.

We first define the countable parallel composition $\|_\omega A$ as follows.

▶ **Definition 100.** If $\mathsf{q}$ is an ees, then $\|_\omega \mathsf{q} = (\mathbb{N} \times |\mathsf{q}|, \leq_{\|_\omega \mathsf{q}})$ with

$$(i, a_1) \leq_{\|_\omega \mathsf{q}} (j, a_2) \quad \Leftrightarrow \quad i = j \ \&\ a_1 \leq_{\mathsf{q}} a_2.$$

If $A$ is an arena, then we additionally have $\mathrm{pol}_{\|_\omega A}((i, a)) = \mathrm{pol}_A(a)$.

On objects, the action of our countable tensor power will simply be this countable parallel composition, *i.e.* $\otimes^\omega A = \|_\omega A$.

#### A.6.3.2 Action on strategies.

Now, we extend the definition above on arenas into a functor $\otimes^\omega : \mathsf{Det} \to \mathsf{Det}$.

▶ **Definition 101.** Let $\sigma : A^\perp \parallel B$. Then, we define

$$\otimes^\omega \sigma = \gamma * (\|_\omega \sigma) : (\|_\omega A)^\perp \parallel (\|_\omega B)$$

where $\gamma : \|_\omega (A^\perp \parallel B) \cong (\|_\omega A)^\perp \parallel (\|_\omega B)$ is the obvious isomorphism.

As expected, this operation on strategies is functorial.

▶ Proposition 102. The operation above yields an order-enriched functor:

$$\otimes^\omega : \mathsf{Det} \to \mathsf{Det}$$

**Proof.** Direct adaptation of the proof of Lemma 89 and Proposition 90. ◀

# B $\Sigma$-strategies

A **signature** is a pair $\Sigma = (\Sigma_f, \Sigma_p)$, with $\Sigma_f$ a countable set of **function symbols** ($\mathsf{f}, \mathsf{g}, \mathsf{h}$, *etc.* range over function symbols), and $\Sigma_p$ a countable set of **predicate symbols** ($\mathsf{P}, \mathsf{Q}$, *etc.* range over predicate symbols). There is an **arity function** $\mathrm{ar} : \Sigma_f \uplus \Sigma_p \to \mathbb{N}$ where $\uplus$ is the usual set-theoretic union, where the argument sets are disjoint.

If $\mathcal{V}$ is a set of **variable names**, we write $\mathsf{Tm}_\Sigma(\mathcal{V})$ for the set of first-order terms on $\Sigma$ with free variables in $\mathcal{V}$. We use variables $t, s, u, v, \ldots$ to range over terms. **Literals** have the form $\mathsf{P}(t_1, \ldots, t_n)$ or $\neg\mathsf{P}(t_1, \ldots, t_n)$, where $\mathsf{P}$ is a $n$-ary predicate symbol and the $t_i$s are terms. **Formulas** are also closed under quantifiers, and the connectives $\vee$ and $\wedge$. **Negation** is not considered a logical connective: the negation $\varphi^\perp$ of $\varphi$ is obtained by De Morgan rules. We write $\mathsf{Form}_\Sigma(\mathcal{V})$ for the set of **first-order formulas** on $\Sigma$ with free variables in $\mathcal{V}$, and use $\varphi, \psi, \ldots$ to range over them. We also write $\mathsf{QF}_\Sigma(\mathcal{V})$ for the set of **quantifier-free** formulas. Finally, we write $\mathrm{fv}(\varphi)$ or $\mathrm{fv}(t)$ for the set of free variables in a formula $\varphi$ or a term $t$. Formulas are considered up to $\alpha$-conversion and assumed to satisfy Barendregt's convention.

## B.1 $\Sigma$-labelled event structures

### B.1.1 Preliminary definitions

In this first subsection, we give the formulations of matching and unification on which the construction of our category of $\Sigma$-strategies will rely. For this subsection, we fix a signature $\Sigma$. From now on, sets referred to as *sets of variables* will be any finite sets. Sets denoted by $\mathcal{V}, \mathcal{V}', \mathcal{V}_i$ *etc* will always denote such sets of variables.

▶ **Definition 103.** A **substitution from $\mathcal{V}_1$ to $\mathcal{V}_2$** is a function

$$\gamma : \mathcal{V}_2 \to \mathsf{Tm}_\Sigma(\mathcal{V}_1),$$

written $\gamma : \mathcal{V}_1 \overset{\mathcal{S}}{\to} \mathcal{V}_2$.

If $t \in \mathsf{Tm}_\Sigma(\mathcal{V}_2)$ is a term and $\gamma : \mathcal{V}_1 \overset{\mathcal{S}}{\to} \mathcal{V}_2$ is a substitution, the **substitution of $t$ by** $\gamma$, written $t[\gamma]$, is defined as usual by induction on $t$. If $\gamma : \mathcal{V}_1 \overset{\mathcal{S}}{\to} \mathcal{V}_2$ and $\delta : \mathcal{V}_2 \overset{\mathcal{S}}{\to} \mathcal{V}_3$ are substitution, the **composition** $\delta \circ \gamma : \mathcal{V}_1 \overset{\mathcal{S}}{\to} \mathcal{V}_3$ is defined as usual:

$$\begin{array}{rcrcl} \delta \circ \gamma & : & \mathcal{V}_3 & \to & \mathsf{Tm}_\Sigma(\mathcal{V}_1) \\ & & \mathsf{x} \in \mathcal{V}_3 & \mapsto & \delta(\mathsf{x})[\gamma] \end{array}$$

Moreover, for all set of variables $\mathcal{V}$, the **identity substitution** associates any $\mathsf{x} \in \mathcal{V}$ to the term $\mathsf{x} \in \mathsf{Tm}_\Sigma(\mathcal{V})$. Together, these data and operations form a cartesian category $\mathrm{Subst}_\Sigma$ (written just Subst when the signature can be recovered from the context).

Matching and unification problems may be phrased in terms of this category. For instance, given $\gamma : \mathcal{V} \overset{\mathcal{S}}{\to} \mathcal{W}$ and $\gamma' : \mathcal{V}' \overset{\mathcal{S}}{\to} \mathcal{W}$, we say that $\gamma$ **subsumes** $\gamma'$ iff $\gamma$ can be further instantiated to match $\gamma'$, *i.e.* there is $\mu : \mathcal{V}' \overset{\mathcal{S}}{\to} \mathcal{V}$ such that $\gamma \circ \mu = \gamma'$. This is a preorder on the set of substitutions with codomain $\mathcal{W}$.

### B.1.1.1 Equalizers in Subst.

Two substitutions $\gamma, \delta : \mathcal{V}_1 \overset{\mathcal{S}}{\to} \mathcal{V}_2$ may not always have an equalizer – imagine $\mathcal{V}_2 = \{\mathsf{x}\}$ with $\gamma(\mathsf{x}) = \mathsf{c}$ and $\delta(\mathsf{x}) = \mathsf{d}$ with $\mathsf{c}$ and $\mathsf{d}$ distinct constant symbols in $\Sigma$. Observe that here $\gamma(\mathsf{x})$ and $\delta(\mathsf{x})$ are not unifyable. And indeed, we recall that there is [Goguen, 89] a correspondence between unification and equalizers in this category Subst. We state below the main theorem of first-order unification, in its categorical formulation from [Goguen, 89]:

▶ **Theorem 104** (Herbrand-Robinson). *Let $\gamma, \delta : \mathcal{V}' \overset{\mathcal{S}}{\to} \mathcal{V}$ be two parallel substitutions. If there is $\mu : \mathcal{V}'' \overset{\mathcal{S}}{\to} \mathcal{V}'$ such that $\gamma \circ \mu = \delta \circ \mu$, then $\gamma, \delta$ have an equalizer.*

**Sketch.** Recall that the pair $\gamma, \delta$ can be regarded as a unification problem:

$$(\gamma(\mathsf{x}) \doteq \delta(\mathsf{x}))_{\mathsf{x} \in \mathcal{V}}$$

If it is solvable, then the Herbrand-Robinson theorem states that it has a most general solution $\mu : \mathcal{V}'' \to \mathcal{V}'$, *i.e.* $\gamma \circ \mu = \delta \circ \mu$, and any other solution can be obtained by substitution from $\mu$. If $\mathcal{V}''$ is chosen without any unused variables (which we can always do), then it also follows that this mediating substitution is unique, and $(\mathcal{V}'', \mu)$ is an equalizer of $\gamma$ and $\delta$. ◀

### B.1.1.2 Further structure of Subst.

We recall that Subst is cartesian: its terminal object is the empty set of variables, and the product of $\mathcal{V}_1, \mathcal{V}_2$ sets of variables is their disjoint union, written $\mathcal{V}_1 \parallel \mathcal{V}_2 = \{1\} \times \mathcal{V}_1 \uplus \{2\} \times \mathcal{V}_2$ here so as to match the notation with the parallel composition as event structures – as events will serve the role of variables later on.

We will, in particular use later on the corresponding bifunctorial action: if $\gamma : \mathcal{V}_1 \overset{\mathcal{S}}{\to} \mathcal{V}_2$ and $\gamma' : \mathcal{V}_1' \overset{\mathcal{S}}{\to} \mathcal{V}_2'$, then

$$\begin{array}{rcrcl} \gamma \parallel \gamma' & : & \mathcal{V}_1 \parallel \mathcal{V}_1' & \overset{\mathcal{S}}{\to} & \mathcal{V}_2 \parallel \mathcal{V}_2' \\ & & (1, \mathsf{x}) & \mapsto & \gamma(\mathsf{x})[(1, \mathsf{y})/\mathsf{y} \mid \mathsf{y} \in \mathcal{V}_1] \\ & & (2, \mathsf{x}) & \mapsto & \gamma'(\mathsf{x})[(2, \mathsf{y})/\mathsf{y} \mid \mathsf{y} \in \mathcal{V}_2] \end{array}$$

We will use later on the following compatibility between the product and equalizer structure.

▶ **Lemma 105.** *Let $\gamma_1, \gamma_2 : \mathcal{V}_1 \overset{\mathcal{S}}{\to} \mathcal{V}_2$, and $\gamma'_1, \gamma'_2 : \mathcal{V}'_1 \overset{\mathcal{S}}{\to} \mathcal{V}'_2$, with have respectively equalizers $\mu : \mathcal{V}_0 \overset{\mathcal{S}}{\to} \mathcal{V}_1$, and $\mu' : \mathcal{V}'_0 \overset{\mathcal{S}}{\to} \mathcal{V}'_1$.*

*Then, $\gamma_1 \parallel \gamma'_1$ and $\gamma_2 \parallel \gamma'_2$ have an equalizer $\mu \parallel \mu'$.*

**Proof.** General property of finitely complete categories. ◀

### B.1.2 Definitions and properties of $\Sigma$-labelled event structures

Our $\Sigma$-strategies will incorporate both a causal structure and a term assignment, and composing them will involve finding an "agreement" between the two compounds, both in terms of causal dependency (as in the previous section) and term annotations (as in unification).

We start by enriching the elementary event structures of the previous section with term annotations.

▶ **Definition 106.** A **$\Sigma$-labelled elementary event structure ($\Sigma$-ees)** is a pair $\mathbf{q} = (\mathsf{q}, \lambda_\mathsf{q})$ where $\lambda_\mathsf{q}$ is a *labeling function*

$$\lambda_\mathsf{q} : (a \in |\mathsf{q}|) \to \mathsf{Tm}_\Sigma([a]_\mathsf{q})$$

which, to any event $a$, associates a term that may use events in the causal dependency of $a$ as variables.

If $\mathbf{q}$ is a $\Sigma$-ees, then for any $x \in \mathscr{C}(\mathsf{q})$ we obtain a substitution

$$\lambda^x_\sigma : x \overset{\mathcal{S}}{\to} x$$

by restriction of $\lambda_\sigma$ to $x$.

▶ **Example 107.** (i) The following diagram represents a $\Sigma$-ees, where the term annotation is written as a superscript.

$$\begin{array}{c} e_0^{\mathsf{c}} \\ \downarrow \\ e_1^{\mathsf{f}(e_0,e_1)} \end{array}$$

(ii) Any ees $\mathsf{q}$ can be regarded as a $\Sigma$-ees by adjoining to it the identity annotation, defined as $\lambda(e) = e$ for all $e \in |\mathsf{q}|$. Whenever we use an ees in a context where a $\Sigma$-ees is expected, it is understood that it is coerced into a $\Sigma$-ees by adjoining the trivial annotation. This should not cause any confusion.

We started Section A by introducing a partial order on ees, for which meets provided the formal basis for interaction of strategies. Likewise here we will later define $\Sigma$-strategies as certain $\Sigma$-ees; and their interaction will be computed as greatest lower bounds for the following partial order.

▶ **Definition 108.** Let $\mathbf{q}, \mathbf{p}$ be two $\Sigma$-ees. We write $\mathbf{q} \preccurlyeq \mathbf{p}$ iff $\mathscr{C}(\mathsf{q}) \subseteq \mathscr{C}(\mathsf{p})$, and for all $x \in \mathscr{C}(\mathsf{q})$, $\lambda^x_\mathsf{p}$ subsumes $\lambda^x_\mathsf{q}$.

By definition this means that $\mathsf{q} \preccurlyeq \mathsf{p}$ holds between the underlying ees, and that furthermore term annotations in $\mathbf{p}$ are more general than those in $\mathbf{p}$: they can be instantiated further to match those in $\mathbf{q}$.

▶ **Example 109.** (i) For any $\mathbf{q} = (\mathsf{q}, \lambda_\mathsf{q})$ we have $\mathbf{q} \preccurlyeq \mathsf{q}$ (where $\mathsf{q}$ is here regarded as the $\Sigma$-ees with identity annotation).

(ii) We have:

$$\begin{array}{c} e_0^{\mathsf{c}} \\ \downarrow \\ e_1^{\mathsf{g}(\mathsf{f}(e_0,e_1))} \end{array} \quad \preccurlyeq \quad e_0^{e_0} \qquad e_1^{\mathsf{g}(e_1)}$$

(iii) We have

$$\begin{array}{ccc} e_0^{\mathsf{c}} & & e_1^{\mathsf{d}} \\ & \searrow \quad \swarrow & \\ & e_2^{e_0} & \end{array} \quad \preccurlyeq \quad \begin{array}{ccc} e_0^{\mathsf{c}} & & e_1^{\mathsf{d}} \\ & \searrow \quad \swarrow & \\ & e_2^{e_1} & \end{array}$$

and, by symmetry, the other way around.

From this last example, it appears that $\preccurlyeq$ is not a partial order, but merely a preorder (it is clearly reflexive and transitive). As we will eventually aim to compute interactions between $\Sigma$-strategies as greatest lower bounds for $\preccurlyeq$, we need first to restrict the $\Sigma$-ees to a case where their meet is defined on the nose, rather than up to equivalence – in other words, finding sufficient conditions for $\preccurlyeq$ to be a partial order.

▶ **Definition 110.** Let $\mathbf{q}$ be a $\Sigma$-ees. We say that $\mathbf{q}$ is **idempotent** iff for all $x \in \mathscr{C}(\mathsf{q})$, $\lambda_{\mathsf{q}}^x$ is idempotent, *i.e.* $\lambda_{\mathsf{q}}^x \circ \lambda_{\mathsf{q}}^x = \lambda_{\mathsf{q}}^x$.

Idempotence implies a certain distinction between events in a $\Sigma$-ees. Certain events are to be considered as variables only: those events $a$ invariant under $\lambda_{\mathsf{q}}$, *i.e.* such that $\lambda_{\mathsf{q}}(a) = a$. Other events may be annotated only by terms involved those event-variables, as if $\lambda_{\mathsf{q}}(a)$ includes as free variable a non-invariant event, we will immediately have a failure of idempotency. In particular, observe that none of the diagrams pictured in Examples 107 and 109 are idempotent.

Idempotent $\Sigma$-ees are better behaved with respect to $\preccurlyeq$:

▶ Proposition 111. The preorder $\preccurlyeq$ is a partial order on idempotent $\Sigma$-ees.

**Proof.** Assume that $\mathbf{q} \preccurlyeq \mathbf{q}'$ and $\mathbf{q}' \preccurlyeq \mathbf{q}$. In particular, $\mathscr{C}(\mathsf{q}) = \mathscr{C}(\mathsf{q}')$ so $\mathsf{q} = \mathsf{q}'$. We need to check that they also have the same annotations. Let $e \in |\mathsf{q}|$, and $x \in \mathscr{C}(\mathsf{q})$ such that $e \in x$. We know that $\lambda_{\mathsf{q}'}^x$ subsumes $\lambda_{\mathsf{q}}^x$, so there is $\mu_x$ such that $\lambda_{\mathsf{q}}^x = \mu_x \circ \lambda_{\mathsf{q}'}^x$. Likewise, $\lambda_{\mathsf{q}}^x$ subsumes $\lambda_{\mathsf{q}'}^x$, so there is $\mu'_x$ such that $\lambda_{\mathsf{q}'}^x = \mu'_x \circ \lambda_{\mathsf{q}}^x$. Therefore, $\lambda_{\mathsf{q}}^x = \mu_x \circ \mu'_x \circ \lambda_{\mathsf{q}}^x$. Hence,

$$\lambda_{\mathsf{q}}(e)[\mu_x \circ \mu'_x] = \lambda_{\mathsf{q}}(e)$$

for all $e \in x$. It follows necessarily that for all $a \in \mathrm{fv}(\lambda_{\mathsf{q}}(e))$, $a[\mu_x \circ \mu'_x] = a$, so $\mu_x(a)$ must be a variable $a' \in x$. Since $\lambda_{\mathsf{q}}^x$ is idempotent, we have $\lambda_{\mathsf{q}}(e)[\lambda_{\mathsf{q}}^x] = \lambda_{\mathsf{q}}(e)$, so $\lambda_{\mathsf{q}}(a) = a$. It follows that $\lambda_{\mathsf{q}'}(a) = a'$. But we also have that $\lambda_{\mathsf{q}'}(a) \in \mathsf{Tm}_\Sigma([a]_{\mathsf{q}})$, so $a' \leq_{\mathsf{q}} a$. Symmetrically, $a \leq_{\mathsf{q}} a'$ as well, hence $a = a'$. Therefore, $\mu_x(a) = a$ for all $a \in \mathrm{fv}(\lambda_{\mathsf{q}}(e))$. Hence, $\lambda_{\mathsf{q}}(e) = \lambda_{\mathsf{q}'}(e)$, thus $\mathbf{q} = \mathbf{q}'$. ◀

Therefore, if two idempotent $\Sigma$-ees $\mathbf{q}$ and $\mathbf{p}$ have a meet, it is *unique*. It remains to see whether sufficiently many such meets exist to compute the interaction of $\Sigma$-strategies. Our first remark in this direction is that idempotent $\Sigma$-ees may still have no meet.

▶ **Example 112.** The $\Sigma$-ees $\mathbf{q}_1 = \begin{array}{ccc} e_1^{e_1} & & e_2^{e_2} \\ & \nwarrow \quad \nearrow & \\ & e_3^{\mathsf{f}(e_1)} & \end{array}$ and $\mathbf{q}_2 = \begin{array}{ccc} e_1^{e_1} & & e_2^{e_2} \\ & \nwarrow \quad \nearrow & \\ & e_3^{\mathsf{f}(e_2)} & \end{array}$ have no meet.

Assume they have a meet $\mathbf{p}$. Necessarily, since $e_1^{e_1}\ e_2^{e_2} \preccurlyeq \mathbf{q}_1, \mathbf{q}_2$, $\mathbf{p}$ must comprise the events-with-annotations $e_1^{e_1}$ and $e_2^{e_2}$. But we also have

$$e_1^{\mathsf{c}} \rightsquigarrow e_3^{\mathsf{f(c)}} \leftsquigarrow e_2^{\mathsf{c}} \qquad \preccurlyeq \mathbf{q}_1, \mathbf{q}_2$$

for any constant symbol $\mathsf{c}$. Therefore, $\mathbf{p}$ must also include event-with-annotation $e_3^t$. But $t$ must be an instance of $f(e_1), f(e_2)$; and must instantiate to $\mathsf{f(c)}$ for all constant symbol $\mathsf{c}$. So $t$ must have the form $\mathsf{f}(e)$ for some $e \in [e_3]$, *i.e.* $e \in \{e_1, e_2, e_3\}$. It is direct to check that none of those options gives a $\Sigma$-ees that is below both $\mathbf{q}_1$ and $\mathbf{q}_2$ for $\preccurlyeq$.

We now introduce a sufficient condition for a pair of $\Sigma$-ees to have a meet. Rather than aiming for maximal generality, we will simply prove meets exist in a situation covering interaction of $\Sigma$-strategies.

▶ **Definition 113.** Let $\mathbf{q}$ and $\mathbf{p}$ be two idempotent $\Sigma$-ees. We say that they are **orthogonal** iff for all $e \in |\mathsf{q} \wedge \mathsf{p}|$, either $\lambda_{\mathsf{q}}(e) = e$ or $\lambda_{\mathsf{p}}(e) = e$.

Let $\mathsf{q}$ be an idempotent $\Sigma$-ees. We say that $e \in |q|$ is a **q-variable** iff $\lambda_{\mathsf{q}}(e) = e$. For any $e$ we have that either it is a $\mathbf{q}$-variable, or its labeling is $\lambda_{\mathsf{q}}(e) = t \in \mathsf{Tm}_\Sigma([e]_{\mathsf{q}})$, where in fact the free variables in $t$ must be $\mathbf{q}$-variables by idempotency – so in particular they cannot involve $e$ itself. With these conventions in mind, the orthogonality of idempotent $\mathbf{q}$ and $\mathbf{p}$ simply states that any $e \in |\mathsf{q} \wedge \mathsf{p}|$ needs to a variable for at least $\mathbf{q}$ or $\mathbf{p}$. An event $e \in |\mathsf{q} \wedge \mathsf{p}|$ may also be a variable for *both* $\mathbf{q}$ and $\mathbf{p}$ – in which case we call it a $(\mathbf{q}, \mathbf{p})$-variable.

First of all, we prove the following lemma.

▶ **Lemma 114.** *For any two orthogonal idempotent $\Sigma$-ees $\mathbf{q}$ and $\mathbf{p}$, for any $x \in \mathscr{C}(\mathsf{q} \wedge \mathsf{p})$, the substitutions*

$$\lambda_{\mathsf{q}}^x, \lambda_{\mathsf{p}}^x : x \overset{\mathcal{S}}{\to} x$$

*have a* unique *equalizer $\nu_x : x' \overset{\mathcal{S}}{\to} x$ such that for all $e \in x$ a $(\mathbf{q}, \mathbf{p})$-variable, $\nu_x(e) = e$. This choice is monotonic: for any $x \subset y \in \mathscr{C}(\mathsf{q} \wedge \mathsf{p})$, $\nu_y \upharpoonright x = \nu_x$.*

**Proof.** *Existence.* First, we prove by well-founded induction on $-\!\!\subset$ that for all $x \in \mathscr{C}(\mathsf{q} \wedge \mathsf{p})$, there exists $\mu_x : x' \to x$ such that $\lambda_{\mathsf{q}}^x \circ \mu_x = \lambda_{\mathsf{p}}^x \circ \mu_x$. It is clearly true for $x = \emptyset$. Assume it is true for $x$, and take $x \overset{e}{-\!\!\subset} y \in \mathscr{C}(\mathsf{q} \wedge \mathsf{p})$. Since $\mathbf{q}$ and $\mathbf{p}$ are orthogonal, we have $\lambda_{\mathsf{q}}(e) = e$ or $\lambda_{\mathsf{p}}(e) = e$. Suppose first that it is both. Then, we extend $\mu_x : x' \overset{\mathcal{S}}{\to} x$ to $\mu_y : y' \overset{\mathcal{S}}{\to} y$ by setting $\mu_y(e) = e$ (setting $y' = x' \cup \{e\}$, assuming *w.l.o.g.* up to renaming, that $e \notin x'$); which still satisfies $\lambda_{\mathsf{q}}^y \circ \mu_y = \lambda_{\mathsf{p}}^y \circ \mu_y$ as required. Otherwise, exactly one of $\lambda_{\mathsf{q}}(e)$ and $\lambda_{\mathsf{p}}(e)$ is $e$. Assume *w.l.o.g.* that $\lambda_{\mathsf{q}}(e) = e$ and $\lambda_{\mathsf{p}}(e) = t$, with $t \in \mathsf{Tm}_\Sigma([e)_{\mathsf{p}})$. Then, we extend $\mu_x : x' \overset{\mathcal{S}}{\to} x$ to $\mu_y : y' \overset{\mathcal{S}}{\to} y$ by defining $\mu_y(e) = t[\mu_x]$. By induction hypothesis we have $\lambda_{\mathsf{q}}^y \circ \mu_y(e') = \lambda_{\mathsf{p}}^y \circ \mu_y(e')$ for any $e' \in x$, and

$$\begin{aligned}
\lambda_{\mathsf{q}}^y \circ \mu_y(e) &= \lambda_{\mathsf{q}}^y(e)[\mu_y] \\
&= e[\mu_y] \\
&= \mu_y(e) \\
&= t[\mu_x] \\
&= (\lambda_{\mathsf{p}}(e))[\mu_y] \\
&= \lambda_{\mathsf{p}}^y \circ \mu_y(e),
\end{aligned}$$

therefore $\lambda^y_{\mathsf{q}} \circ \mu_y = \lambda^y_{\mathsf{p}} \circ \mu_y$. Since for all $x \in \mathscr{C}(\mathsf{q} \wedge \mathsf{p})$ there is $\mu_x : x' \overset{\mathcal{S}}{\to} x$ such that $\lambda^x_{\mathsf{q}} \circ \mu_x = \lambda^x_{\mathsf{p}} \circ \mu_x$, by Theorem 104, $\lambda^x_{\mathsf{q}}$ and $\lambda^x_{\mathsf{p}}$ have an equalizer:

$$\nu_x : x' \overset{\mathcal{S}}{\to} x$$

We prove that $\nu_x$ can *w.l.o.g.* be assumed to preserve $(\mathbf{q},\mathbf{p})$-variables. For all $x \in \mathscr{C}(\mathsf{q} \wedge \mathsf{p})$, $\nu_x$ subsumes $\mu_x$. By construction, if $e \in |\mathsf{q} \wedge \mathsf{p}|$ is a $(\mathbf{q},\mathbf{p})$-variable), then $\mu_x(e) = e$. Therefore $\nu_x(e) = e' \in x'$ is a variable as well. Hence, $\nu_x$ induces an injection of $(\mathbf{q},\mathbf{p})$-variables into $x'$. That means that by renaming we can always get $\nu'_x : x'' \overset{\mathcal{S}}{\to} x$ such that $(\mathbf{q},\mathbf{p})$-variables are included in $x''$ and that for all $(\mathbf{q},\mathbf{p})$-variable $e$, we have $\nu'_x(e) = e$. As obtained from an equalizer by renaming, $\nu'_x$ is still an equalizer, and preserves $(\mathbf{q},\mathbf{p})$-variables as desired.

*Uniqueness.* Recall from [Goguen] that if $\mu : \mathcal{V}'' \overset{\mathcal{S}}{\to} \mathcal{V}'$ is an equalizer of $\gamma, \delta : \mathcal{V}' \overset{\mathcal{S}}{\to} \mathcal{V}$, then

$$\mathcal{V}'' = \cup\{\mathrm{fv}(\mu(\mathsf{x})) \mid \mathsf{x} \in \mathcal{V}'\},$$

a failure of that leading easily to a contradiction to the universal property. In our case, that means that for an equalizer $\nu_x : x' \overset{\mathcal{S}}{\to} x$ satisfying the required condition is by necessity such that $x'$ is exactly the set of $(\mathbf{p},\mathbf{p})$-variables of $x$.

We prove by induction on the well-founded relation $\multimap\!\!\subset$ that for all $x \in \mathscr{C}(\mathsf{q} \wedge \mathsf{p})$, for all $e \in x$, we have $\nu_x(e) = \mu_x(e)$. Take $x \overset{e}{\multimap\!\!\subset} y$. Separate two cases as above: if $e$ is a $(\mathbf{q},\mathbf{p})$-variable, then by hypothesis we have $\nu_y(e) = e$. We observe that restricting $\nu_y$ to:

$$\nu_y \upharpoonright x \ : \ y' \setminus \{e\} \ \overset{\mathcal{S}}{\to} \ x$$

we get an equalizer of $\lambda^x_{\mathsf{q}}$ and $\lambda^x_{\mathsf{p}}$. Indeed, any $\gamma_x : x'' \overset{\mathcal{S}}{\to} x$ such that $\lambda^x_{\mathsf{q}} \circ \gamma_x = \lambda^x_{\mathsf{p}} \circ \gamma_x$ extends by setting $\gamma_y(e) = e$ such that (same reasoning as for *existence*) $\lambda^y_{\mathsf{q}} \circ \gamma_y = \lambda^x_{\mathsf{p}} \circ \gamma_y$. The unique factorization preserves $e$ as well by necessity, and restricts to a unique factorization $\delta : x'' \overset{\mathcal{S}}{\to} y' \setminus \{e\}$ such that $(\nu_y \upharpoonright x) \circ \delta = \gamma_x$. This proves that $\nu_y \upharpoonright x$ is an equalizer of $\lambda^x_{\mathsf{q}}$ and $\lambda^x_{\mathsf{p}}$ which satisfies the required conditions, so by induction hypothesis it is equal to $\mu_x$. Hence, $\nu_y$ is equal to $\mu_y$ as well. Finally, if $e$ is not a $(\mathbf{q},\mathbf{p})$-variable, then *w.l.o.g.* say that $\lambda_{\mathsf{q}}(e) = e$ and $\lambda_{\mathsf{p}}(e) = t \in \mathsf{Tm}_\Sigma([e]_{\mathsf{p}})$. Then, as above, the restriction:

$$\nu_y \upharpoonright x \ : \ y' \overset{\mathcal{S}}{\to} x$$

yields (by a reasoning completely analogous to the one just above) an equalizer of $\lambda^x_{\mathsf{q}}$ and $\lambda^x_{\mathsf{p}}$ preserving $(\mathbf{q},\mathbf{p})$-variables. By induction hypothesis, $\nu_y \upharpoonright x = \nu_x$, and the same calculation as above, the requirement that $\lambda^y_{\mathsf{q}} \circ \nu_y(e) = \lambda^y_{\mathsf{p}} \circ \nu_y(e)$ then amounts to $\nu_y(e) = t[\nu_x]$, which shows that $\nu_y = \mu_y$ as required.

The monotonicity follows trivially by construction of $\mu_x$. ◀

Before we conclude from this that the meet of orthogonal idempotent $\Sigma$-ees always exist, we note the following further property of the equalizer in the lemma above.

▶ **Lemma 115.** *Let $\mathbf{q}$ and $\mathbf{p}$ be two idempotent orthogonal $\Sigma$-ees, $x \in \mathscr{C}(\mathsf{q} \wedge \mathsf{p})$, and $\nu_x : x' \overset{\mathcal{S}}{\to} x$ their unique equalizer given by Lemma 114. Then,*

$$\nu_x = \lambda^x_{\mathsf{q}} \circ \nu_x = \lambda^x_{\mathsf{p}} \circ \nu_x.$$

**Proof.** Direct from the construction of $\nu_x$ in Lemma 114. ◀

▶ **Lemma 116.** *Any two orthogonal idempotent $\Sigma$-ees $\mathbf{q}$ and $\mathbf{p}$ have a meet. Moreover, it is idempotent.*

**Proof.** We set their meet candidate $\mathbf{q} \wedge \mathbf{p}$ as based on ees $\mathsf{q} \wedge \mathsf{p}$, with labeling, for $e \in |\mathsf{q} \wedge \mathsf{p}|$,

$$\lambda_{\mathsf{q}\wedge\mathsf{p}}\, e = \nu_x\, e$$

for any $x \in \mathscr{C}(\mathsf{q} \wedge \mathsf{p})$ such that $e \in x$, and $\nu_x : x' \stackrel{\mathcal{S}}{\to} x$ is the unique equalizer preserving $(\mathbf{q}, \mathbf{p})$-variables given by Lemma 114 – note that from the monotonicity property, the choice of $x$ does not matter.

It is a lower bound of $\mathbf{q}$ and $\mathbf{p}$: the causal condition is clear, and for all $x \in \mathscr{C}(\mathsf{q} \wedge \mathsf{p})$, $\lambda^x_{\mathsf{q}\wedge\mathsf{p}}$ is by Lemma 115 subsumed by both $\lambda^x_{\mathsf{q}}$ and $\lambda^x_{\mathsf{p}}$. To prove that it is the greatest lower bound, take $\mathbf{q}' \preccurlyeq \mathbf{q}, \mathbf{p}$. For $x \in \mathscr{C}(\mathbf{q}')$, $\lambda^x_{\mathsf{q}'}$ is subsumed by both $\lambda^x_{\mathsf{q}}$ and $\lambda^x_{\mathsf{p}}$. Therefore there are $\gamma_1, \gamma_2 : x \stackrel{\mathcal{S}}{\to} x$ such that $\lambda^x_{\mathsf{q}'} = \lambda^x_{\mathsf{q}} \circ \gamma_1 = \lambda^x_{\mathsf{p}} \circ \gamma_2$. But then we calculate:

$$\begin{array}{rcl}
\lambda^x_{\mathsf{q}} \circ \lambda^x_{\mathsf{q}'} & = & \lambda^x_{\mathsf{q}} \circ \lambda^x_{\mathsf{q}} \circ \gamma_1 \\
& = & \lambda^x_{\mathsf{q}} \circ \gamma_1 \\
& = & \lambda^x_{\mathsf{p}} \circ \gamma_2 \\
& = & \lambda^x_{\mathsf{p}} \circ \lambda^x_{\mathsf{p}} \circ \gamma_2 \\
& = & \lambda^x_{\mathsf{p}} \circ \lambda^x_{\mathsf{q}}
\end{array}$$

Since $\nu_x$ is an equalizer of $\lambda^x_{\mathsf{q}}$ and $\lambda^x_{\mathsf{p}}$, it follows that there is a (unique) $\gamma : x \stackrel{\mathcal{S}}{\to} x'$ such that $\lambda^x_{\mathsf{q}'} = \nu_x \circ \gamma$. Extending $\gamma$ to $\gamma' : x \stackrel{\mathcal{S}}{\to} x$ in an arbitrary way (by setting *e.g.* $\gamma(e) = e$ for any $e \in x \setminus x'$), we get a factorization $\lambda^x_{\mathsf{q}'} = \lambda^x_{\mathsf{q}\wedge\mathsf{p}} \circ \gamma'$; so $\lambda^x_{\mathsf{q}\wedge\mathsf{p}}$ subsumes $\lambda_{\mathsf{q}'}$ as required.

Finally, it is easy to see that it is idempotent from the construction of $\nu_x$ in Lemma 114 (as the variables used in term annotations are only the $(\mathbf{q}, \mathbf{p})$-variables, preserved by $\nu_x$ by construction). ◀

Before exploiting this to define Σ-strategies and their interactions, we briefly show a few further constructions and properties of Σ-ees.

▶ **Definition 117.** Let $\mathbf{q}, \mathbf{p}$ be Σ-ees. Then, we define their parallel composition $\mathbf{q} \parallel \mathbf{p}$ as having the underlying ees $\mathsf{q} \parallel \mathsf{p}$, with the inherited annotation:

$$\begin{array}{rcl}
\lambda_{\mathsf{q}\parallel\mathsf{p}}(1, a) & = & \lambda_{\mathsf{q}}(a)[(1, e)/e \mid e \in [a]_{\mathsf{q}}] \\
\lambda_{\mathsf{q}\parallel\mathsf{p}}(2, b) & = & \lambda_{\mathsf{p}}(b)[(2, e)/e \mid e \in [b]_{\mathsf{p}}]
\end{array}$$

We prove the statement analogous to Lemma 59 in the presence of labeling.

▶ **Lemma 118.** *Let $\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2$ be idempotent Σ-ees, such that $\mathbf{q}_1$ and $\mathbf{p}_1$ are orthogonal, and $\mathbf{q}_2$ and $\mathbf{p}_2$ are orthogonal. Then $\mathbf{q}_1 \parallel \mathbf{q}_2$ and $\mathbf{p}_1 \parallel \mathbf{p}_2$ are orthogonal as well, and we have:*

$$(\mathbf{q}_1 \parallel \mathbf{q}_2) \wedge (\mathbf{p}_1 \parallel \mathbf{p}_2) = (\mathbf{q}_1 \wedge \mathbf{p}_1) \parallel (\mathbf{q}_2 \wedge \mathbf{p}_2)$$

**Proof.** Immediate consequence of Lemma 59 together with Lemma 105. ◀

## B.2 A category of Σ-strategies

### B.2.1 Definitions and basic properties of Σ-strategies

We now define Σ-strategies. As in Section A, we start by defining *uncovered* Σ-strategies playing on *uncovered* arenas; and their interactions. In a second stage, we will specialize them to the covered case.

▶ **Definition 119.** Let $A$ be an uncovered arena. An **uncovered $\Sigma$-strategy** on $A$, written $\boldsymbol{\sigma} : A$, is a $\Sigma$-ees $\boldsymbol{\sigma} = (\sigma, \lambda_\sigma)$ where $\sigma : A$ is an uncovered strategy on $A$, and we have:

- $\Sigma$-*receptivity.* For any $a^- \in |\sigma|, \lambda_\sigma(a) = a$,
- $\Sigma$-*courtesy.* For any $a^p \in |\sigma|, \lambda_\sigma(a) \in \mathsf{Tm}_\Sigma([a]^-_\Sigma)$.

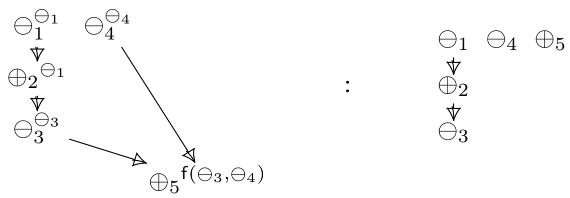
where $a^p$ denoted any event of polarity 0 or +, and:

$$[a]^-_\sigma = \{\ a' \in |\sigma| \ \mid\ \mathrm{pol}_A(a') = - \ \&\ a' \leq_\sigma a\ \}.$$

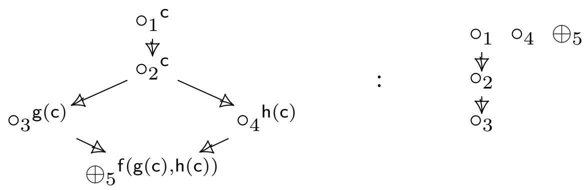
The names $\Sigma$-receptivity and $\Sigma$-courtesy are there to emphasize an analogy with receptivity and courtesy (regarding the roles played by those conditions in proofs), and the fact that they are concerned with term assignments.

▶ **Example 120.** The following diagrams represent (uncovered) $\Sigma$-strategies on the displayed (uncovered) arenas. Each event is displayed following its polarity.

- Firstly, we show a covered $\Sigma$-strategy on a covered arena.



- Secondly, we show an uncovered $\Sigma$-strategy with some neutral events.



where $\Sigma = \{\mathsf{c}, \mathsf{f}, \mathsf{g}, \mathsf{h}\}$ with $\mathsf{c}$ of arity 0, $\mathsf{g}, \mathsf{h}$ of arity 1, and $\mathsf{f}$ of arity 2.

We observe the following immediate consequence of the definition:

▶ **Lemma 121.** *Let $\boldsymbol{\sigma} : A$ be an (uncovered) strategy. Then, it is idempotent.*

**Proof.** Let $x \in \mathscr{C}(\sigma)$, and let $a \in x$. If $\mathrm{pol}_A(a) = -$, then by $\Sigma$-receptivity we have that $\lambda_\sigma(a) = a$, and clearly $\lambda_\sigma(a)[\lambda^x_\sigma] = \lambda_\sigma(a) = a$. If $\mathrm{pol}_A(a) \neq -$, then the free variables of $\lambda_\sigma(a)$ have all negative polarity. By $\Sigma$-receptivity again, it follows that $\lambda_\sigma(a)[\lambda^x_\sigma] = \lambda_\sigma(a)$, so $\lambda^x_\sigma$ is indeed idempotent. ◀

### B.2.2 Interaction

The *interaction* of $\Sigma$-strategies is then obtained as described below.

▶ Proposition 122. Let $\boldsymbol{\sigma} : A^\perp \parallel B$ and $\boldsymbol{\tau} : B^\perp \parallel C$ be uncovered strategies, where $B$ is an arena (*i.e.* is covered). Then, the **interaction** $\boldsymbol{\tau} \circledast \boldsymbol{\sigma}$ is defined as the meet:

$$\boldsymbol{\tau} \circledast \boldsymbol{\sigma} = (\boldsymbol{\sigma} \parallel C) \wedge (A \parallel \boldsymbol{\tau})$$

which always exists, and is idempotent.

Note that for now this only defines the interaction as a $\Sigma$-ees.

**Proof.** Note that in the definition above, $C$ and $A$ are implicitely coerced into $\Sigma$-ees by adjoining them the identity annotation, as explained in Example 107. Furthermore, by construction those are clearly idempotent.

By Lemma 121, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are idempotent. Moreover, idempotent $\Sigma$-ees are by construction stable under parallel composition, so $\boldsymbol{\sigma} \parallel C$ and $A \parallel \boldsymbol{\tau}$ are idempotent as well. Finally, they are orthogonal: if $(1, a) \in |\tau \circledast \sigma|$, then $\lambda_{A\parallel\tau}((1, a)) = (1, a)$ by construction, and likewise for $(3, c) \in |\tau \circledast \sigma|$. If $(2, b) \in |\tau \circledast \sigma|$, then $(2, b) \in |\sigma|$ and $(1, b) \in |\tau|$, and since $B$ is covered,

$$\mathrm{pol}_{A^\perp\parallel B}(2, b) = -\mathrm{pol}_{B^\perp\parallel C}(1, b) \neq 0$$

Let us say *w.l.o.g.* that $\mathrm{pol}_{A^\perp\parallel B}(2, b) = -$. Then, by $\Sigma$-receptivity, $\lambda_{\sigma\parallel C}(2, b) = (2, b)$; hence $\boldsymbol{\sigma} \parallel C$ and $A \parallel \boldsymbol{\tau}$ are orthogonal. By Lemma 116 they must therefore have a meet $\boldsymbol{\tau} \circledast \boldsymbol{\sigma}$, which furthermore is idempotent. ◀

Interaction of uncovered $\Sigma$-strategies always yields an uncovered $\Sigma$-strategy.

▶ Proposition 123. Let $\boldsymbol{\sigma} : A^\perp \parallel B$ and $\boldsymbol{\tau} : B^\perp \parallel C$ be uncovered strategies, where $B$ is an arena (*i.e.* is covered). Then, $\boldsymbol{\tau} \circledast \boldsymbol{\sigma} : A^\perp \parallel B^0 \parallel C$ is an uncovered $\Sigma$-strategy.

**Proof.** We already know that $\tau \circledast \sigma$ is an uncovered strategy on $A^\perp \parallel B^0 \parallel C$ and that $\boldsymbol{\tau} \circledast \boldsymbol{\sigma}$ is an idempotent $\Sigma$-ees. It remains to check that it is $\Sigma$-receptive and $\Sigma$-courteous.

For that; observe from the definition of $\lambda_{\boldsymbol{\tau}\circledast\boldsymbol{\sigma}}$ and Lemma 114, the $e \in |\tau \circledast \sigma|$ such that $\lambda_{\boldsymbol{\tau}\circledast\boldsymbol{\sigma}}(e) = e$ are precisely the $(\boldsymbol{\sigma} \parallel C, A \parallel \boldsymbol{\tau})$-variables (one direction follows from Lemma 114, while the other is straighforward). By $\Sigma$-receptivity and $\Sigma$-courtesy of $\boldsymbol{\sigma}$, the $(\boldsymbol{\sigma} \parallel C)$-variables are either in $C$ or negative in $A^\perp \parallel B$ and symmetrically for $(A \parallel \boldsymbol{\tau})$-variables. Hence, the $(\boldsymbol{\sigma} \parallel C, A \parallel \boldsymbol{\tau})$-variables are exactly the events negative in $A^\perp \parallel B^0 \parallel C$. It follows that $\boldsymbol{\tau} \circledast \boldsymbol{\sigma}$ is $\Sigma$-receptive, and it follows as well that it is $\Sigma$-courteous by the observation that for any $\mathbf{q}$ a $\Sigma$-ees, the free variables of term annotations in $\mathbf{q}$ are $\mathbf{q}$-variables, hence here negative events. ◀

As before, we aim now for a first associative notion of composition for uncovered $\Sigma$-strategies, without hiding. We say that an uncovered $\Sigma$-strategy **from** $A$ **to** $B$ (both covered arenas) is an uncovered $\Sigma$-strategy $\boldsymbol{\sigma} : A^\perp \parallel N^0 \parallel B$. Then we have:

▶ Proposition 124. Interaction of uncovered $\Sigma$-strategies is associative.

**Proof.** The proof follows exactly as for Proposition 69, using Lemma 118 and the associativity of the greatest lower bound. ◀

### B.2.3 Hiding

In this subsection, we define the hiding operation on uncovered $\Sigma$-strategies, and prove a few lemmas.

▶ **Definition 125.** Let $\boldsymbol{\sigma} : A$ be an uncovered $\Sigma$-strategy on an uncovered arena $A$. Its **hiding** is $\boldsymbol{\sigma}_\downarrow$, defined as having $\sigma_\downarrow$ as underlying ees, and $\lambda_{\sigma_\downarrow}$ is the restriction of $\lambda_\sigma$ to $|\sigma_\downarrow|$.

▶ **Lemma 126.** *For any $\boldsymbol{\sigma} : A$ an uncovered $\Sigma$-strategy on uncovered arena $A$, $\boldsymbol{\sigma}_\downarrow : A_\downarrow$ is a covered $\Sigma$-strategy on covered arena $A_\downarrow$.*

**Proof.** We need to prove that $\boldsymbol{\sigma}_\downarrow$ is a $\Sigma$-ees (*i.e.* that for all $a \in |\sigma_\downarrow|$, we have that $\lambda_{\sigma_\downarrow}(a) \in \mathsf{Tm}_\Sigma([a]_{\sigma_\downarrow}))$, that it is $\Sigma$-receptive and $\Sigma$-courteous.

By $\Sigma$-receptivity and $\Sigma$-courtesy of $\boldsymbol{\sigma}$, we have

$$\begin{array}{rcl}\lambda_\sigma(a^-) & = & a \\ \lambda_\sigma(a^p) & \in & \mathsf{Tm}_\Sigma([a]^-_\sigma)\end{array}$$

so in particular, if $a$ has non-neutral polarity, $\lambda_\sigma(a) \in \mathsf{Tm}([a]_{\sigma_\downarrow})$, hence $\boldsymbol{\sigma}_\downarrow$ is a $\Sigma$-ees. Its $\Sigma$-receptivity and $\Sigma$-courtesy are immediate consequences of that of $\boldsymbol{\sigma}$ expressed as above. ◀

Given $\boldsymbol{\sigma} : A^\perp \parallel B$ and $\boldsymbol{\tau} : B^\perp \parallel C$ two covered $\Sigma$-strategies between covered arenas, this concludes the definition of $\boldsymbol{\tau} \odot \boldsymbol{\sigma}$.

### B.2.4 Associativity

Now, we prove that composition of $\Sigma$-strategies is associative. As for the plain case in the previous sections, this relies on the following lemma.

▶ **Lemma 127.** *Let $\boldsymbol{\sigma} : A^\perp \parallel M^0 \parallel B$ and $\boldsymbol{\tau} : B^\perp \parallel N^0 \parallel C$ be uncovered $\Sigma$-strategies, where $A, B, C$ are covered games. Then,*

$$(\boldsymbol{\tau}_\downarrow \circledast \boldsymbol{\sigma}_\downarrow)_\downarrow = (\boldsymbol{\tau} \circledast \boldsymbol{\sigma})_\downarrow$$

**Proof.** By Lemma 76, we know that these two $\Sigma$-strategies have the same underlying strategy. To finish the proof, we need to check that for any $e \in |\boldsymbol{\tau}_\downarrow \circledast \boldsymbol{\sigma}_\downarrow| \subseteq |\boldsymbol{\tau} \circledast \boldsymbol{\sigma}|$,

$$\lambda_{\boldsymbol{\tau}_\downarrow \circledast \boldsymbol{\sigma}_\downarrow}(e) = \lambda_{\boldsymbol{\tau} \circledast \boldsymbol{\sigma}}(e).$$

The proof is direct by well-founded induction on the order $\leq_{\boldsymbol{\tau}_\downarrow \circledast \boldsymbol{\sigma}_\downarrow}$. If $e$ is negative in $A^\perp \parallel B^0 \parallel C$, then it is a $(\boldsymbol{\sigma} \parallel C, A \parallel \boldsymbol{\tau})$-variable, and its labeling is $e$ in both cases. Otherwise, *w.l.o.g.*, $e$ is positive for $\boldsymbol{\sigma}$, and is a $(A \parallel \boldsymbol{\tau})$-variable. Then, $\lambda_\sigma(e) = \lambda_{\sigma_\downarrow}(e) \in \mathsf{Tm}_\Sigma([e]^-_\sigma)$. Events in $[e]^-_\sigma$ are in $|\boldsymbol{\tau}_\downarrow \circledast \boldsymbol{\sigma}_\downarrow|$, and are strictly lower than $e$. By induction hypothesis, for each $e' \in [e]^-_\sigma$, we have $\lambda_{\boldsymbol{\tau}_\downarrow \circledast \boldsymbol{\sigma}_\downarrow}(e') = \lambda_{\boldsymbol{\tau} \circledast \boldsymbol{\sigma}}(e')$. The required equality follows from Lemma 115. ◀

▶ Proposition 128. Composition of covered $\Sigma$-strategies is associative.

**Proof.** Same proof as for Proposition 77, using Proposition 124 and Lemma 127. ◀

### B.2.5 Global and local renaming

As before, we start by defining the global renaming operation, defined at first on $\Sigma$-ees on an arena $A$.

▶ **Definition 129.** If $A$ is an arena, we say that $\mathbf{q}$ is a **$\Sigma$-ees on** $A$ if it is a $\Sigma$-ees with the underlying $\mathsf{q}$ an ees on $A$, *i.e.* $\mathscr{C}(\mathsf{q}) \subseteq \mathscr{C}(A)$.

Then, we define the global renaming operation.

▶ **Definition 130.** Let $\mathbf{q}$ be a $\Sigma$-ees on arena $A$ and $f : A \cong A'$ be an isomorphism of arenas. We define $f * \mathbf{q}$ the **global renaming of q by** $f$ as having underlying ees $f * \mathsf{q}$, and labeling:

$$\lambda_{f*\mathsf{q}}(f\,a) \quad = \quad \lambda_\mathsf{q}(a)[f]$$

where $f : A \cong A'$ is regarded as the obvious substitution.

By definition, it is immediate that $f * \mathbf{q}$ is a $\Sigma$-ees on $A$.

As in the case without terms, we prove here compatibility of renaming with the meet operation.

▶ **Lemma 131.** *Let* $\mathbf{q}$ *and* $\mathbf{p}$ *be two orthogonal idempotent* $\Sigma$*-ees on arena* $A$*, and* $f : A \cong A'$ *be an arena isomorphism. Then* $f * \mathbf{q}$ *and* $f * \mathbf{p}$ *are still orthogonal, and*

$$(f * \mathbf{q}) \wedge (f * \mathbf{p}) = f * (\mathbf{q} \wedge \mathbf{p})$$

**Proof.** We already know by Lemma 81 that the equality holds at the level of the underlying strategies. From the stability of equalizers by isomorphisms, $\lambda^{f\,x}_{\gamma*(\mathsf{q}\wedge\mathsf{p})}$ is an equalizer of $\lambda^{f\,x}_{\gamma*\mathsf{q}}$ and $\lambda^{f\,x}_{\gamma*\mathsf{p}}$, and preserves $(\gamma * \mathbf{q}, \gamma * \mathbf{p})$-variables. Hence, by Lemma 114, we have the announced equality. ◀

#### B.2.5.1 Local renaming.

To study the effect of the composition of $\Sigma$-strategies with copycat $\Sigma$-strategies, we extend to $\Sigma$-strategies the renaming operation.

▶ **Definition 132.** Let $\boldsymbol{\sigma} : A^\perp \parallel B$ be a covered $\Sigma$-strategies, and let $f : A' \cong A$ and $g : B \cong B'$ be two isomorphisms of arenas. Then, we define $g \cdot \boldsymbol{\sigma} \cdot f$ to be $(f^{-1} \parallel g) * \boldsymbol{\sigma}$.
By definition, it is immediate that $g \cdot \boldsymbol{\sigma} \cdot f : A'^\perp \parallel B'$ is a $\Sigma$-strategy.

As for plain strategies, this defines actions of isomorphisms of games onto $\Sigma$-strategies.

▶ **Lemma 133.** *For* $\boldsymbol{\sigma} : A_1^\perp \parallel B_1$ *a covered* $\Sigma$*-strategy and* $f_1 : A_1 \cong A_2, f_2 : A_2 \cong A_3, g_1 : B_1 \cong B_2, g_2 : B_2 \cong B_3$ *isomorphisms of arenas, we have:*

$$\begin{array}{rcl} \mathrm{id}_{B_1} \cdot \boldsymbol{\sigma} \cdot \mathrm{id}_{A_1} & = & \boldsymbol{\sigma} \\ g_2 \cdot (g_1 \cdot \boldsymbol{\sigma}) & = & (g_2 \circ g_1) \cdot \boldsymbol{\sigma} \\ (\boldsymbol{\sigma} \cdot f_1) \cdot f_2 & = & \boldsymbol{\sigma} \cdot (f_2 \circ f_1) \end{array}$$

**Proof.** Direct from Lemma 84 and the definition of renamings. ◀

### B.2.6 Composition with copycat strategies

The first step is to define copycat $\Sigma$-strategies.

▶ **Definition 134.** Let $f : A \cong B$ be an isomorphism of arenas. Then, we turn the strategy $\text{æ}_f : A^\perp \parallel B$ into a $\Sigma$-strategy by equipping it with the following labeling.

$$\begin{array}{rcl} \lambda_{\text{æ}_f}((1,a)^-) & = & (1,a) \\ \lambda_{\text{æ}_f}((2,b)^-) & = & (2,b) \\ \lambda_{\text{æ}_f}((1,a)^+) & = & (2,f\,a) \\ \lambda_{\text{æ}_f}((2,f\,a)^+) & = & (1,a) \end{array}$$

▶ **Lemma 135.** *For any isorphism of arenas* $f : A \cong B$*, we have* $\text{æ}_f : A^\perp \parallel B$ *a* $\Sigma$*-strategy.*

**Proof.** By Proposition 66, we observe that negative events of $\text{æ}_A$ are labeled by themselves, and positive events by their unique immediate dependency on the other side – $\Sigma$-receptivity and $\Sigma$-courtesy follow. ◀

Finally, we relate renamings with compositions by $\Sigma$-strategies.

▶ **Lemma 136.** *Let* $\boldsymbol{\sigma} : A^\perp \parallel B$ *be a covered* $\Sigma$*-strategy, and* $g : B \cong B'$*,* $f : A \cong A'$ *be isomorphisms. Then,*

$$\begin{array}{rcl} æ_g \odot \boldsymbol{\sigma} & = & g \cdot \boldsymbol{\sigma} \\ \boldsymbol{\sigma} \odot æ_f & = & \boldsymbol{\sigma} \cdot f \end{array}$$

**Proof.** We only prove the first one, the other one being symmetric. By Lemma 85, we know that the underlying strategies are the same, we only have to check that they share the same labelings. Let $e \in |æ_g \odot \boldsymbol{\sigma}|$. In particular, $e \in |æ_g \circledast \boldsymbol{\sigma}|$. If $e$ is negative, then its labeling is forced in both cases by $\Sigma$-receptivity. If $e$ is positive, then there are two cases. If $e = (3, g\, b) \in |æ_g \circledast \boldsymbol{\sigma}|$, then:

$$\begin{array}{rcl} \lambda_{æ_g \circledast \boldsymbol{\sigma}}((3, g\, b)) & = & (A \parallel \lambda_{æ_g})(3, g\, b)[\lambda_{æ_g \circledast \boldsymbol{\sigma}}] \\ & = & \lambda_{æ_g \circledast \boldsymbol{\sigma}}((2, b)) \\ & = & (\lambda_\sigma \parallel B')((2, b))[\lambda_{æ_g \circledast \boldsymbol{\sigma}}] \\ & = & (\lambda_\sigma \parallel B')((2, b))[(A \parallel \lambda_{æ_g})][\lambda_{æ_g \circledast \boldsymbol{\sigma}}] \\ & = & (\lambda_\sigma \parallel B')((2, b))[(A \parallel \lambda_{æ_g})] \end{array}$$

where the first and third equalities hold by virtue of Lemma 115. For the fourth one, remark that $(\lambda_\sigma \parallel B')((2, b))$ has for free variables negative events in $A^\perp \parallel B$. Hence, after further subtitution by $A \parallel \lambda_{æ_g}$, the free variables are negative in $A^\perp \parallel B^0 \parallel B'$, and therefore invariant under $\lambda_{æ_g \circledast \boldsymbol{\sigma}}$.

From the above, unfolding the definitions, it follows that:

$$\lambda_{æ_g \odot \boldsymbol{\sigma}}((2, g\, b)) = \lambda_\sigma((2, g\, b))[(2, g\, b')/(2, b')]$$

which is what we needed to prove. The last case, where $e = (1, a)$ positive, is similar but slightly easier. ◀

For $A$ a covered arena, write $æ_A = æ_{\mathrm{id}_A} : A^\perp \parallel A$ for the copycat $\Sigma$-strategy on $A$. From all of the above, we imemdiately deduce:

▶ **Corollary 137.** *There is a category* $\Sigma$-Det *with covered arenas as objects,* $\Sigma$*-strategies* $\boldsymbol{\sigma} : A^\perp \parallel B$ *as morphisms from* $A$ *to* $B$*, and the* $\Sigma$*-strategies* $æ_A : A^\perp \parallel A$ *as identities.*

**Proof.** We know that composition is associative by Proposition 128. Finally, for $\boldsymbol{\sigma} : A^\perp \parallel B$, we have $æ_B \odot \sigma = \mathrm{id}_B \cdot \sigma = \sigma$ where the first equality follows from Lemma 136, and the second by definition. Likewise $\sigma \odot æ_A = \sigma$, and we have a category. ◀

In the next subsection, we continue and finish the construction of the category of $\Sigma$-strategies by developping its compact closed structure.

## B.3 Compact closed structure

In this subsection, we investigate the further structure of $\Sigma$-Det, and we show that akin to Det, it is compact closed.

### B.3.1 Symmetric monoidal structure

#### B.3.1.1 Tensor.

First of all, we extend the tensor of Det to a bifunctor:

$$- \otimes - : \Sigma\text{-Det} \times \Sigma\text{-Det} \to \Sigma\text{-Det}$$

As Det and Σ-Det have the same objects, there is no need to redefine ⊗ on objects. We start by defining its bifunctorial action.

▶ **Definition 138.** Let $\boldsymbol{\sigma}_1 : A_1^\perp \parallel B_1$ and $\boldsymbol{\sigma}_2 : A_2^\perp \parallel B_2$ be strategies. We define:

$$\boldsymbol{\sigma}_1 \otimes \boldsymbol{\sigma}_2 = \gamma * (\boldsymbol{\sigma}_1 \parallel \boldsymbol{\sigma}_2) : (A_1 \otimes A_2)^\perp \parallel (B_1 \otimes B_2)$$

with $\gamma : (A_1^\perp \parallel B_1) \parallel (A_2^\perp \parallel B_2) \to (A_1^\perp \parallel A_2^\perp) \parallel (B_1 \parallel B_2)$ the obvious isomorphism.

It is obvious from the definition that this yields a Σ-strategy. First, we check that it preserves the identities.

▶ **Lemma 139.** *Let $A, B$ be arenas. Then, $\text{æ}_{A \otimes B} = \text{æ}_A \otimes \text{æ}_B$.*

**Proof.** This holds for the underlying strategies by Lemma 89. To establish that the term labeling coincide, we need to check that positive events in $\text{æ}_A \otimes \text{æ}_B$ are labeled by their immediate predecessor on the other side, which is direct. ◀

We prove bifunctoriality.

▶ Proposition 140. Let $\boldsymbol{\sigma}_1 : A_1^\perp \parallel B_1, \boldsymbol{\sigma}_2 : A_2^\perp \parallel B_2, \boldsymbol{\tau}_1 : B_1^\perp \parallel C_1$ and $\boldsymbol{\tau}_2 : B_2^\perp \parallel C_2$. Then,

$$(\boldsymbol{\tau}_1 \odot \boldsymbol{\sigma}_1) \otimes (\boldsymbol{\tau}_2 \odot \boldsymbol{\sigma}_2) = (\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_2) \odot (\boldsymbol{\sigma}_1 \otimes \boldsymbol{\sigma}_2)$$

**Proof.** The proof, done equationally at the level of interactions, is the same as for Proposition 90; first unfolding definitions, by Lemma 131, then Lemma 118. The desired statement on composition follows immediately by hiding. ◀

#### B.3.1.2 Symmetric monoidal structure.

We perform as in the case without terms – we have introduced the same technology: copycat Σ-strategies, along with Lemma 136 that allows us to relate renaming of Σ-strategies with composition by copycat strategies. As before the structural isomorphisms for the symmetric monoidal closed structure are all copycat Σ-strategies, and their coherence and naturality all follow directly from Lemma 136. The proofs are the same.

We finished the proof of:

▶ Proposition 141. The tuple $(\Sigma\text{-Det}, \otimes, 1)$ is a symmetric monoidal category.

### B.3.2 Compact closure

Finally, we show that just as Det, Σ-Det is compact closed. First we define the unit and co-unit.

▶ **Definition 142.** Let $A$ be an arena. Then we have two Σ-strategies:

$$\eta_A : 1^\perp \parallel (A^\perp \otimes A) \qquad \epsilon_A : (A \otimes A^\perp)^\perp \parallel 1$$

defined as renamings of $\text{æ}_A : A^\perp \parallel A$.

We can finally state and prove the main result of this section.

▶ Proposition 143. The category Σ-Det is compact closed.

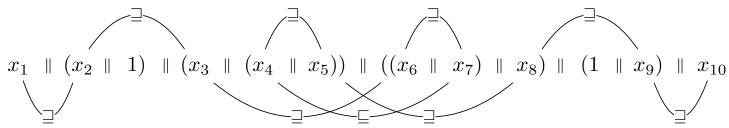
**Proof.** We only have to check the two equations for duals in a compact closed category, *i.e.* for any arena $A$, we have:

$$\begin{array}{rcl} \text{æ}_A & = & \lambda_A \odot (\epsilon_A \otimes \text{æ}_A) \odot \alpha^{-1}_{A,A^\perp,A} \odot (\text{æ}_A \otimes \eta_A) \odot \rho^{-1}_A \\ \text{æ}_{A^\perp} & = & \rho_{A^\perp} \odot (\text{æ}_{A^\perp} \otimes \epsilon_A) \odot \alpha_{A^\perp,A,A^\perp} \odot (\eta_A \otimes \text{æ}_{A^\perp}) \odot \lambda^{-1}_{A^\perp} \end{array}$$

We focus on the first, as the two equations are similar. By repeated applications of Lemma 127, the right hand side of the equation is equal to

$$(\lambda_A \circledast (\epsilon_A \otimes \text{æ}_A) \circledast \alpha^{-1}_{A,A^\perp,A} \circledast (\text{æ}_A \otimes \eta_A) \circledast \rho^{-1}_A)_\downarrow$$

The underlying strategies are all copycat strategies. As observed in the proof of Proposition 95, the configurations of the corresponding interaction have the form below:




We need to prove that the labeling of the compound Σ-strategy coincides with copycat, *i.e.* that each positive event is labeled by its unique negative immediate predecessor on the other side. We prove this using Lemma 115 repeatedly. Take $a^+_{10} \in x_{10}$ above. By Lemma 115, its labeling must be the same as that of $a^0_9 \in x_9$. By Lemma 115 again, its labeling must be the same as that of $a^0_8$ – and so on and so forth, until we reach $a^-_1 \in x_1$, which is a variable. The case of $a^+_1 \in x_1$ is similar. ◀

This concludes the construction of the compact closed category Σ-Det, which will perform the heavy lifting in our model construction, and the bulk of the witness extraction when interpreting classical proofs. Next, we adjoin it with winning conditions and prove that we get a ∗-autonomous category.

## B.4 Further structure of Σ-Det

Now, we lift to Σ-Det the further structure introduced in Section A.6.

### B.4.1 Order-enrichment

As for Section A.6.1, we first prove that Σ-Det is an order-enriched category. The order between Σ-strategies will simply be the order $\preccurlyeq$ from Definition 108, which was used to define (as greatest lower bounds) the interaction of Σ-strategies.

First of all, we characterise $\preccurlyeq$ on Σ-strategies.

▶ **Lemma 144.** *Let $\boldsymbol{\sigma}, \boldsymbol{\tau} : A$ be Σ-strategies on $A$. Then, $\boldsymbol{\sigma} \preccurlyeq \boldsymbol{\tau}$ iff $\sigma \preccurlyeq \tau$, and for all $a \in |\sigma|$, $\lambda_{\boldsymbol{\sigma}}(a) = \lambda_{\boldsymbol{\tau}}(a)$.*

**Proof.** *If.* Obvious.

*Only if.* We know that $\sigma \preccurlyeq \tau$. Take $a \in x \in \mathscr{C}(\sigma)$. By definition, we know that $\lambda^x_{\boldsymbol{\tau}}$ subsumes $\lambda^x_{\boldsymbol{\sigma}}$. Hence, there is $\gamma : x \overset{\mathcal{S}}{\to} x$ such that $\lambda^x_{\boldsymbol{\sigma}} = \lambda^x_{\boldsymbol{\tau}} \circ \gamma$. For $a^- \in x$, this entails that $\gamma(a) = a$. But for $a^+ \in x$, variables in $\lambda^x_{\boldsymbol{\tau}}(a)$ are negative, hence $\lambda^x_{\boldsymbol{\tau}}(a)[\gamma] = \lambda^x_{\boldsymbol{\tau}}(a)$, therefore $\lambda_{\boldsymbol{\sigma}}(a) = \lambda_{\boldsymbol{\tau}}(a)$ as required. ◀

▶ Proposition 145. $\Sigma$-Det is an order-enriched category: for arenas $A$ and $B$, $\Sigma\text{-Det}(A, B)$ is the partial order of $\Sigma$-strategies from $A$ to $B$, ordered by $\preccurlyeq$.

**Proof.** We know by Proposition 111 that $\preccurlyeq$ is a partial order on idempotent $\Sigma$-ees, so it is in particular a partial order on $\Sigma$-strategies.

We first make a few observations, that are easy to check: the partial order $\preccurlyeq$ is preserved (on idempotent $\Sigma$-ees) by parallel composition of $\Sigma$-ees, and by renaming of $\Sigma$-ees on arena $A$ through an iso $f : A \cong A'$. From that, it follows that the tensor of $\Sigma$-strategies preserves $\preccurlyeq$. Finally, as the interaction of $\Sigma$-strategies is a meet for $\preccurlyeq$, it clearly preserves $\preccurlyeq$. Finally, it is clearly preserved by hiding as well, hence by composition. ◀

As observed before, this order-enrichment will only play a minor role in the development, when we will study in what restricted sense cut elimination is preserved by our interpretation.

## B.4.2 Substitution

Since $\Sigma$-strategies carry terms, these terms may have some additional free variables that we may wish to substitute. As we expect in such situations, $\Sigma$-strategies may be organized as an indexed category on top of the category of substitutions.

▶ **Definition 146.** If $\mathcal{V}$ is a (finite) set of free variables we define the compact closed category $\Sigma\text{-Det}(\mathcal{V})$ to be simply $(\Sigma \uplus \mathcal{V})$-Det.

In other terms, the $\Sigma$-strategies in $\Sigma\text{-Det}(\mathcal{V})$ may use in their term labeling not only terms in signature $\Sigma$, but also potentially using free variables in $\mathcal{V}$.

In this section, the main operation we are interested in is the *substitution* operation on $\Sigma$-strategies. We define it first on $\Sigma$-ees.

▶ **Definition 147.** Let $\mathbf{q}$ be a $(\Sigma \uplus \mathcal{V}_2)$-ees on arena $A$, and $\gamma : \mathcal{V}_1 \xrightarrow{\mathcal{S}} \mathcal{V}_2$ be a substitution. Then, we define $\mathbf{q}[\gamma]$ as having same components as $\mathbf{q}$, except for

$$\lambda_{\mathbf{q}[\gamma]}(a) = \lambda_{\mathbf{q}}(a)[\gamma]$$

for $a \in |\mathsf{q}|$. It is clear that this yields a $(\Sigma \uplus \mathcal{V}_1)$-ees on arena $A$.

From that definition, it follows that:

▶ **Lemma 148.** *Let $\boldsymbol{\sigma} : A$ be a $(\Sigma \uplus \mathcal{V}_2)$-strategy, and $\gamma : \mathcal{V}_1 \xrightarrow{\mathcal{S}} \mathcal{V}_2$ be a substitution. Then, $\boldsymbol{\sigma}[\gamma]$ us a $(\Sigma \uplus \mathcal{V}_1)$-strategy on arena $A$.*

**Proof.** We only need to check $\Sigma$-receptivity and $\Sigma$-courtesy. For $\Sigma$-receptivity, take $a^- \in |\sigma|$. We know that $\lambda_{\boldsymbol{\sigma}}(a) = a$, so in particular without variables in $\mathcal{V}_2$. Therefore, $\lambda_{\boldsymbol{\sigma}[\gamma]}(a) = a$ as well as required.

For $\Sigma$-courtesy, take $a^+ \in |\sigma|$. We have $\lambda_{\boldsymbol{\sigma}}(a) \in \mathsf{Tm}_{\Sigma \uplus \mathcal{V}_2}([a]_{\sigma}^-) = \mathsf{Tm}_{\Sigma}([a]_{\sigma}^- \uplus \mathcal{V}_2)$ by $\Sigma$-courtesy (we use, here, the implicit assumption that sets of free variables are always disjoint from the sets of events in a game – this can be easily ensured *w.l.o.g.*). Therefore, $\lambda_{\boldsymbol{\sigma}}(a)[\gamma] \in \mathsf{Tm}_{\Sigma}([a]_{\sigma}^- \uplus \mathcal{V}_1)$ as required. ◀

We wish to prove that substitution extends into a strict compact closed functor. Of all our proof obligations for that, only the preservation of composition is non-trivial – this relies on the lemma below.

▶ **Lemma 149.** *Let $\mathbf{q}, \mathbf{p}$ be orthogonal idempotent $(\Sigma \uplus \mathcal{V}_2)$-ees on arena $A$. Then,*

$$(\mathbf{q}[\gamma] \wedge \mathbf{p}[\gamma]) = (\mathbf{q} \wedge \mathbf{p})[\gamma]$$

**Proof.** First, $\mathbf{q}[\gamma]$ and $\mathbf{p}[\gamma]$ are still idempotent and orthogonal: the $\mathbf{p}$-variables and $\mathbf{p}[\gamma]$-variables are the same. Let $x \in \mathscr{C}(\mathsf{q} \wedge \mathsf{p})$. Recall that $\lambda^x_{\mathbf{q}\wedge\mathbf{p}}$ is defined as the equalizer:

$$x' \xrightarrow{\lambda^x_{\mathbf{q}\wedge\mathbf{p}}} x \underset{\lambda^x_{\mathbf{p}}}{\overset{\lambda^x_{\mathbf{q}}}{\rightrightarrows}} x$$

with the proviso that it is then considered as a substitution from $x$ to $x$ – here $x' \subseteq x$ contains events that actually appear in terms, to ensure the equalizer universal property. This equalizer is computed in $\mathrm{Subst}_{\Sigma \uplus \mathcal{V}_2}$.

Our first observation is that $\mathrm{Subst}_{\Sigma \uplus \mathcal{V}_2}$ can be presented as the sub-category of $\mathrm{Subst}_\Sigma$ with objects all $\mathcal{V} \parallel \mathcal{V}_2$ for $\mathcal{V}$ any (finite) set of variables, and morphisms those of the form

$$\langle \mu, \pi_2 \rangle : \mathcal{V} \parallel \mathcal{V}_2 \stackrel{\mathcal{S}}{\to} \mathcal{V}' \parallel \mathcal{V}_2$$

where the expression above refers to the cartesian structure of $\mathrm{Subst}_\Sigma$. The equalizer above in $\mathrm{Subst}_{\Sigma \uplus \mathcal{V}_2}$ defining $\lambda^x_{\mathbf{q}\wedge\mathbf{p}}$ corresponds accordingly to an equalizer in $\mathrm{Subst}_\Sigma$ (though we only get by hypothesis the equalizer universal property *w.r.t.* the subcategory described above, it follows from an elementary reasoning using the cartesian structure that it holds in fact for $\mathrm{Subst}_\Sigma$):

$$x' \parallel \mathcal{V}_2 \xrightarrow{\langle \lambda^x_{\mathbf{q}\wedge\mathbf{p}}, \pi_2 \rangle} x \parallel \mathcal{V}_2 \underset{\langle \lambda^x_{\mathbf{p}}, \pi_2 \rangle}{\overset{\langle \lambda^x_{\mathbf{q}}, \pi_2 \rangle}{\rightrightarrows}} x \parallel \mathcal{V}_2$$

We then consider the diagram below.

$$\begin{array}{ccccc}
x' \parallel \mathcal{V}_2 & \xrightarrow{\langle \lambda^x_{\mathbf{q}\wedge\mathbf{p}}, \pi_2 \rangle} & x \parallel \mathcal{V}_2 & \underset{\langle \lambda^x_{\mathbf{p}}, \pi_2 \rangle}{\overset{\langle \lambda^x_{\mathbf{q}}, \pi_2 \rangle}{\rightrightarrows}} & x \parallel \mathcal{V}_2 \\
{\scriptstyle x' \parallel \gamma} \uparrow & & {\scriptstyle x \parallel \gamma} \uparrow & & \uparrow {\scriptstyle x \parallel \gamma} \\
x' \parallel \mathcal{V}_1 & \xrightarrow{\langle \lambda^x_{(\mathbf{q}\wedge\mathbf{p})[\gamma]}, \pi_2 \rangle} & x \parallel \mathcal{V}_1 & \underset{\langle \lambda^x_{\mathbf{p}[\gamma]}, \pi_2 \rangle}{\overset{\langle \lambda^x_{\mathbf{q}[\gamma]}, \pi_2 \rangle}{\rightrightarrows}} & x \parallel \mathcal{V}_1
\end{array}$$

Observing that $\lambda^x_{\mathbf{q}[\gamma]} = \lambda^x_{\mathbf{q}} \circ (x \parallel \gamma)$ by definition, it is clear that this diagram commutes. Through a general reasoning using the cartesian structure, it follows as well that all squares are pullbacks. Therefore, the bottom part of the diagram is an equalizer in $\mathrm{Subst}_\Sigma$. For the same reason as above, this means that $\lambda^x_{(\mathbf{q}\wedge\mathbf{p})[\gamma]}$ is an equalizer of $\lambda^x_{\mathbf{q}[\gamma]}$ and $\lambda^x_{\mathbf{p}[\gamma]}$ in $\mathrm{Subst}_{\Sigma \uplus \mathcal{V}_1}$, concluding the proof. ◀

From this, we easily obtain:

▶ **Corollary 150.** *Arenas and $\Sigma$-strategies can be organized into an indexed (order-enriched) compact closed category,* i.e. *the definitions above yield a functor*

$$\Sigma\text{-}\mathsf{Det}(-) : \mathrm{Subst}^{\mathrm{op}} \to \mathsf{CompClosed}$$

*where* $\mathsf{CompClosed}$ *is the category of (order-enriched) compact closed categories and (order-preserving) strict compact closed functors.*

**Proof.** First, we prove that substitution preserves composition. Take $\boldsymbol{\sigma} : A^\perp \parallel B$ and $\boldsymbol{\tau} : B^\perp \parallel C$ be $(\Sigma \uplus \mathcal{V}_2)$-strategies, and $\gamma : \mathcal{V}_1 \xrightarrow{\mathcal{S}} \mathcal{V}_2$. By Lemma 149 (and using that regarding an arena $A$ as a trivial $(\Sigma \uplus \mathcal{V}_2)$-ees we have $A[\gamma] = A$ as each event is labeled by itself):

$$(\boldsymbol{\tau} \circledast \boldsymbol{\sigma})[\gamma] = (\boldsymbol{\tau}[\gamma]) \circledast (\boldsymbol{\sigma}[\gamma])$$

Hence, the same holds after hiding and substitution preserves composition.

It is immediate by definition that substitution preserves the tensor of strategies. Finally, it preserves all copycat strategies, as those have no variables from $\mathcal{V}_2$ in their term annotations. So overall, we have proved that there is a strict compact closed identity-on-objects functor:

$$-[\gamma] : \Sigma\text{-}\mathsf{Det}(\mathcal{V}_2) \to \Sigma\text{-}\mathsf{Det}(\mathcal{V}_1)$$

Finally, it is obvious by definition that this assignment of such functors to substitutions is itself functorial, yielding an indexed structure as announced. ◀

### B.4.3 Functorial shifts

We now lift the functorial shifts of Section A.6.2 to Σ-strategies. As the category Σ-Det has the same objects as Det (namely, arenas), there is nothing to add to the definition of shifts on arenas of Section A.6.2; we only need to define the functorial action of ↑, ↓ and check that this indeed defines functors.

There is, however, an important distinction with respect to the shifts in Det: the shifts are eventually intended to give an interpretation for quantifiers. In particular, they are binders. Now that our strategies carry terms, this binding operation should become visible. Hence, we will annotate shifts with a variable they bind, and we expect them to be functors:

$$\underset{\mathcal{V},\mathsf{x}}{\uparrow}, \underset{\mathcal{V},\mathsf{x}}{\downarrow} : \Sigma\text{-}\mathsf{Det}(\mathcal{V} \uplus \{\mathsf{x}\}) \to \Sigma\text{-}\mathsf{Det}(\mathcal{V})$$

We give the definition below.

▶ **Lemma 151.** *Let $\boldsymbol{\sigma} : A^\perp \parallel B$ be a $(\Sigma \uplus \mathcal{V} \uplus \{\mathsf{x}\})$-strategy. Then, we define $\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}$ to have underlying strategy $\uparrow \sigma$, and labeling:*

$$\begin{array}{rcl}
\lambda_{\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}}((2,\bullet)) &=& (2,\bullet)\\
\lambda_{\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}}((1,\bullet)) &=& (2,\bullet)\\
\lambda_{\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}}(e) &=& \lambda_{\boldsymbol{\sigma}}(e)[(2,\bullet)/\mathsf{x}] \quad (\text{if } e \in |A^\perp \parallel B|)
\end{array}$$

*Then, $\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}$ is a $(\Sigma \uplus \mathcal{V})$-strategy. Likewise, $\downarrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}$, defined dually, is a $(\Sigma \uplus \mathcal{V})$-strategy.*

**Proof.** By definition, term labelings in $\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}$ only involve free variables in $\Sigma \uplus \mathcal{V}$. All the required conditions (Σ-ees, Σ-receptivity, Σ-courtesy) follow from the definition and the observation that $(2,\bullet)$ is minimal in $\uparrow \sigma$. ◀

With that, we prove:

▶ Proposition 152. The operations

$$\underset{\mathcal{V},\mathsf{x}}{\uparrow}, \underset{\mathcal{V},\mathsf{x}}{\downarrow} : \Sigma\text{-}\mathsf{Det}(\mathcal{V} \uplus \{\mathsf{x}\}) \to \Sigma\text{-}\mathsf{Det}(\mathcal{V})$$

yield order-enriched functors.

**Proof.** It is direct to check that $\uparrow_{\mathcal{V},\mathsf{x}}, \downarrow_{\mathcal{V},\mathsf{x}}$ preserve $\preccurlyeq$. We already know from Proposition 99 that $\uparrow_{\mathcal{V},\mathsf{x}}, \downarrow_{\mathcal{V},\mathsf{x}}$ preserve copycat at the causal level; and it is clear by definition that the new positive event is as needed labelled by its negative counterpart. Since the labeling in copycat does not involve $\mathsf{x}$, the substitution in the definition of $\uparrow_{\mathcal{V},\mathsf{x}} \text{æ}_A$ has no effect.

Let us now check that $\uparrow_{\mathcal{V},\mathsf{x}}$ preserves composition. By Proposition 99, for $\boldsymbol{\sigma} : A^\perp \parallel B$ and $\boldsymbol{\tau} : B^\perp \parallel C$, we already know that $\uparrow(\tau \odot \sigma) = \uparrow \tau \odot \uparrow \sigma$ (without the term annotations). We need to show that they also share the same term annotations. For that, we need to characterise the term annotation of events $e \in \uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\tau} \circledast \uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}$. First, we observe:

$$\begin{array}{rcl} \lambda_{\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\tau} \circledast \uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}}(3, \bullet) & = & \lambda_{\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\tau} \circledast \uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}}(2, \bullet) \\ & = & \lambda_{\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\tau} \circledast \uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}}(1, \bullet) \\ & = & (3, \bullet) \end{array}$$

which is direct by definition of the functorial action of $\uparrow_{\mathcal{V},\mathsf{x}}$ and Lemma 115.

Now, we prove by well-founded induction on $\leq_{\tau \circledast \sigma}$ that for all $e \in |\tau \circledast \sigma|$,

$$\lambda_{\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\tau} \circledast \uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}}(e) = \lambda_{\boldsymbol{\tau} \circledast \boldsymbol{\sigma}}(e)[(3, \bullet)/\mathsf{x}]$$

If $e$ is in $A, C$ and is negative for $A^\perp \parallel C$, then it is clear as in both cases, $e$ is labeled by itself. Otherwise $e$ is positive for $\sigma$ or for $\tau$, *w.l.o.g.* say for $\sigma$. Then, we can reason equationally (as usual, leaving some renamings implicit):

$$\begin{array}{rcl} \lambda_{\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\tau} \circledast \uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}}(e) & = & \lambda_{\boldsymbol{\sigma}}(e)[\lambda_{\uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\tau} \circledast \uparrow_{\mathcal{V},\mathsf{x}} \boldsymbol{\sigma}}] \\ & = & \lambda_{\boldsymbol{\sigma}}(e)[\lambda_{\boldsymbol{\tau} \circledast \boldsymbol{\sigma}}][(3, \bullet)/\mathsf{x}] \\ & = & \lambda_{\boldsymbol{\tau} \circledast \boldsymbol{\sigma}}(e)[(3, \bullet)/\mathsf{x}] \end{array}$$

where we use Lemma 115 for the first and third equations, and the second is by induction hypothesis – this exploits that $e$ is assumed positive for $\boldsymbol{\sigma}$, so that the free variables in $\lambda_{\boldsymbol{\sigma}}(e)$ are strictly below $e$ for $\leq_{\tau \circledast \sigma}$ by $\Sigma$-courtesy of $\boldsymbol{\sigma}$.

This concludes the proof that $\uparrow_{\mathcal{V},\mathsf{x}}$ preserves composition, and hence that it is an order-enriched functor. The reasoning for $\downarrow_{\mathcal{V},\mathsf{x}}$ is entirely symmetric. ◀

## B.4.4 Introduction of shifts

Now that we have an indexed compact closed category with shifts being functors between fibres, it is natural to ask whether this data forms – as one would expect – an adjunction. We will see here that it is not the case. However, while investigating this structure we will introduce constructions that will serve later in defining the interpretation of introduction rules for quantifiers.

### B.4.4.1 Introduction of positive shift.

We first define a basic $\Sigma$-strategy introducing a term on a positive shift on the right, and then playing as copycat.

▶ Proposition 153. Let $A$ be an arena, and $t \in \mathsf{Tm}_\Sigma(\mathcal{V})$. We define a tuple $(|A^\perp \parallel \downarrow A|, \leq_{\exists^t_A}, \lambda_{\exists^t_A})$, where $\leq_{\exists^t_A}$ includes $\leq_{\text{æ}_A}$, plus dependencies:

$$\{((2, \bullet), (2, a)) \mid a \in A\} \cup \{((2, \bullet), (1, a)) \mid \exists a_0^- \in A,\ a_0 \leq_A a\}$$

and term assignment $\lambda_{\text{æ}_A}$, plus $\lambda_{\exists^t_A}((2, \bullet)) = t$.

This is a $(\Sigma \uplus \mathcal{V})$-strategy $\exists^t_A : A^\perp \parallel \downarrow A$.

**Proof.** This follows from elementary verifications, observing that the dependencies given are a partial order with immediate causality relation that of $\mathrm{æ}_A$ with, additionally, $(2, \bullet) \rightarrowtriangle (2, a)$ for $a \in A$ minimal in $A$. ◀

As the reader might guess, the notation $\exists^t_A$ for this construction comes from the fact that it will serve as the semantic construction corresponding to the introduction rule for the existential quantifier.

### B.4.4.2 Introduction of negative shift.

Symmetrically, we now introduce the semantic construction that will match introduction of the universal quantifier.

▶ Proposition 154. Let $\boldsymbol{\sigma}$ be a $(\Sigma \uplus \mathcal{V} \uplus \{\mathsf{x}\})$-strategy on $A^\perp \parallel B$. We define a tuple $(|\boldsymbol{\sigma}| \uplus \{(2, \bullet)\}, \leq_{\forall\mathrm{I}^{\mathsf{x}}_{A,B}}, \lambda_{\forall\mathrm{I}^{\mathsf{x}}_{A,B}})$ where

$$\leq_{\forall\mathrm{I}^{\mathsf{x}}_{A,B}(\boldsymbol{\sigma})} \;=\; \leq_{\boldsymbol{\sigma}} \;\cup \{((2,\bullet), s) \mid s \in {\uparrow} B \;\vee\; \exists s' \leq_{\boldsymbol{\sigma}} s.\, \mathsf{x} \in \mathrm{fv}(\lambda_{\boldsymbol{\sigma}}(s'))\}$$

and $\lambda_{\forall\mathrm{I}^{\mathsf{x}}_{A,B}(\boldsymbol{\sigma})}((2,\bullet)) = (2,\bullet)$, and $\lambda_{\forall\mathrm{I}^{\mathsf{x}}_{A,B}(\boldsymbol{\sigma})}(s) = \lambda_{\boldsymbol{\sigma}}(s)[(2,\bullet)/\mathsf{x}]$ for $s \in |\boldsymbol{\sigma}|$.

Then, $\forall\mathrm{I}^{\mathsf{x}}_{A,B}(\boldsymbol{\sigma}) : A^\perp \parallel {\uparrow} B$ is a $(\Sigma \uplus \mathcal{V})$-strategy. Moreover, the operation $\forall\mathrm{I}^{\mathsf{x}}_{A,B}(-)$ preserves the order $\preccurlyeq$ on strategies.

**Proof.** Again, this follows from elementary verifications. ◀

Intuitively, $\forall\mathrm{I}^{\mathsf{x}}_{A,B}(\sigma)$ waits for the new negative move before playing any event whose annotation includes $\mathsf{x}$. It might, however, play a positive move on $A$ whose annotation does not include $\mathsf{x}$. Additionally, in the term annotations, the variable $\mathsf{x}$ is changed to refer instead to the new negative event $(2, \bullet)$ corresponding to the negative shift on the right.

### B.4.4.3 Cancellation of introductions.

As a first consistency check, we verify that the two introduction rules given satisfy the expected equality, corresponding to a cut between a introduction rule for $\forall$ and $\exists$.

▶ **Lemma 155.** *Let $\boldsymbol{\sigma} : A^\perp \parallel B$ be a $(\Sigma \uplus \mathcal{V} \uplus \{\mathsf{x}\})$-strategy. Then, we have:*

$$(\exists^{\mathsf{x}}_B)^\perp \odot \Sigma\text{-}\mathsf{Det}(\mathrm{w}_{\mathcal{V},\mathsf{x}})(\forall_{\mathcal{V},\mathsf{x}}(\boldsymbol{\sigma})) = \boldsymbol{\sigma}$$

*where $\mathrm{w}_{\mathcal{V},\mathsf{x}} : \mathcal{V} \uplus \{\mathsf{x}\} \overset{\mathcal{S}}{\to} \mathcal{V}$ is the weakening substitution.*

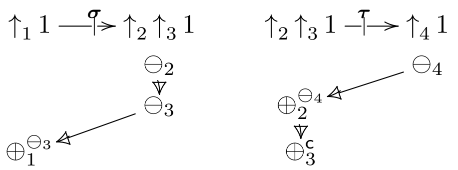
**Proof.** The verification is direct. The synchronization between the two shifts is minimal in the interaction, and causes a renaming of $\bullet$ to $\mathsf{x}$. After it is played, we get an interaction between $\boldsymbol{\sigma}$ and copycat on $B$. ◀

In fact, to the trained eye this looks like a part of an adjunction, expected from categorical logic:

$$\mathcal{V}\text{-}\mathsf{Det} \underset{{\uparrow}_{\mathcal{V},\mathsf{x}}}{\overset{\Sigma\text{-}\mathsf{Det}(\mathrm{w}_{\mathcal{V},\mathsf{x}})}{\underset{\perp}{\rightleftarrows}}} \mathcal{V} \uplus \{\mathsf{x}\}\text{-}\mathsf{Det}$$

But unfortunately, this does not hold. In fact, the operation of Proposition 154, which one would expect to be a natural bijection, is neither a bijection not natural. The counter-example for both aspects is the same, given below.

▶ **Example 156.** Consider $\boldsymbol{\sigma} : \uparrow_1 1 \to\!\!\!\!\!\!+\!\!\!> \uparrow_2 \uparrow_3 1$ and $\boldsymbol{\tau} : \uparrow_2 \uparrow_3 1 \to\!\!\!\!\!\!+\!\!\!> \uparrow_4 1$ two $\Sigma$-strategies (we use $\boldsymbol{\sigma} : A \to\!\!\!\!\!\!+\!\!\!> B$ as an alternative notation for $\boldsymbol{\sigma} : A^\perp \parallel B$):



where we omit the annotation of negative events, forced by $\Sigma$-receptivity.

Note that $\boldsymbol{\tau} = \forall\!\!\!\mathrm{I}^{\mathsf{x}}_{(\uparrow_2 \uparrow_3 1),1}(\oplus_2^{\mathsf{x}} \rightharpoonup \oplus_3^{\mathsf{c}})$. On the other hand, $\boldsymbol{\tau} \odot \boldsymbol{\sigma} = \ominus_4 \rightharpoonup \oplus_1^{\mathsf{c}}$, which cannot be of the form $\forall\!\!\!\mathrm{I}^{\mathsf{x}}_{\uparrow_1 1,1}$ – this construction would put no causal link from $\ominus_4$ to $\oplus_1^{\mathsf{c}}$, since $\mathsf{c}$ does not involve the variable $\mathsf{x}$.

The intuition behind this failure is that $\forall\!\!\!\mathrm{I}^{\mathsf{x}}_{A,B}$ only introduces causal links that follow occurrences of a variable $\mathsf{x}$. However, after composition, we may end up with $\Sigma$-strategies that are not *minimal*, *i.e.* they have immediate causal links not reflecting directly a syntactic dependency. In other words, in order to get an adjunction as one would expect, only the term information would have to be retained – but our interpretation remembers more.

Despite this failure, we can nonetheless prove the following.

▶ Proposition 157. For $\boldsymbol{\tau} : B^\perp \parallel C$ a $(\Sigma \uplus \mathcal{V} \uplus \{\mathsf{x}\})$-strategy and $\boldsymbol{\sigma} : A^\perp \parallel B$ a $(\Sigma \uplus \mathcal{V})$-strategy, then we have:

$$\forall\!\!\!\mathrm{I}^{\mathsf{x}}_{B,C}(\boldsymbol{\tau}) \odot \boldsymbol{\sigma} \preccurlyeq \forall\!\!\!\mathrm{I}^{\mathsf{x}}_{A,C}(\boldsymbol{\tau} \odot \boldsymbol{\sigma}).$$

**Proof.** From Lemma 155 it follows that the two $\Sigma$-strategies have the same residual after the minimal shift. Hence, they can only differ on which events can depend on the initial shift. If an event depends on the shift in the right hand side, then its labeling contains $\mathsf{x}$, and so it must depend on the shift on the left hand side as well. From that observation the inequality follows. ◀

In fact, though we are not going to use it, it turns out that we have a lax 2-adjunction $\Sigma\text{-}\mathsf{Det}(\mathrm{w}_{\mathcal{V},\mathsf{x}}) \dashv \uparrow_{\mathcal{V},\mathsf{x}}$ – this follows from Lemma 155, Proposition 157, and the observation that $\forall\!\!\!\mathrm{I}^{\mathsf{x}}_{A,\uparrow A}((\exists^{\mathsf{x}}_A)^\perp) = \boldsymbol{\alpha}_{\uparrow A}$ for all $A, \mathsf{x}$.

## B.4.5 Countable tensor power

In this section, we lift to $\Sigma$-Det the countable tensor powers from Section A.6.3. As above, since Det and $\Sigma$-Det share the same objects, we only need to define the functorial action and to prove functoriality.

▶ **Definition 158.** Let $\boldsymbol{\sigma} : A^\perp \parallel B$ be a $\Sigma$-strategy. Then, as in Definition 100 we define

$$\otimes^\omega \boldsymbol{\sigma} = \gamma * (\parallel_\omega \boldsymbol{\sigma}) : (\parallel_\omega A)^\perp \parallel (\parallel_\omega B)$$

where $\gamma : \parallel_\omega (A^\perp \parallel B) \cong (\parallel_\omega A)^\perp \parallel (\parallel_\omega B)$ is the obvious isomorphism, and using the renaming operation of Definition 130.

By definition, $\otimes^\omega \boldsymbol{\sigma}$ is a $\Sigma$-strategy. Finally, we have:

▶ Proposition 159. The operation above yields an order-enriched functor:

$$\otimes^\omega : \Sigma\text{-}\mathsf{Det} \to \Sigma\text{-}\mathsf{Det}$$

**Proof.** Direct adaptation of Lemma 139 and Proposition 140. ◀

# C Winning $\Sigma$-strategies

Now, we enrich our arenas with winning conditions, explaining when a $\Sigma$-strategy is *winning*. First, we will give a few preliminaries on infinite formulas; useful in defining the winning conditions. Then we will construct the games with winning proper.

## C.1 Preliminaries on infinite formulas

We start with a few definitions. We assume define, given a signature $\Sigma$ and a set of free variables $\mathcal{V}$, the set $\mathsf{Tm}_\Sigma(\mathcal{V})$ of terms with free variables in $\mathcal{V}$.

▶ **Definition 160.** A **literal** with free variables in $\mathcal{V}$ has the form $\mathsf{P}(t_1, \ldots, t_n)$ or $\neg\mathsf{P}(t_1, \ldots, t_n)$, where $\mathsf{P}$ is a $n$-ary predicate symbol.

We denote the set of literals with free variables in $\mathcal{V}$ by $\mathsf{Lit}_\Sigma(\mathcal{V})$.

In our model construction, there will be a particular focus in *infinitary quantifier-free formulas*. We define them formally, and prove a few useful implications and equivalences below.

▶ **Definition 161.** The **infinitary quantifier-free formula** with variables in $\mathcal{V}$ are given by the grammar below.

$$\begin{array}{lll} \varphi, \psi & ::= & \mathsf{P}(t_1, \ldots, t_n) \mid \neg\mathsf{P}(t_1, \ldots, t_n) \quad (\in \mathsf{Lit}_\Sigma(\mathcal{V})) \\ & & \bot \mid \top \mid \bigwedge_{i \in I} \varphi_i \mid \bigvee_{i \in I} \psi_i \end{array}$$

where $I$ is any at most countable set.

We write $\mathsf{QF}^\infty_\Sigma(\mathcal{V})$ for the set of infinitary quantifier-free formulas on set of variables $\mathcal{V}$. We write $\mathsf{QF}_\Sigma(\mathcal{V})$ for the corresponding set of *finite* formulas, where all disjunctions and conjunctions are finite.

Let us recall quickly their formal Tarskian semantics. If $\mathcal{V}$ is a set of free variables, a **$\mathcal{V}$-valuation** is a function

$$\rho : \mathsf{Lit}_\Sigma(\mathcal{V}) \to \{\bot, \top\}$$

assigning a value to any literal in a way compatible with negation, *i.e.* $\rho(\neg\mathsf{P}(t_1, \ldots, t_n)) = \neg\rho(\mathsf{P}(t_1, \ldots, t_n))$.

▶ **Definition 162.** If $\varphi \in \mathsf{QF}^\infty_\Sigma(\mathcal{V})$ and $\rho$ is a $\mathcal{V}$-valuation, then we define $(\!|\varphi|\!)_\rho$, the evaluation of $\varphi$ following the Tarskian semantics:

$$\begin{array}{rcl} (\!|\top|\!)_\rho & = & \top \\ (\!|\bot|\!)_\rho & = & \bot \\ (\!|\mathsf{P}(t_1, \ldots, t_n)|\!)_\rho & = & \rho(\mathsf{P}(t_1, \ldots, t_n)) \\ (\!|\neg\mathsf{P}(t_1, \ldots, t_n)|\!)_\rho & = & \rho(\neg\mathsf{P}(t_1, \ldots, t_n)) \\ (\!|\bigwedge_{i \in I} \varphi_i|\!)_\rho & = & \left\{ \begin{array}{ll} \top & \text{if for all } i \in I,\ (\!|\varphi_i|\!)_\rho = \top \\ \bot & \text{otherwise} \end{array} \right. \\ (\!|\bigvee_{i \in I} \varphi_i|\!)_\rho & = & \left\{ \begin{array}{ll} \top & \text{if for some } i \in I,\ (\!|\varphi_i|\!)_\rho = \top \\ \bot & \text{otherwise} \end{array} \right. \end{array}$$

Using that we can define *tautologies*:

▶ **Definition 163.** Let $\varphi \in \mathsf{QF}^\infty_\Sigma(\mathcal{V})$ be a (potentially) infinite quantifier-free formula. We say that $\varphi$ is a **tautology** iff for all $\mathcal{V}$-valuation $\rho$, we have $(\!|\varphi|\!)_\rho = \top$.

We write $\models \varphi$ to indicate that $\varphi$ is a tautology.

Finally, we consider defined the **negation** operation on quantifier-free formulas, which to any $\varphi \in \mathsf{QF}^{\infty}_{\Sigma}(\mathcal{V})$ associates $\varphi^{\perp} \in \mathsf{QF}^{\infty}_{\Sigma}(\mathcal{V})$ by induction of formulas, using De Morgan laws, and $\mathsf{P}(t_1, \dots, t_n)^{\perp} = \neg\mathsf{P}(t_1, \dots, t_n)$ and $(\neg\mathsf{P}(t_1, \dots, t_n))^{\perp} = \mathsf{P}(t_1, \dots, t_n)$ on literals.

## C.2 Games and winning $\Sigma$-strategies

From now on, as we approach our objective of modeling logic with those games, we will use intercheangably $\exists/+$ and $\forall/-$ for the polarities and ∃loïse/Player and ∀bélard/Opponent for the two players.

From now on, all the strategies we consider will be $\Sigma$-strategies, *i.e.* associated with terms. Therefore, to alleviate notations and terminology, we might refer to them simply as *strategies* – if we mean to say that a strategy has no term assignment and only the causal structure, we will say so explicitely. Likewise, we will drop the bold font $\boldsymbol{\sigma} : A$ that was useful to emphasize the distinction between strategies and $\Sigma$-strategies, and only refer to $\Sigma$-strategies as $\sigma : A$.

### C.2.1 Games and strategies

#### C.2.1.1 Games.

We give the definition of games, and illustrate it with an example.

▶ **Definition 164.** A **game** $\mathcal{A}$ is an arena $A$ along with a function:

$$\mathcal{W}_{\mathcal{A}} : (x \in \mathscr{C}^{\infty}(A)) \to \mathsf{QF}^{\infty}_{\Sigma}(x)$$

Intuitively, moves of an arena correspond to, depending on their polarity, either an existential or universal quantifier. Hence, a configuration of the arena informs a set of quantifiers which have been visited. It may unlock quantifier-free remnants of the formulas, which are then displayed by $\mathcal{W}_{\mathcal{A}}$. Note that we use the events of the game themselves as free variables.

▶ **Example 165.** Consider the formula DF (for the “Drinker formula”):

$$\exists \mathsf{x} \forall \mathsf{y} \neg \mathsf{P}(\mathsf{x}) \vee \mathsf{P}(\mathsf{y})$$

As we will see later, it will be interpreted by a game, comprising an arena

$$\begin{array}{cccc} \exists_1 & \dots & \exists_n & \dots \\ \vdots & & \vdots & \\ \forall_1 & & \forall_n & \end{array}$$

along with some winning conditions.

The winning function associates the empty configuration with the formula $\perp$ since we are waiting for ∀bélard's input. Similarly, for any $i \in \mathbb{N}$, $\{\exists_i\}$ is associated with $\top$ as now we are waiting for ∀bélard's input. A configuration $\{\exists_i, \forall_i\}$, where both players have played, is associated to the quantifier-free formula $\neg\mathsf{P}(\exists_i) \vee \mathsf{P}(\forall_i)$, a copy of the quantifier-free part of the original formula where variables have been replaced by the corresponding moves in the game.

But the arena comprises countably many copies of $\forall, \exists$ – we give below a few more cases for the winning conditions:

$$\begin{array}{rcl} \mathcal{W}_{[\![DF]\!]}(\{\exists_3, \forall_3, \exists_6, \forall_6\}) & = & (\neg\mathsf{P}(\exists_3) \vee \mathsf{P}(\forall_3)) \vee \\ & & (\neg\mathsf{P}(\exists_6) \vee \mathsf{P}(\forall_6)) \\ \mathcal{W}_{[\![DF]\!]}(\{\exists_3, \forall_3, \exists_6\}) & = & (\neg\mathsf{P}(\exists_3) \vee \mathsf{P}(\forall_3)) \vee \top \end{array}$$

In general, the winning conditions associate to a configuration a $\vee$-expansion of the original formula where variables have been replaced by the moves of the game matching the copy of the quantifier.

Of course, all of that will be made formal later on with the definition of the interpretation of arbitrary formulas of first-order logic as games.

#### C.2.1.2 Winning $\Sigma$-strategies.

If $\mathcal{A}$ is a game, we give the definition of what it means for a $\Sigma$-strategy $\sigma : A$ to be *winning*, and illustrate it with examples.

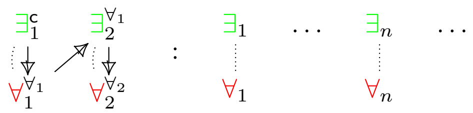
▶ **Definition 166.** If $\sigma : A$ is a $\Sigma$-strategy and $x \in \mathscr{C}^\infty(\sigma)$, we say that $x$ is **tautological** in $\sigma$ if the formula

$$\mathcal{W}_{\mathcal{A}}(x)[\lambda_\sigma]$$

corresponding to the substitution of $\mathcal{W}_{\mathcal{A}}(x) \in \mathsf{QF}^\infty_\Sigma(x)$ by $\lambda_\sigma : x \to \mathsf{Tm}_\Sigma(x)$, is a (possibly infinite) tautology.

A $\Sigma$-strategy $\sigma : A$ is **winning** if any $x \in \mathscr{C}^\infty(\sigma)$ that is **$\exists$-maximal** (*i.e.* $x \in \mathscr{C}^\infty(\sigma)$ such that for all $a \in |\sigma|$ with $x \cup \{a\} \in \mathscr{C}^\infty(\sigma)$, $\mathrm{pol}_A(a) = \forall$) is tautological. We write then $\sigma : \mathcal{A}$.

▶ **Example 167.** For some constant symbol $\mathsf{c} \in \Sigma$, the following diagram denotes a $\Sigma$-strategy on the arena for DF outlined in Example 165.



Moreover, this $\Sigma$-strategy is **winning**: its $\exists$-maximal configurations are:

$$\begin{array}{rcl} \mathcal{W}_{DF}(\{\exists_1\})[\lambda] &=& \top \\ \mathcal{W}_{DF}(\{\exists_1, \forall_1, \exists_2\})[\lambda] &=& (\neg\mathsf{P}(\mathsf{c}) \vee \mathsf{P}(\forall_1)) \vee \top \\ \mathcal{W}_{DF}(\{\exists_1, \forall_1, \exists_2, \forall_2\})[\lambda] &=& (\neg\mathsf{P}(\mathsf{c}) \vee \mathsf{P}(\forall_1)) \vee (\neg\mathsf{P}(\forall_1) \vee \mathsf{P}(\forall_2)) \end{array}$$

which are all tautologies.

### C.2.2 Constructions on games

In the presence of winning conditions, the *parallel composition* operation on arenas used in Sections A and B as the operation on objects yielding the symmetric monoidal structure will split here into two distinct constructions, the *tensor* and the *par*.

▶ **Definition 168.** Let $\mathcal{A}$ and $\mathcal{B}$ be games. Their **tensor** $\mathcal{A} \otimes \mathcal{B}$ and their **par** $\mathcal{A} \parr \mathcal{B}$ have both $A \parallel B$ as underlying arena, and winning conditions, for $x_A \parallel x_B \in \mathscr{C}^\infty(A \parallel B)$:

$$\begin{array}{rcl} \mathcal{W}_{\mathcal{A}\otimes\mathcal{B}}(x_A \parallel x_B) &=& \mathcal{W}_{\mathcal{A}}(x_A) \wedge \mathcal{W}_{\mathcal{B}}(x_B) \\ \mathcal{W}_{\mathcal{A}\parr\mathcal{B}}(x_A \parallel x_B) &=& \mathcal{W}_{\mathcal{A}}(x_A) \vee \mathcal{W}_{\mathcal{B}}(x_B) \end{array}$$

Note that in both cases, we use here the syntactic constructions $\vee$ and $\wedge$ on formulas. The winning conditions yield uninterpreted syntactic objects. Note also that there is an implicit renaming going on there: in $\mathcal{W}_{\mathcal{A}}(x_A) \wedge \mathcal{W}_{\mathcal{B}}(x_B)$ the free variables are in $x_A \cup x_B$, whereas we need them to be in $x_A \parallel x_B$, with the tagged union – we keep such renamings implicit most of the time in the sequel.

Likewise, we extend the notion of the *dual* of a game in the presence of winning conditions.

▶ **Definition 169.** Let $\mathcal{A}$ be a game. We define its **dual**, $\mathcal{A}^\perp$, as having arena $A^\perp$ and as winning conditions, for $x \in \mathscr{C}^\infty(A^\perp)$,

$$\mathcal{W}_{\mathcal{A}^\perp}(x) = \mathcal{W}_{\mathcal{A}}(x)^\perp$$

Note that this definition yields the expected duality between the tensor and the par.

▶ **Lemma 170.** *For any games $\mathcal{A}$ and $\mathcal{B}$, we have:*

$$\begin{array}{rcl}(\mathcal{A} \otimes \mathcal{B})^\perp & = & \mathcal{A}^\perp ⅋ \mathcal{B}^\perp \\ (\mathcal{A} ⅋ \mathcal{B})^\perp & = & \mathcal{A}^\perp \otimes \mathcal{B}^\perp\end{array}$$

**Proof.** Straightforward by definition and the De Morgan laws on formulas. ◀

## C.3 $*$-autonomous structure

In this subsection, we prove that we have a symmetric linearly distributive category with negation – or equivalently, a $*$-autonomous category.

### C.3.1 A category of winning $\Sigma$-strategies

First of all, we need to define winning $\Sigma$-strategies from a game $\mathcal{A}$ to another.

▶ **Definition 171.** Let $\mathcal{A}$ and $\mathcal{B}$ be games. A **winning $\Sigma$-strategy from $\mathcal{A}$ to $\mathcal{B}$** is a winning $\Sigma$-strategy $\sigma : \mathcal{A}^\perp ⅋ \mathcal{B}$.

We also write $\sigma : \mathcal{A} \rightarrow\!\!\!\!+\!\!\!\!> \mathcal{B}$.

First of all, we give snecessary conditions for copycat strategies to be winning.

▶ **Lemma 172.** *Let $\mathcal{A}, \mathcal{B}$ be two games, and $f : A \cong B$ be an isomorphism of arenas. Assume moreover that $f$* preserves winning, *in the sense that for all $x \in \mathscr{C}(A)$, we have*

$$\models \mathcal{W}_{\mathcal{A}}(x) \implies \mathcal{W}_{\mathcal{B}}(f\,x)[f^{-1}]$$

*(we then write $f : \mathcal{A} \to \mathcal{B}$) where $\varphi \implies \psi$ is as usual defined as $\varphi^\perp \vee \psi$.*

*Then, $\text{œ}_f : \mathcal{A} \rightarrow\!\!\!\!+\!\!\!\!> \mathcal{B}$ is a winning $\Sigma$-strategy from $\mathcal{A}$ to $\mathcal{B}$.*

**Proof.** Let $x_A \parallel x_B \in \mathscr{C}^\infty(\text{œ}_f)$ be $+$-maximal. By $+$-maximality though, it follows that $x_B = f\,x_A$. Indeed, if there was *e.g.* $f\,a \in x_B$ with $a \notin x_A$, then $(x_A \cup \{a\}) \parallel x_B$ would be a positive extension of $x_A \parallel x_B$ still in $\mathscr{C}(\text{œ}_f)$.

Then, we observe that we have:

$$\mathcal{W}_{\mathcal{A}^\perp ⅋ \mathcal{B}}(x_A \parallel f\,x_A) = \mathcal{W}_{\mathcal{A}}(x_A) \implies \mathcal{W}_{\mathcal{B}}(f\,x_A)$$

To check that $x_A \parallel x_B$ is tautological, we need to check that the substitution of the above by $\lambda_{\text{œ}_f}$ yields a tautology. Recall that $\lambda_{\text{œ}_f}$ leaves negative events unchanged, and replaces positive events with their negative counterpart on the other side. Hence, we have:

$$\begin{array}{rcl}\mathcal{W}_{\mathcal{A}^\perp ⅋ \mathcal{B}}(x_A \parallel f\,x_A)[\lambda_{\text{œ}_f}] & = & \mathcal{W}_{\mathcal{A}}(x_A)[f\,a/a \mid a^- \in A] \\ & & \implies \mathcal{W}_{\mathcal{B}}(f\,x_A)[a/f\,a \mid a^+ \in A]\end{array}$$

Applying the global renaming exchanging $a$ and $f\,a$ for $a^- \in A$ (which preserves and reflects tautological status), we get:

$$\mathcal{W}_A(x_A) \implies \mathcal{W}_B(f\,x_A)[f^{-1}]$$

a tautology by assumption – in fact, we have proved that preserving winning in the sense above is a necessary and sufficient so that the corresponding copycat strategy is winning, though we shall only use the sufficient part in the sequel. ◀

In particular, it follows from this lemma that for all game $\mathcal{A}$, $\text{cc}_{\mathcal{A}} : \mathcal{A} \rightarrow\!\!\!\!+\!\!\!> \mathcal{A}$ is a winning strategy. We now prove that winning strategies are stable under composition.

▶ **Lemma 173.** *Let $\sigma : \mathcal{A} \rightarrow\!\!\!\!+\!\!\!> \mathcal{B}$ and $\tau : \mathcal{B} \rightarrow\!\!\!\!+\!\!\!> \mathcal{C}$ be two winning strategies. Then, $\tau \odot \sigma : \mathcal{A} \rightarrow\!\!\!\!+\!\!\!> \mathcal{C}$ is still a winning strategy.*

**Proof.** Let $x_A \parallel x_C \in \mathscr{C}^\infty(\tau \odot \sigma)$ be a +-maximal configuration, and consider its witness

$$x_A \parallel x_B \parallel x_C = [x_A \parallel \emptyset \parallel x_C]_{\tau \circledast \sigma} \in \mathscr{C}^\infty(\tau \circledast \sigma)$$

We know that $x_A \parallel x_B \in \mathscr{C}^\infty(\sigma)$ and $x_B \parallel x_C \in \mathscr{C}^\infty(\tau)$. Unfortunately, these two configurations might not be +-maximal in the respective strategies. However, consider the set of all witnesses for $x_A \parallel x_C \in \mathscr{C}^\infty(\tau \odot \sigma)$, *i.e.* the set

$$\{x_A \parallel x'_B \parallel x_C \in \mathscr{C}^\infty(\tau \circledast \sigma)$$

partially ordered by inclusion. It has suprema of all chains – as it is stable under unions – therefore, by Zorn's lemma, it has a maximal element:

$$x_A \parallel x_B^{\max} \parallel x_C \in \mathscr{C}^\infty(\tau \circledast \sigma)$$

Then, $x_A \parallel x_B^{\max} \in \mathscr{C}^\infty(\sigma)$ and $x_B^{\max} \parallel x_C \in \mathscr{C}^\infty(\tau)$ are +-maximal. Indeed, if for instance $x_A \parallel x_B^{\max}$ was *not* +-maximal, it would have a positive extension either in $A$, contradicting +-maximality of $x_A \parallel x_C \in \mathscr{C}^\infty(\tau \odot \sigma)$, or in $B$, which would be matched by an extension of $x_B^{\max} \parallel x_C \in \mathscr{C}^\infty(\tau)$ by receptivity, contradicting the maximality of $x_A \parallel x_B^{\max} \parallel x_C$ amongst interactions witnessing $x_A \parallel x_C \in \mathscr{C}^\infty(\tau \odot \sigma)$.

From the fact that $\sigma$ and $\tau$ are winning, we get then:

$$\begin{array}{ll} \models & (\mathcal{W}_{\mathcal{A}}(x_A) \implies \mathcal{W}_{\mathcal{B}}(x_B^{\max}))[\lambda_\sigma] \\ \models & (\mathcal{W}_{\mathcal{B}}(x_B^{\max}) \implies \mathcal{W}_{\mathcal{C}}(x_C))[\lambda_\tau] \end{array}$$

Since tautologies are stable under substitution, it follows:

$$\begin{array}{ll} \models & (\mathcal{W}_{\mathcal{A}}(x_A) \implies \mathcal{W}_{\mathcal{B}}(x_B^{\max}))[(\lambda_\sigma \parallel C) \circ \lambda_{\tau \circledast \sigma}] \\ \models & (\mathcal{W}_{\mathcal{B}}(x_B^{\max}) \implies \mathcal{W}_{\mathcal{C}}(x_C))[(A \parallel \lambda_\tau) \circ \lambda_{\tau \circledast \sigma}] \end{array}$$

(where there is, again, an implicit renaming of the free variables so that the expression typechecks). However, by Lemma 115, both substitutions are equal to $\lambda_{\tau \circledast \sigma}$; therefore, by transitivity of implication,

$$\models (\mathcal{W}_{\mathcal{A}}(x_A) \implies \mathcal{W}_{\mathcal{C}}(x_C))[\lambda_{\tau \circledast \sigma}]$$

but this is the same as $(\mathcal{W}_{\mathcal{A}}(x_A) \implies \mathcal{W}_{\mathcal{C}}(x_C))[\lambda_{\tau \odot \sigma}]$, and hence $\tau \odot \sigma$ is as required a winning strategy. ◀

Overall, we have proved:

▶ **Corollary 174.** *There is a category* $\Sigma$-Gam *of games and winning* $\Sigma$*-strategies.*

### C.3.2 Bifunctors for $\otimes$ and $⅋$

Here, we prove that the tensor of strategies defined in the previous sections extends to bifunctorial actions for both tensor and par.

▶ **Lemma 175.** *For any* $\sigma : \mathcal{A}_1 \multimap\!\!\!\to \mathcal{B}_1$ *and* $\tau : \mathcal{A}_2 \multimap\!\!\!\to \mathcal{B}_2$*, we have*

$$\sigma \otimes \tau : \mathcal{A}_1 \otimes \mathcal{A}_2 \multimap\!\!\!\to \mathcal{B}_1 \otimes \mathcal{B}_2 \qquad \sigma ⅋ \tau : \mathcal{A}_1 ⅋ \mathcal{A}_2 \multimap\!\!\!\to \mathcal{B}_1 ⅋ \mathcal{B}_2$$

*a winning strategy.*

**Proof.** Let $(x_{A_1} \parallel x_{A_2}) \parallel (x_{B_1} \parallel x_{B_2}) \in \mathscr{C}^\infty(\sigma \otimes \tau)$ be $+$-maximal. By definition of $\sigma \otimes \tau$, we have

$$(x_{A_1} \parallel x_{B_1}) \parallel (x_{A_2} \parallel x_{B_2}) \in \mathscr{C}^\infty(\sigma \parallel \tau)$$

which is $+$-maximal as well, *i.e.* $x_{A_1} \parallel x_{B_1} \in \mathscr{C}^\infty(\sigma)$ and $x_{A_2} \parallel x_{B_2} \in \mathscr{C}^\infty(\tau)$ are $+$-maximal. Since $\sigma$ and $\tau$ are winning, it follows that:

$$\begin{array}{ll} \models & (\mathcal{W}_{\mathcal{A}_1}(x_{A_1}) \implies \mathcal{W}_{B_1}(x_{B_1}))[\lambda_\sigma] \\ \models & (\mathcal{W}_{\mathcal{A}_2}(x_{A_2}) \implies \mathcal{W}_{B_2}(x_{B_2}))[\lambda_\tau] \end{array}$$

Therefore, we have:

$$\begin{array}{ll} \models & (\mathcal{W}_{\mathcal{A}_1}(x_{A_1})[\lambda_\sigma] \wedge (\mathcal{W}_{\mathcal{A}_2}(x_{A_2})[\lambda_\tau])) \\ & \quad \implies (\mathcal{W}_{\mathcal{B}_1}(x_{B_1})[\lambda_\sigma] \wedge \mathcal{W}_{\mathcal{B}_2}(x_{B_2})[\lambda_\tau]) \end{array}$$

by monotonicity of $\wedge$. But that is the same as:

$$\models ((\mathcal{W}_{\mathcal{A}_1}(x_{A_1}) \wedge \mathcal{W}_{\mathcal{A}_2}(x_{A_2})) \implies (\mathcal{W}_{\mathcal{B}_1}(x_{B_1}) \wedge \mathcal{W}_{\mathcal{B}_2}(x_{B_2})))[\lambda_{\sigma\otimes\tau}]$$

(leaving as usual some renamings implicit) as required.

Likewise, the strategy for $\sigma ⅋ \tau$ is the same as for $\sigma \otimes \tau$ (as defined in Section B.3.1), only typed differently. The proof that $\sigma ⅋ \tau : \mathcal{A}_1 ⅋ \mathcal{A}_2 \multimap\!\!\!\to \mathcal{B}_1 ⅋ \mathcal{B}_2$ is the same as above, critically using monotonicity of $\vee$ instead. ◀

Since we already know that these operations preserve identities, we have finished constructing the bifunctorial action of $\otimes$ and $⅋$:

$$\begin{array}{lcl} - \otimes - & : & \Sigma\text{-}\mathsf{Gam} \times \Sigma\text{-}\mathsf{Gam} \to \Sigma\text{-}\mathsf{Gam} \\ - ⅋ - & : & \Sigma\text{-}\mathsf{Gam} \times \Sigma\text{-}\mathsf{Gam} \to \Sigma\text{-}\mathsf{Gam} \end{array}$$

### C.3.3 A symmetric linearly distributive category

We now equip the category $\Sigma$-$\mathsf{Gam}$, together with the two bifunctors $\otimes$ and $⅋$, with the structure of a symmetric linearly distributive category. First, we introduce the necessary units.

▶ **Definition 176.** We define the **games** $1$ **and** $\perp$ to share as arena the empty arena, and with respective winning conditions:

$$\begin{array}{lcl} \mathcal{W}_1(\emptyset) & = & \top \\ \mathcal{W}_\perp(\emptyset) & = & \perp \end{array}$$

With that in place, we state and prove the main result of this subsection.

▶ Proposition 177. The category Σ-Gam is a symmetric linearly distributive category.

**Proof.** All necessary structural strategies are copycat strategies, obtained using Lemma 172 and the observation that the following canonical isomorphisms of arenas all preserve winning, and so do their inverse.

$$\begin{array}{rcll}
\rho^{\otimes}_{\mathcal{A}} & : & \mathcal{A} \otimes 1 & \to & \mathcal{A} \\
\lambda^{\otimes}_{\mathcal{A}} & : & 1 \otimes \mathcal{A} & \to & \mathcal{A} \\
s^{\otimes}_{\mathcal{A},\mathcal{B}} & : & \mathcal{A} \otimes \mathcal{B} & \to & \mathcal{B} \otimes \mathcal{A} \\
\alpha^{\otimes}_{\mathcal{A},\mathcal{B},\mathcal{C}} & : & (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} & \to & \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \\
\rho^{\parr}_{\mathcal{A}} & : & \mathcal{A} \parr \bot & \to & \mathcal{A} \\
\lambda^{\parr}_{\mathcal{A}} & : & \bot \parr \mathcal{A} & \to & \mathcal{A} \\
s^{\parr}_{\mathcal{A},\mathcal{B}} & : & \mathcal{A} \parr \mathcal{B} & \to & \mathcal{B} \parr \mathcal{A} \\
\alpha^{\parr}_{\mathcal{A},\mathcal{B},\mathcal{C}} & : & (\mathcal{A} \parr \mathcal{B}) \parr \mathcal{C} & \to & \mathcal{A} \parr (\mathcal{B} \parr \mathcal{C})
\end{array}$$

That these preserve winning boils down to the fact that they correspond to equivalences at the propositional level. There is a faithful forgetful functor from Σ-Gam to Σ-Det sending 1 and ⊥ to 1, both ⊗ and ⅋ to ⊗ in the strict sense; and each of the copycat strategies above to the corresponding structural morphism for the symmetric monoidal structure of Σ-Det. It follows automatically that they satisfy the required coherence and naturality conditions, equipping Σ-Gam with two symmetric monoidal structures $(\Sigma\text{-}\mathsf{Gam}, \otimes, 1)$ and $(\Sigma\text{-}\mathsf{Gam}, \parr, \bot)$.

Finally, we check that there is a *linear distributivity* natural transformation. We notice that the associativity isomorphism of arenas also *preserves winning*:

$$\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}} : \mathcal{A} \otimes (\mathcal{B} \parr \mathcal{C}) \to (\mathcal{A} \otimes \mathcal{B}) \parr \mathcal{C}$$

which boils down to the fact that for any $\varphi_1, \varphi_2, \varphi_3 \in \mathsf{QF}^{\infty}_{\Sigma}(\mathcal{V})$,

$$\models \varphi_1 \wedge (\varphi_2 \vee \varphi_3) \implies (\varphi_1 \wedge \varphi_2) \vee \varphi_3$$

Note that unlike previously, the inverse of this isomorphism of arenas does not preserve winning. The coherence [?] and naturality conditions are again direct hrough the strict monoidal faithful forgetful operation. ◀

### C.3.4 Negation

Finally, we show that this symmetric linearly distributive category has negation. Again, we will do that by showing that the units and co-units for the compact closed structure of Σ-Det can be enriched with winning conditions.

▶ **Lemma 178.** *Let $\mathcal{A}$ be any game. Then, the following are winning strategies:*

$$\eta_A : 1 \twoheadrightarrow \mathcal{A}^{\perp} \parr \mathcal{A} \qquad \epsilon_A : \mathcal{A} \otimes \mathcal{A}^{\perp} \twoheadrightarrow \bot$$

**Proof.** The proof follows the exact same lines as for Lemma 172. By a direct analysis of +-maximal configurations and their labelings, the lemma boils down to the fact that for any formula $\varphi$, the following are tautologies.

$$\begin{array}{ll}
\models & \top \implies \varphi^{\perp} \vee \varphi \\
\models & \varphi \wedge \varphi^{\perp} \implies \bot
\end{array}$$

Through the same faithful forgetful operation as above, the required equations follow from the corresponding equations for the compact closure of Σ-Det. ◀

Together, we have finished the proof of:

▶ **Corollary 179.** *The category* Σ-Gam *is a symmetric linearly distributive category with negation, or equivalently [?] a ∗-autonomous category.*

## C.4 Further structure of $\Sigma$-Gam

First, we observe that just as $\Sigma$-Det, $\Sigma$-Gam is an order-enriched category. There is nothing to prove: the partial order $\preccurlyeq$ on $\Sigma$-strategies is of course also a partial order on *winning* $\Sigma$-strategies, and we already know that all operations on $\Sigma$-strategies are compatible with it.

On games and winning $\Sigma$-strategies we focus on three elements of further structure: first, we construct a version of our $*$-autonomous category over each set of free variables $\mathcal{V}$ and we show that this is functorial. Then, we enrich the shifts of Section B.4.3 to a interpretation of quantifiers, which allow to move between fibres. Finally, we introduce exponential modalities and prove some of their properties.

### C.4.1 Substitution

We show that the constructions of Section B.4.2 extend with winning.

▶ **Definition 180.** Let $\mathcal{V}$ be a (finite) set of variables. We define the $*$-autonomous order-enriched category:

$$\Sigma\text{-}\mathsf{Gam}(\mathcal{V}) = (\Sigma \uplus \mathcal{V})\text{-}\mathsf{Gam}$$

and refer to its objects as **$\mathcal{V}$-games**.

As usual, in such situations, substitution can be used to transport between fibres functorially. But unlike in Section B.4.2, substitution now have an action on $\mathcal{V}$-games as well.

▶ **Definition 181.** Let $\gamma : \mathcal{V}_1 \overset{\mathcal{S}}{\to} \mathcal{V}_2$, and $\mathcal{A}$ be a $\mathcal{V}_2$-game. Then the substituted $\mathcal{V}_1$-game $\mathcal{A}[\gamma]$ has unchanged arena $A$, and winning conditions for $x \in \mathscr{C}^\infty(A)$:

$$\mathcal{W}_{\mathcal{A}[\gamma]}(x) = \mathcal{W}_{\mathcal{A}}(x)[\gamma] \in \mathsf{QF}^\infty_{\Sigma \uplus \mathcal{V}_1}(x)$$

It is obvious from this definition that substitution commutes with all operations on $\mathcal{V}$-games, *i.e.* $(-)^\perp, \otimes$ and $\parr$. Substitution also preserves winning:

▶ **Lemma 182.** *Let $\gamma : \mathcal{V}_1 \overset{\mathcal{S}}{\to} \mathcal{V}_2$ be a substitution, $\mathcal{A}$ a $\mathcal{V}_2$-game, and $\sigma : \mathcal{A}$ a winning $(\Sigma \uplus \mathcal{V}_2)$-strategy. Then,*

$$\sigma[\gamma] : \mathcal{A}[\gamma]$$

*is a winning $(\Sigma \uplus \mathcal{V}_1)$-strategy.*

**Proof.** Let $x \in \mathscr{C}^\infty(\sigma)$ be $\exists$-maximal. Since $\sigma$ is winning,

$$\models \mathcal{W}_{\mathcal{A}}(x)[\lambda_\sigma].$$

Tautologies are stable under substitution, so $\mathcal{W}_{\mathcal{A}}(x)[\lambda_\sigma][\gamma]$ is a tautology as well. But that is the same as $\mathcal{W}_{\mathcal{A}}(x)[\gamma][\lambda_{\sigma[\gamma]}]$ as substitution by $\gamma$ does not create variables in $x$. Hence, $\sigma[\gamma]$ is winning as required. ◀

From that plus the developments of Section B.4.2, we get:

▶ **Corollary 183.** *Games and winning $\Sigma$-strategies can be organized into an indexed order-enriched) $*$-autonomous category,* i.e. *a functor*

$$\Sigma\text{-}\mathsf{Gam}(-) : \mathrm{Subst}^{\mathrm{op}} \to *\text{-}\mathsf{Aut}$$

*where $*$-Aut is the category of (order-enriched) $*$-autonomous categories and (oder-preserving) strict $*$-autonomous functors.*

**Proof.** Substitution is defined on games above, and substitution as defined on $\Sigma$-strategies in Section B.4.2 preserves winning by Lemma 182. All further verifications are obvious or independent of winning, in which case they follow from Corollary 150. ◀

## C.4.2 Linear Quantifiers

We enrich with winning the shifts of Sections B.4.3 and B.4.4. This way we will get a notion of quantifiers, though without the exponentials to come, they will remain linear and will not support contraction.

### C.4.2.1 Functorial quantifiers.

The first step is to add winning to the definition of the shifts.

▶ **Definition 184.** For $\mathcal{A}$ a $(\mathcal{V} \uplus \{\mathsf{x}\})$-game, the $\mathcal{V}$-game $\forall \mathsf{x}.\mathcal{A}$ and its dual $\exists \mathsf{x}.\mathcal{A}$ have arenas $\forall.A$ and $\exists.A$ respectively, and:

$$\begin{array}{rclrcl} \mathcal{W}_{\forall \mathsf{x}.\mathcal{A}}(\emptyset) & = & \top & \mathcal{W}_{\forall \mathsf{x}.\mathcal{A}}(\forall.x_A) & = & \mathcal{W}_{\mathcal{A}}(x_A)[\bullet/\mathsf{x}] \\ \mathcal{W}_{\exists \mathsf{x}.\mathcal{A}}(\emptyset) & = & \bot & \mathcal{W}_{\exists \mathsf{x}.\mathcal{A}}(\exists.x_A) & = & \mathcal{W}_{\mathcal{A}}(x_A)[\bullet/\mathsf{x}] \end{array}$$

In other words, on a universal quantifier, Opponent is supposed to start: if he doesn't, that is a win for Player regardless of the rest of $\mathcal{A}$. Dually, on an existential quantifier Player is supposed to provide a witness and loses if he fails to do so. In both cases, once the initial move has been played, we continue on $\mathcal{A}$ with the variable $\mathsf{x}$ replaced with the newly introduced witness $\bullet$.

Now that this is introduced, we only have to check that all our operations on $\Sigma$-strategies involving lift (functorial action, introduction of existential and universal quantifiers) still typecheck in the presence of winning.

▶ **Lemma 185.** *Let $\sigma : \mathcal{A}^\perp \parr \mathcal{B}$ be a winning $(\Sigma \uplus \mathcal{V} \uplus \{\mathsf{x}\})$-strategy between $(\mathcal{V} \uplus \{\mathsf{x}\})$-games. Then,*

$$\underset{\mathcal{V},\mathsf{x}}{\uparrow} \sigma : (\forall \mathsf{x}.\,\mathcal{A})^\perp \parr (\forall \mathsf{x}.\,\mathcal{B}) \qquad \underset{\mathcal{V},\mathsf{x}}{\downarrow} \sigma : (\exists \mathsf{x}.\,\mathcal{A})^\perp \parr (\exists \mathsf{x}.\,\mathcal{B})$$

*are winning $(\Sigma \uplus \mathcal{V})$-strategies between $\mathcal{V}$-games. We rename them $\forall_{\mathcal{V},\mathsf{x}}\sigma$ and $\exists_{\mathcal{V},\mathsf{x}}\sigma$ respectively.*

**Proof.** We only prove the first, the other one being symmetric.

Let $x = x_{\uparrow A} \parallel x_{\uparrow B} \in \mathscr{C}^\infty(\uparrow_{\mathcal{V},\mathsf{x}} \sigma)$ be $+$-maximal. If $x$ is empty, then $\mathcal{W}_{(\forall \mathsf{x}.\,\mathcal{A})^\perp \parr (\forall \mathsf{x}.\,\mathcal{B})}(\emptyset) = \bot \vee \top$ is a tautology. Otherwise, $x$ has the form

$$x = (\{\bullet_A\} \cup x_A) \parallel (\{\bullet_B\} \cup x_B)$$

with $x_A \parallel x_B \in \mathscr{C}^\infty(\sigma)$ $+$-maximal as well. Furthermore, we have

$$\begin{array}{rl} & \mathcal{W}_{(\forall \mathsf{x}.\,\mathcal{A})^\perp \parr (\forall \mathsf{x}.\,\mathcal{B})}(x)[\lambda_{\uparrow_{\mathcal{V},\mathsf{x}}\sigma}] \\ = & (\mathcal{W}_{\mathcal{A}}(x_A)^\perp[\bullet_A/\mathsf{x}] \vee \mathcal{W}_{\mathcal{B}}(x_B)[\bullet_B/\mathsf{x}])[\bullet_B/\bullet_A][\lambda_\sigma][\bullet_B/\mathsf{x}] \\ = & (\mathcal{W}_{\mathcal{A}}(x_A)^\perp \vee \mathcal{W}_{\mathcal{B}}(x_B))[\lambda_\sigma][\bullet_B/\mathsf{x}] \end{array}$$

which is a tautology. ◀

From this along with the developments of Section B.4.3, we immediately get the counterpart with winning of Proposition 152.

▶ Proposition 186. The operations

$$\forall_{\mathcal{V},\mathsf{x}}, \exists_{\mathcal{V},\mathsf{x}} : \Sigma\text{-}\mathsf{Gam}(\mathcal{V} \uplus \{\mathsf{x}\}) \to \Sigma\text{-}\mathsf{Gam}(\mathcal{V})$$

yield order-enriched functors.

#### C.4.2.2 Introduction rules.

Now, we show that the semantic constructions of Section B.4.4 for the introduction of shifts also extend in the presence of winning conditions, to get semantic counterparts of the introduction rules for quantifiers.

▶ **Lemma 187.** *Let $A$ be an arena, and $t \in \mathsf{Tm}_\Sigma(\mathcal{V})$. Then,*

$$\exists^t_A : \mathcal{A}^\perp[t/\mathsf{x}] \parr \exists \mathsf{x}\mathcal{A}$$

*is a winning $(\Sigma \uplus \mathcal{V})$-strategy.*

**Proof.** Let $x_A \parallel x_{\downarrow A} \in \mathscr{C}^\infty(\exists^t_A)$ be a +-maximal configuration. As $\bullet$ is minimal in $\exists^t_A$, and by property of +-maximal configurations of copycat, $x$ necessarily has the form $x_A \parallel (\{\bullet\} \cup x_A)$ where $x_A \in \mathscr{C}^\infty(A)$. It follows by definition of $\lambda_{\exists^t_A}$ that:

$$\begin{array}{rl} & \mathcal{W}_{\mathcal{A}^\perp[t/\mathsf{x}]\parr\exists\mathsf{x}\mathcal{A}}(x)[\lambda_{\exists^t_A}] \\ = & (\mathcal{W}_A(x_A)[t/\mathsf{x}]^\perp \vee \mathcal{W}_A(x_A)[\bullet/\mathsf{x}])[t/\bullet] \\ = & (\mathcal{W}_A(x_A)^\perp \vee \mathcal{W}_A(x_A))[t/\mathsf{x}] \end{array}$$

which is a tautology. ◀

Now, we also prove the semantic correctness of the introduction of the universal quantifier.

▶ **Lemma 188.** *Let $\sigma : \mathcal{A}[\mathrm{w}_{\mathcal{V},\mathsf{x}}]^\perp \parr \mathcal{B}$ be a winning $(\Sigma \uplus \mathcal{V} \uplus \{\mathsf{x}\})$-strategy. Then,*

$$\forall I^{\mathsf{x}}_{A,B}(\sigma) : \mathcal{A}^\perp \parr \forall \mathsf{x}\mathcal{B}$$

*is also winning.*

**Proof.** Let $x = x_A \parallel x_{\uparrow B} \in \mathscr{C}^\infty(\forall\mathrm{I}^{\mathsf{x}}_{A,B}(\sigma))$. We distinguish two cases. If $\bullet \notin x_{\uparrow B}$, $x = x_A \parallel \emptyset$, and

$$\mathcal{W}_{\mathcal{A}^\perp\parr\forall\mathsf{x}\mathcal{B}}(x)[\lambda_{\forall\mathrm{I}^{\mathsf{x}}_{A,B}(\sigma)}] = \mathcal{W}_{\mathcal{A}}(x_A)[\lambda_{\forall\mathrm{I}^{\mathsf{x}}_{A,B}(\sigma)}]^\perp \vee \top$$

is a tautology.

Otherwise, $x = x_A \parallel \{\bullet\} \cup x_B$ where $x_A \parallel x_B \in \mathscr{C}^\infty(\sigma)$ is +-maximal, and

$$\begin{array}{rl} & \mathcal{W}_{\mathcal{A}^\perp\parr\forall\mathsf{x}\mathcal{B}}(x)[\lambda_{\forall\mathrm{I}^{\mathsf{x}}_{A,B}(\sigma)}] \\ = & (\mathcal{W}_{\mathcal{A}}(x_A)^\perp \vee \mathcal{W}_{\mathcal{B}}(x_B)[\bullet/\mathsf{x}])[\lambda_\sigma][\bullet/\mathsf{x}] \\ = & (\mathcal{W}_{\mathcal{A}}(x_A)^\perp[\bullet/\mathsf{x}] \vee \mathcal{W}_{\mathcal{B}}(x_B)[\bullet/\mathsf{x}])[\lambda_\sigma][\bullet/\mathsf{x}] \\ = & (\mathcal{W}_{\mathcal{A}[\mathrm{w}_{\mathcal{V},\mathsf{x}}]}(x_A)^\perp \vee \mathcal{W}_{\mathcal{B}}(x_B))[\lambda_\sigma][\bullet/\mathsf{x}] \end{array}$$

where the second equality uses that $\mathcal{A}$ is a $\mathcal{V}$-game, so that $\mathsf{x}$ cannot appear in $\mathcal{W}_{\mathcal{A}}(x_A)$. The formula we obtain is a tautology, since $\sigma$ is winning. ◀

We now know that these constructions preserve winning. It is important to note that besides this, we can use with no further proof all laws proved in Sections B.4.3 and B.4.4, as the data as not changed (we have only learned some further property with respect to winning).

In particular, we have (again, we note that as an aside, this will not be used) as in Section B.4.4 that there is a lax 2-adjunction:

$$\Sigma\text{-}\mathsf{Gam}(\mathcal{V}) \underset{\forall_{\mathcal{V},\mathsf{x}}}{\overset{\Sigma\text{-}\mathsf{Gam}(\mathrm{w}_{\mathcal{V},\mathsf{x}})}{\rightleftarrows}} \Sigma\text{-}\mathsf{Gam}(\mathcal{V} \uplus \{\mathsf{x}\}) \quad (\perp)$$

As things stands, this is *not* an adjunction, as illustrated by Example 156. The quotient required to make the missing equality hold is not in general respected by composition. We expect it *is* respected by composition with interpretations of proofs though, so hope is not lost – however investigating this is left for future work.

### C.4.3 Exponentials

Now, we introduce the final ingredient of the model construction: the exponentials, the integration of which will allow us to interpret the contraction rule. There will be two exponentials ! and ?, which are dual enrichments with winning of the countable tensor power of Section A.6.3.

It is worth noting here that unfortunately, those will not satisfy all the laws and equations expected of models of the Linear Logic exponentials. In an adequate extension of the presented work with symmetry, we expect we could prove that games of the form $!\mathcal{A}$ are comonoids and objects of the form $?\mathcal{A}$ are monoids. However, with the interpretation in mind it would not help much, since strategies coming from proofs do not respect these monoid and comonoid structures, not even in a lax way. Hence our only aim (and proof obligation) in this section is to define the strategies, show that they are winning, and show that all operations involved in the interpretation of proofs preserve winning. We do not aim to prove only equations, only the existence of adequate winning strategies.

The developments of this section are for the most part independent of the signature $\Sigma$ and the underlying set of variables $\mathcal{V}$, so we will often omit these annotations and speak only of games, winning strategies etc.

▶ **Definition 189.** Let $\mathcal{A}$ be a $\mathcal{V}$-game. We define two new $\mathcal{V}$-games $!\mathcal{A}$ and $?\mathcal{A}$ with arena $\|_\omega A$, and winning conditions:

$$\begin{array}{rcl}\mathcal{W}_{!\mathcal{A}}(\|_{i\in\omega} x_i) & = & \bigwedge_{i\in\omega} \mathcal{W}_{\mathcal{A}}(x_i)\\ \mathcal{W}_{?\mathcal{A}}(\|_{i\in\omega} x_i) & = & \bigvee_{i\in\omega} \mathcal{W}_{\mathcal{A}}(x_i)\end{array}$$

Note here that those are countable conjunctions and disjunctions, with a component for every copy index. Formally we get an infinite formula even for finite configurations, those it is finite if one reasons up to idempotency.

We prove that the action of ! and ? can be extended to strategies.

▶ **Lemma 190.** *Let $\sigma : \mathcal{A}^\perp \parr \mathcal{B}$ be a winning strategy. Then,*

$$!\sigma : (!\mathcal{A})^\perp \parr !\mathcal{B} \qquad ?\sigma : (?\mathcal{A})^\perp \parr ?\mathcal{B}$$

*are also winning, where both are defined as $\|^\omega \sigma$ (Definition 158).*

**Proof.** Let $x = (\|_i x_A^i) \parallel (\|_i x_B^i) \in \mathscr{C}^\infty(\|^\omega \sigma)$ be +-maximal. By definition, we have that for all $i \in \omega$, $x_A^i \parallel x_B^i \in \mathscr{C}(\sigma)$ is +-maximal in $\sigma$. Therefore, $(\mathcal{W}_{\mathcal{A}}(x_A^i)^\perp \vee \mathcal{W}_{\mathcal{B}}(x_B^i))[\lambda_\sigma]$ is a tautology. It is then elementary to prove that both $\mathcal{W}_{(!\mathcal{A})^\perp \parr !\mathcal{B}}(x)[\lambda_{\|^\omega\sigma}]$ and $\mathcal{W}_{(?\mathcal{A})^\perp \parr ?\mathcal{B}}(x)[\lambda_{\|^\omega\sigma}]$ are tautologies as well. ◀

#### C.4.3.1 Basic winning strategies.

First of all, we notice that there is always a winning strategy for the weakening.

▶ **Lemma 191.** *For any $\mathcal{A}$, there is a winning $\mathcal{A} \rightarrow\!\!\!\!+\!\!\!\!\rightarrow 1$.*

**Proof.** The strategy is simply the minimal one, just closed under receptivity (hence comprising the minimal negative events of $A^\perp$, labeled by themselves). It is clearly winning by definition of winning on 1. ◀

We obtain important winning strategies as particular copycat strategies.

▶ **Lemma 192.** *Let $\mathcal{A}$ be a $\mathcal{V}$-game. Then the following isomorphisms of arenas preserve winning, yielding copycat winning strategies:*

$$\begin{array}{rcl} !\mathcal{A} & \to & !!\mathcal{A} \\ (\langle i,j\rangle, a) & \mapsto & (i,(j,a)) \end{array} \qquad \begin{array}{rcl} !\mathcal{A} & \to & !\mathcal{A}\otimes !\mathcal{A} \\ (2i,a) & \mapsto & (1,(i,a)) \\ (2i+1,a) & \mapsto & (2,(i,a)) \end{array}$$

$$\begin{array}{rcl} !\mathcal{A}\otimes !\mathcal{B} & \to & !(\mathcal{A}\otimes\mathcal{B}) \\ (j,(i,a)) & \mapsto & (i,(j,a)) \end{array} \qquad \begin{array}{rcl} !\mathcal{A}\parr !\mathcal{B} & \to & !(\mathcal{A}\parr\mathcal{B}) \\ (j,(i,a)) & \mapsto & (i,(j,a)) \end{array}$$

$$\begin{array}{rcl} ?!\mathcal{A} & \to & !?\mathcal{A} \\ (i,(j,a)) & \mapsto & (j,(i,a)) \end{array}$$

**Proof.** Elementary verifications. ◀

Note in passing that not all of these copycat strategies have an inverse that is winning as well. At this point, ! satisfies all the data for a linear exponential comonad (we will not prove any equation – drawing free inspiration from homotopy type theory we might say that it is a *truncated linear exponential comonad*, in the sense that we only care about the inhabitation of certain games, and not about the identity of the inhabitants), and by duality we obtain similar data for ?, dual to !.

#### C.4.3.2 Non-copycat winning strategies.

We introduce the final winning strategies used in our model construction. First, we introduce dereliction. Its behaviour is the same as a copycat strategy, but it does not follow an isomorphism.

▶ **Lemma 193.** *For any game $\mathcal{A}$, there is a winning strategy (dereliction):*

$$!\mathcal{A} \,{\to\!\!\!\!+\!\!\!>}\, \mathcal{A}$$

**Proof.** The causal structure of the strategy is defined through Proposition 61, by setting the corresponding configuration-strategy to comprise the finite configurations of the form

$$\|_{i\in\omega}\, x_i \parallel y \in \mathscr{C}(\|^\omega\, A \parallel A)$$

such that $y \sqsubseteq_A x_0$, and for all $i > 0$, $\mathrm{pol}_A(x_i) \subseteq \{-\}$. In other words, the strategy plays like copycat on the 0th copy, and is only closed under receptivity for all the other copies.

By the same reasoning as in Proposition 66, it follows that the causal history of a positive move $(2,a)^+ \in \|^\omega\, A \parallel A$ comprises $(1,(0,a))^- \in \|^\omega\, A \parallel A$ and the causal history of a positive move $(1,(0,a))^+ \in \|^\omega\, A \parallel A$ comprises $(2,a)^- \in \|^\omega\, A \parallel A$ – we set as annotations $\lambda((2,a)^+) = (1,(0,a))^-$ and $\lambda((1,(0,a))^+) = (2,a)^-$.

The $+$-maximal configurations of the strategy have the form

$$\|_{i\in\omega}\, x_i \parallel x_0.$$

From the labeling above, it yields the formula (up to bijective renaming):

$$(\bigwedge_{i\in\omega} \mathcal{W}_{\mathcal{A}}(x_i))^{\perp} \vee \mathcal{W}_{\mathcal{A}}(x_0)$$

which is a tautology. ◀

Finally, the last winning strategy we construct describes a distribution of the existential quantifier over the exponential, modulo an infinitary explosion.

▶ **Lemma 194.** *Let $\mathcal{A}$ be a $(\mathcal{V} \uplus \{\mathsf{x}\})$-game. Then, there is a winning $(\Sigma \uplus \mathcal{V})$-strategy:*

$$\exists \mathsf{x} ! \mathcal{A} \multimap ! \exists \mathsf{x} \mathcal{A}.$$

**Proof.** First, we describe its causal structure via the set of finite configurations, relying on Proposition 61. Its finite configurations are those of the form

$$x = x_{\downarrow !A} \parallel (\parallel_{i\in\omega} y^i_{\downarrow A}) \in \mathscr{C}((\downarrow !A)^{\perp} \parallel ! \downarrow A)$$

such that if there is $i \in \omega$ such that $\bullet \in x^i_{\downarrow A}$, then $\bullet \in x_{\downarrow !A}$; and such that

$$x \cap \mathscr{C}((!A)^{\perp} \parallel !A) = (\parallel_{i\in\omega} x^i_A) \parallel (\parallel_{i\in\omega} y^i_A) \in \mathscr{C}(\alpha_{!A}).$$

It is a direct verification that this defines a configuration-strategy. Its causal dependency is (the transitive closure of) that of the game, enriched with the immediate causal dependencies of $\alpha_{!A}$ plus those of the form $(1, \bullet) \rightarrowtriangle (2, (i, \bullet))$ for every $i \in \omega$.

This is lifted to a $(\Sigma \uplus \mathcal{V})$-strategy by setting as labeling that of $\alpha_{!\mathcal{A}}$, plus (necessarily) $\lambda((1, \bullet)) = (1, \bullet)$; along with $\lambda((2, (i, \bullet))) = (1, \bullet)$ for all $i \in \omega$.

It remains to check that it is winning. Its $+$-maximal configurations are either the empty set (clearly tautological), plus those of the form:

$$(\{\bullet\} \cup (\parallel_{i\in\omega} x^i_A)) \parallel (\parallel_{i\in\omega} \{\bullet\} \cup x^i_A).$$

By definition of the winning conditions and the labeling, this yields (a bijective renaming of) the formula

$$((\bigwedge_{i\in\omega} \mathcal{W}_{\mathcal{A}}(x^i_A))^{\perp} \vee (\bigwedge_{i\in\omega} \mathcal{W}_{\mathcal{A}}(x^i_A)))[\bullet/\mathsf{x}]$$

which is a tautology. ◀

Intuitively, this strategy waits for Opponent to play the existential quantifier on the left. When it is played, the strategy proceeds to simultaneously forward it to all copies on the right. It then carries on as copycat. This infinitary explosion appears as a clear source of infinitary behaviour in our construction. However, it will actually only be used as an intermediate step, to show that there is a winning contraction strategy on games coming from the interpretation of formulas. While the contraction strategy can also be used to define infinite behaviour (as we will see later, the construction is not obvious), it is hard to imagine a finitary alternative for it without further restriction on the shape of formulas or proofs.

# D Interpretation of the sequent calculus

In this final section, we put together all the ingredients of the previous section, and build an interpretation of a first-order classical sequent calculus using winning concurrent strategies.

First we develop the interpretation of first-order MLL and prove a lax soundness result with respect to cut elimination, and finally we give the interpretation of unrestricted classical proofs.

**$\mathcal{V}$-MLL**

$$\text{Ax}\ \frac{}{\vdash^{\mathcal{V}} \varphi^{\perp}, \varphi}\ \mathrm{fv}(\varphi) \subseteq \mathcal{V} \qquad \text{Cut}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \varphi^{\perp}, \Delta}{\vdash^{\mathcal{V}} \Gamma, \Delta} \qquad \text{Ex}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi, \Delta}{\vdash^{\mathcal{V}} \Gamma, \psi, \varphi, \Delta}$$

$$1\text{I}\ \frac{}{\vdash^{\mathcal{V}} 1} \qquad \perp\text{I}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V}} \Gamma, \perp} \qquad \otimes\text{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \psi, \Delta}{\vdash^{\mathcal{V}} \Gamma, \varphi \otimes \psi, \Delta} \qquad \parr\text{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi, \Delta}{\vdash^{\mathcal{V}} \Gamma, \varphi \parr \psi, \Delta}$$

**First-order MLL ($\mathbf{MLL}_1$)**

$$\forall\text{I}\ \frac{\vdash^{\mathcal{V} \uplus \{\mathsf{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathsf{x}.\, \varphi}\ \mathsf{x} \notin \mathrm{fv}(\Gamma) \qquad \exists\text{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi[t/\mathsf{x}]}{\vdash^{\mathcal{V}} \Gamma, \exists \mathsf{x}.\, \varphi}\ t \in \mathsf{Tm}_{\Sigma}(\mathcal{V})$$

**Figure 12** Rules for first-order MLL

## D.1 Interpretation of first-order MLL

### D.1.0.1 First-order MLL.

First of all, we define the *formulas* of first-order MLL. Let us fix a signature $\Sigma$.

▶ **Definition 195.** The **first-order MLL formulas** are the terms generated by the grammar below.

$$\begin{aligned} \varphi, \psi \quad ::= &\quad 1 \mid \perp \mid \mathsf{P}(t_1, \ldots, t_n) \mid \neg \mathsf{P}(t_1, \ldots, t_n) \\ &\mid \forall \mathsf{x}.\ \varphi \mid \exists \mathsf{x}.\ \varphi \mid \varphi \otimes \psi \mid \varphi \parr \psi \end{aligned}$$

where $\mathsf{P}$ is a predicate symbol of arity $n$ in $\Sigma$, and $t_1, \ldots, t_n$ are (possibly open) first-order terms in $\Sigma$. If $\varphi$ is a formula, we obtain its free variables as usual. A $\mathcal{V}$-formula is a formula with free variables in $\mathcal{V}$.

The **first-order MLL judgements** have the form

$$\vdash^{\mathcal{V}} \varphi_1, \ldots, \varphi_n$$

where for all $i \in \{1, \ldots, n\}$, $\varphi_i$ is a $\mathcal{V}$-formula.

In Figure 12, we give the rules for first-order MLL.

### D.1.0.2 Interpretation.

We start with the interpretation of formulas. We regard a literal $\varphi$ (*i.e.* $\mathsf{P}(t_1, \ldots, t_n)$ or $\neg\mathsf{P}(t_1, \ldots, t_n)$ with $t_i \in \mathsf{Tm}_{\Sigma}(\mathcal{V})$) as a $\mathcal{V}$-game on arena $\emptyset$, with $\mathcal{W}_{\varphi}(\emptyset) = \varphi$.

▶ **Definition 196.** We define the interpretation function of first-order MLL $\mathcal{V}$-formulas into

$\mathcal{V}$-games by induction on formulas, as follows:

$$
\begin{array}{rcl}
[\![1]\!]_{\mathcal{V}} & = & 1 \\
[\![\bot]\!]_{\mathcal{V}} & = & \bot \\
[\![\exists \mathsf{x}\varphi]\!]_{\mathcal{V}} & = & \exists \mathsf{x}.[\![\varphi]\!]_{\mathcal{V}\uplus\{\mathsf{x}\}} \\
[\![\forall \mathsf{x}\varphi]\!]_{\mathcal{V}} & = & \forall \mathsf{x}.[\![\varphi]\!]_{\mathcal{V}\uplus\{\mathsf{x}\}} \\
[\![\mathsf{P}(t_1,\dots,t_n)]\!]_{\mathcal{V}} & = & \mathsf{P}(t_1,\dots,t_n) \\
[\![\neg\mathsf{P}(t_1,\dots,t_n)]\!]_{\mathcal{V}} & = & \neg\mathsf{P}(t_1,\dots,t_n) \\
[\![\varphi_1 \parr \varphi_2]\!]_{\mathcal{V}} & = & [\![\varphi_1]\!]_{\mathcal{V}} \parr [\![\varphi_2]\!]_{\mathcal{V}} \\
[\![\varphi_1 \otimes \varphi_2]\!]_{\mathcal{V}} & = & [\![\varphi_1]\!]_{\mathcal{V}} \otimes [\![\varphi_2]\!]_{\mathcal{V}}
\end{array}
$$

Then, we give the interpretation of proofs.

▶ **Definition 197.** To each proof $\pi$ of $\vdash^{\mathcal{V}} \varphi_1, \dots, \varphi_n$, we associate a morphism in $\Sigma\text{-}\mathsf{Gam}(\mathcal{V})$:

$$[\![\pi]\!] : 1 \twoheadrightarrow [\![\varphi_1]\!]_{\mathcal{V}} \parr \cdots \parr [\![\varphi_n]\!]_{\mathcal{V}}.$$

Ignoring first the rules for quantifiers, the interpretation follows the standard lines of the interpretation of MLL into a $*$-autonomous category (relying on Corollary 179). Those are detailed in Figure 13. The interpretation of quantifier rules is given in Figure 14. (where, leveraging $*$-autonomy, we silently convert between $1 \twoheadrightarrow A^\perp \parr B$ and $A \twoheadrightarrow B$ following the canonical natural isomorphism).

By definition, this yields, for any proof, a winning strategy in the corresponding game. Furthermore, the structures we have developped on games yield further properties of the interpretation.

▶ **Theorem 198.** *Let $\rightsquigarrow$ denote the standard cut reduction in first-order MLL. Then, for two proofs $\pi_1$ and $\pi_2$ of a sequent $\vdash^{\mathcal{V}} \Gamma$, we have*

$$\pi_1 \rightsquigarrow \pi_2 \implies [\![\pi_2]\!] \preccurlyeq [\![\pi_1]\!]$$

**Proof.** All cut reductions involving only MLL rules are preserved up to equality thanks to the $*$-autonomous structure. The cut reduction for an introduction rule for the existential quantifier against an introduction rule for the universal quantifier is preserved by Lemma 155. Finally, we still have to check the commuting conversions. Commutations of cuts through MLL rules and the introduction of existential quantifier are preserved up to equality thanks to the $*$-autonomy. Finally, commutation of cut with introduction of the universal quantifier is preserved up to $\preccurlyeq$ by Proposition 157. This order is preserved by all operation on strategies, so the result follows. ◀

## D.2 Interpretation of LK

Finally, we give the interpretation of LK. First we will give the interpretation of formulas, and prove some important extra properties of games obtained as the interpretation of formulas. With the help of this structure, we will be able to refine the interpretation of proofs of Section D.1 so as to support the structural rules of contraction and weakening.

$$\left[\!\!\left[\textsc{Ax}\ \frac{}{\vdash^{\mathcal{V}} \varphi^{\perp}, \varphi}\right]\!\!\right] = 1 \xrightarrow{\eta_\varphi} \varphi^{\perp} \parr \varphi \qquad\qquad \left[\!\!\left[1\mathrm{I}\ \frac{}{\vdash^{\mathcal{V}} 1}\right]\!\!\right] = 1 \rightarrow 1$$

$$\left[\!\!\left[\perp\mathrm{I}\ \frac{\dfrac{\pi}{\vdash^{\mathcal{V}} \Gamma}}{\vdash^{\mathcal{V}} \Gamma, \perp}\right]\!\!\right] = 1 \xrightarrow{[\![\pi]\!]} \Gamma \cong \Gamma \parr \perp$$

$$\left[\!\!\left[\textsc{Ex}\ \frac{\dfrac{\pi}{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi, \Delta}}{\vdash^{\mathcal{V}} \Gamma, \psi, \varphi, \Delta}\right]\!\!\right] = 1 \xrightarrow{[\![\pi]\!]} \Gamma \parr \varphi \parr \psi \parr \Delta \cong \Gamma \parr \psi \parr \varphi \parr \Delta$$

$$\left[\!\!\left[\textsc{Cut}\ \frac{\dfrac{\pi_1}{\vdash^{\mathcal{V}} \Gamma, \varphi} \qquad \dfrac{\pi_2}{\vdash^{\mathcal{V}} \varphi^{\perp}, \Delta}}{\vdash^{\mathcal{V}} \Gamma, \Delta}\right]\!\!\right] = 1 \xrightarrow{[\![\pi_1]\!] \otimes [\![\pi_2]\!]}$$

$$(\Gamma \parr \varphi) \otimes (\varphi^{\perp} \parr \Delta) \rightarrow \Gamma \parr (\varphi \otimes \varphi^{\perp}) \parr \Delta \xrightarrow{\Gamma \otimes \epsilon_\varphi \otimes \Delta} \Gamma \parr \perp \parr \Delta \cong \Gamma \parr \Delta$$

$$\left[\!\!\left[\otimes\mathrm{I}\ \frac{\dfrac{\pi_1}{\vdash^{\mathcal{V}} \Gamma, \varphi} \qquad \dfrac{\pi_2}{\vdash^{\mathcal{V}} \psi, \Delta}}{\vdash^{\mathcal{V}} \Gamma, \varphi \otimes \psi, \Delta}\right]\!\!\right] = 1 \xrightarrow{[\![\pi_1]\!] \otimes [\![\pi_2]\!]} (\Gamma \parr \varphi) \otimes (\psi \parr \Delta) \rightarrow \Gamma \parr (\varphi \otimes \psi) \parr \Delta$$

$$\left[\!\!\left[\parr\mathrm{I}\ \frac{\dfrac{\pi}{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi, \Delta}}{\vdash^{\mathcal{V}} \Gamma, \varphi \parr \psi, \Delta}\right]\!\!\right] = 1 \xrightarrow{[\![\pi]\!]} \Gamma \parr \varphi \parr \psi \parr \Delta \cong \Gamma \parr (\varphi \parr \psi) \parr \Delta$$

**Figure 13** Interpretation of MLL

$$\left[\!\!\left[\forall\mathrm{I}\ \frac{\dfrac{\pi}{\vdash^{\mathcal{V} \uplus \{\mathsf{x}\}} \Gamma, \varphi}}{\vdash^{\mathcal{V}} \Gamma, \forall \mathsf{x}.\, \varphi}\ \mathsf{x} \notin \mathrm{fv}(\Gamma)\right]\!\!\right] = \forall\!\!\mathrm{I}^{\mathsf{x}}_{\Gamma,\varphi}([\![\pi]\!])$$

$$\left[\!\!\left[\exists\mathrm{I}\ \frac{\dfrac{\pi}{\vdash^{\mathcal{V}} \Gamma, \varphi[t/\mathsf{x}]}}{\vdash^{\mathcal{V}} \Gamma, \exists \mathsf{x}.\, \varphi}\ t \in \mathsf{Tm}_\Sigma(\mathcal{V})\right]\!\!\right] = \Gamma^{\perp} \xrightarrow{[\![\pi]\!]} \varphi[t/\mathsf{x}] \xrightarrow{\exists^t_\varphi} \exists \mathsf{x} \varphi$$

**Figure 14** Interpretation of quantifier rules in first-order MLL

### D.2.1 Interpretation of formulas

First of all, we need to adjust the interpretation of formulas. In classical logic, a proof (Player) may provide several witnesses for an existential quantifier (hence obtaining a Herbrand disjunction for a lone existential quantifier). By duality, so that we can compose strategies, so can Opponent. The interpretation of quantifiers needs therefore to be adjusted to take account of that, by adding the appropriate exponential (truncated) modalities.

▶ **Definition 199.** To each $\varphi \in \mathsf{Form}_\Sigma(\mathcal{V})$ we associate $[\![\varphi]\!]_\mathcal{V}$ a $\mathcal{V}$-game. The interpretation function $[\![-]\!]_\mathcal{V}$ is defined as in Definition 196 (with $\otimes$ considered as $\wedge$, 1 considered as $\top$ and $⅋$ considered as $\vee$) except for the two clauses:

$$[\![\exists \mathsf{x} \varphi]\!]_\mathcal{V} = ?\exists \mathsf{x}.[\![\varphi]\!]_{\mathcal{V} \uplus \{\mathsf{x}\}} \qquad [\![\forall \mathsf{x} \varphi]\!]_\mathcal{V} = !\forall \mathsf{x}.[\![\varphi]\!]_{\mathcal{V} \uplus \{\mathsf{x}\}}$$

From this definition follows the following important property.

▶ **Lemma 200.** *Formulas coming from the interpretation are* perennializable, i.e. *for each* $\varphi \in \mathsf{Form}_\Sigma(\mathcal{V})$*, there is a winning* $(\Sigma \uplus \mathcal{V})$*-strategy:*

$$[\![\varphi]\!]_\mathcal{V} \twoheadrightarrow ![\![\varphi]\!]_\mathcal{V}$$

**Proof.** This is proved for all $\mathcal{V}$, by induction on formulas.

For units and literals, it is clear ($\wedge$ and $\vee$ are idempotent up to equivalence).

For $\varphi = \varphi_1 \wedge \varphi_2$, we have

$$[\![\varphi_1]\!]_\mathcal{V} \otimes [\![\varphi_2]\!]_\mathcal{V} \twoheadrightarrow ![\![\varphi_1]\!]_\mathcal{V} \otimes ![\![\varphi_2]\!]_\mathcal{V} \twoheadrightarrow !([\![\varphi_1]\!]_\mathcal{V} \otimes [\![\varphi_2]\!]_\mathcal{V})$$

using induction hypothesis, the functorial action of $\otimes$ and Lemma 192. The reasoning is the same $\varphi = \varphi_1 ⅋ \varphi_2$.

For $\varphi = !\forall \mathsf{x} \psi$, it follows immediately from Lemma 192.

Finally, the only case remaining, and the most interesting one: if $\varphi = ?\exists \mathsf{x} \psi$, we use the following composition.

$$?\exists \mathsf{x} \psi \twoheadrightarrow ?\exists \mathsf{x} !\psi \twoheadrightarrow ?!\exists \mathsf{x} \psi \twoheadrightarrow !?\exists \mathsf{x} \psi$$

composing a strategy obtained from induction hypothesis, the winning strategy of Lemma 194, and the distribution between exponentials of Lemma 192 (along with the functorial actions of various constructors). ◀

And from that, we immediately deduce:

▶ **Corollary 201.** *If* $\varphi \in \mathsf{Form}_\Sigma(\mathcal{V})$*, then* $[\![\varphi]\!]_\mathcal{V}$ *is a "truncated comonoid" in* $\Sigma$-$\mathsf{Gam}(\mathcal{V})$*, in the sense that there are winning strategies:*

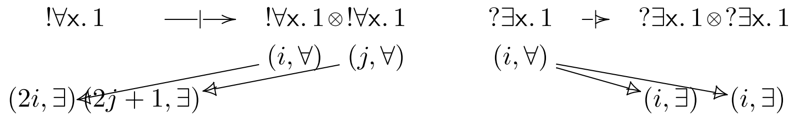
$$[\![\varphi]\!]_\mathcal{V} \twoheadrightarrow 1 \qquad [\![\varphi]\!]_\mathcal{V} \twoheadrightarrow [\![\varphi]\!]_\mathcal{V} \otimes [\![\varphi]\!]_\mathcal{V}$$

**Proof.** The former comes from Lemma 191, while the latter comes from the composition

$$[\![\varphi]\!]_\mathcal{V} \twoheadrightarrow ![\![\varphi]\!]_\mathcal{V} \twoheadrightarrow ![\![\varphi]\!]_\mathcal{V} \otimes ![\![\varphi]\!]_\mathcal{V} \twoheadrightarrow [\![\varphi]\!]_\mathcal{V} \otimes [\![\varphi]\!]_\mathcal{V}$$

using Lemma 200, contraction and dereliction from Lemmas 192 and 193. ◀

While, since we do not aim here to prove any equations satisfied by the interpretation, we do not keep track of the names of all these winning strategies, they are all defined precisely in the paper. It is informative to look at two concrete examples of the contraction strategy above:




For contraction on universally quantified variables, contraction works as expected from standard game semantics, following a bijection between $\omega$ and $\omega + \omega$. In contrast, for existential quantifiers, so as to ensure winning one has to propagate the Opponent input on the left to the *two* components on the right, reflecting the fact that if an introduction rule for $\exists$ is cut against a contraction for the dual $\forall$, then the introduction rule for $\exists$ will be duplicated.

## D.2.2 Interpretation of proofs

We now conclude by providing an interpretation for all classical proofs. The interpretation is mostly informed by the interpretation of first-order MLL above plus the strategies for contraction and weakening of the previous section, however there is a small adjustment to make to the interpretation of introduction rules for quantifiers as we have changed the interpretation of the corresponding formulas by adding exponentials.

We state our final result:

▶ **Theorem 202.** *For all $\mathcal{V}$, there is an interpretation $[\![-]\!]_{\mathcal{V}}$, which to any $\varphi \in \mathsf{Form}_\Sigma(\mathcal{V})$ associates a $\mathcal{V}$-game $[\![\varphi]\!]_{\mathcal{V}}$, and which to any proof $\pi$ of a LK sequent $\vdash^{\mathcal{V}} \varphi_1, \ldots, \varphi_n$ associates a winning strategy:*

$$[\![\varphi]\!]_{\mathcal{V}} : 1 \twoheadrightarrow [\![\varphi_1]\!]_{\mathcal{V}} \parr \cdots \parr [\![\varphi_n]\!]_{\mathcal{V}}$$

**Proof.** We modify the interpretation of MLL by adding interpretations for contraction and weakening, and adjusting the interpretation of introduction rules for quantifiers of Figure 14, as shown below:

$$\left[\!\!\left[ \mathrm{W}\, \frac{\dfrac{\pi}{\vdash^{\mathcal{V}} \Gamma}}{\vdash^{\mathcal{V}} \Gamma, \varphi} \right]\!\!\right] = \Gamma^\perp \cong \Gamma^\perp \parr \perp \twoheadrightarrow \Gamma^\perp \parr \varphi$$

$$\left[\!\!\left[ \mathrm{C}\, \frac{\dfrac{\pi}{\vdash^{\mathcal{V}} \Gamma, \varphi, \varphi}}{\vdash^{\mathcal{V}} \Gamma, \varphi} \right]\!\!\right] = \Gamma^\perp \overset{[\![\pi]\!]}{\twoheadrightarrow} \varphi \parr \varphi \twoheadrightarrow \varphi$$

$$\left[\!\!\left[ \forall\mathrm{I}\, \frac{\dfrac{\pi}{\vdash^{\mathcal{V} \uplus \{\mathsf{x}\}} \Gamma, \varphi}}{\vdash^{\mathcal{V}} \Gamma, \forall \mathsf{x}.\, \varphi} \right]\!\!\right] = \Gamma^\perp \twoheadrightarrow !\Gamma^\perp \overset{!(\forall\mathrm{I}\,([\![\pi]\!]))}{\twoheadrightarrow} !\forall \mathsf{x}.\, \varphi$$

$$\left[\!\!\left[ \forall\mathrm{I}\, \frac{\dfrac{\pi}{\vdash^{\mathcal{V}} \Gamma, \varphi[t/\mathsf{x}]}}{\vdash^{\mathcal{V}} \Gamma, \exists \mathsf{x}.\, \varphi} \right]\!\!\right] = \Gamma^\perp \overset{[\![\pi]\!]}{\twoheadrightarrow} \varphi[t/\mathsf{x}] \overset{\exists^t_\varphi}{\twoheadrightarrow} \exists \mathsf{x}.\varphi \twoheadrightarrow ?\exists \mathsf{x}.\varphi$$

using the (natural) isomorphism between $\Gamma^\perp \twoheadrightarrow A$ and $1 \twoheadrightarrow \Gamma \parr A$, (sometimes dualized) winning strategies from Corollary 201 and Lemmas 200 and 193. ◀

As a final illustration, a proof $\pi$ of $\vdash \exists \mathsf{x} \varphi$ where $\varphi$ is quantifier-free yields by interpretation a winning strategy on $?\exists \mathsf{x} [\![\varphi]\!]_{\{\mathsf{x}\}}$. The corresponding arena has $\omega$ moves, all positive, all comparable. A $\Sigma$-strategy in this arena must therefore play any (possibly infinite) subset of these moves along with annotations by closed terms (as there are no Opponent events to provide free variables). The winning condition ensures exactly that

$$\bigvee_{i \in \omega} \varphi(\lambda((i, \bullet)))$$

is a tautology. Unfortunately it might be infinite, but is effectively computable, so one can always effectively extract a finite Herbrand disjunction from this interpretation.

## D.2.3 Non-finiteness of the interpretation

From the infinitary primitives in the interpretation, it is natural to expect the interpretation to be infinitary. It was surprisingly difficult to find such an example, however one can do so by revisiting standard pathological examples in the proof theory of classical logic, having arbitrarily large normal forms.

More precisely, we construct an LK proof of the formula $\exists \mathsf{x}. \top$ whose interpretation is infinite, despite the fact that there is no move by ∀bélard in the game. Besides showing that the interpretation is infinitary, we also take advantage of the presentation of the example to detail as much as reasonable the interpretation, so that the interested reader can see it at play in a non-trivial case.

Our starting point is the following proof:

$$\varpi_1 = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \varphi, \varphi^\perp}\,\text{Ax} \quad \dfrac{}{\vdash \varphi, \varphi^\perp}\,\text{Ax}}{\vdash \varphi \wedge \varphi, \varphi^\perp, \varphi^\perp}\,\wedge\text{I}}{\vdash \varphi \wedge \varphi, \varphi^\perp}\,\text{C} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \varphi, \varphi^\perp}\,\text{Ax} \quad \dfrac{}{\vdash \varphi, \varphi^\perp}\,\text{Ax}}{\vdash \varphi, \varphi, \varphi^\perp \wedge \varphi^\perp}\,\wedge\text{I}}{\vdash \varphi, \varphi^\perp \wedge \varphi^\perp}\,\text{C}}{\vdash \varphi \wedge \varphi, \varphi^\perp \wedge \varphi^\perp}\,\text{Cut}$$

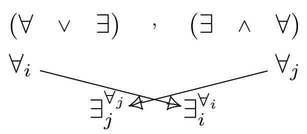
This proof is referred to in [8] as a *structural dilemma*. There are two ways to push the Cut beyond contraction, as the two proofs interact, and try to duplicate one another. This is often used as an example of a proof where unrestricted cut reduction does not necessarily terminate; and which has infinitely large cut-free forms.

In order to construct a proof with an infinite interpretation, we will start with this proof, with $\varphi = \forall \mathsf{x}. \perp \vee \exists \mathsf{y}. \top$, which to shorten notations we will just write as $\forall \vee \exists$.

#### D.2.3.1 Interpretation of $\varpi_1$.

We detail the interpretation of $\varpi_1$. We start from the axioms on the left branch:

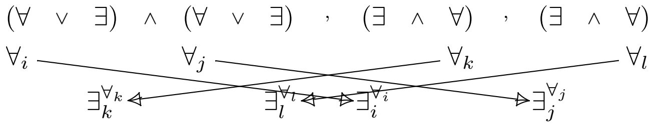
$$\left[\!\!\left[ \dfrac{}{\vdash \varphi, \varphi^\perp}\,\text{Ax} \right]\!\!\right] =$$



The indices $i, j$ are the copy indices for the ! and ? arising from the interpretation of formulas, and we only display the term annotations for ∃loïse's moves. The Σ-strategy above is the copycat Σ-strategy as defined in Definition 22.

Interpreting the introduction rule for $\wedge$ simply has the effect of tensoring two copies of copycat together, obtaining:

$$\left[\!\!\left[\dfrac{\mathrm{Ax}\;\dfrac{}{\vdash \varphi, \varphi^\perp} \qquad \mathrm{Ax}\;\dfrac{}{\vdash \varphi, \varphi^\perp}}{\vdash \varphi \wedge \varphi, \varphi^\perp, \varphi^\perp}\,\wedge\mathrm{I}\right]\!\!\right] =$$

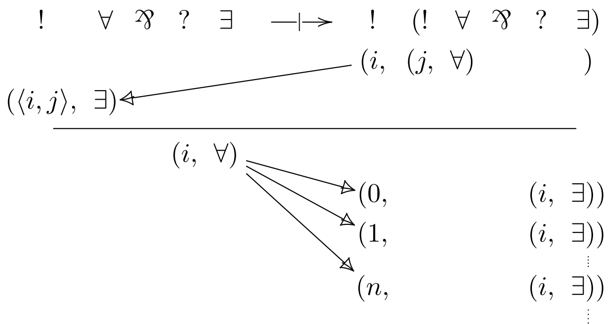



*i.e.* again copycat, in accordance with the functoriality of $\otimes$.

Now, to interpret contraction, we need to compose with $\delta^\perp_{\forall\vee\exists} : (\exists \wedge \forall) \vee (\exists \wedge \forall) \rightarrow\!\!\!\!\!+\!\!\!> \exists \wedge \forall$, where

$$\delta_{\forall\vee\exists} : (!\forall \parr ?\exists) \rightarrow\!\!\!\!\!+\!\!\!> (!\forall \parr ?\exists) \otimes (!\forall \parr ?\exists)$$

is the contraction on $\varphi$. Note that this time, make explicit the exponential modalities. Recall also that this strategy is derived from $co_{\forall\vee\exists} : (!\forall \parr ?\exists) \rightarrow\!\!\!\!\!+\!\!\!> !(!\forall \parr ?\exists)$, which we display below. To display it best we deviate from the representation below by showing exactly the correspondence between copy indices and occurrences of ! and ?, and we omit the terms, which are trivial and always correspond with the unique predecessor for ∃loïse's events. We display the Σ-strategy separating two sub-configurations for clarity; the full Σ-strategy is obtained by taking their union.

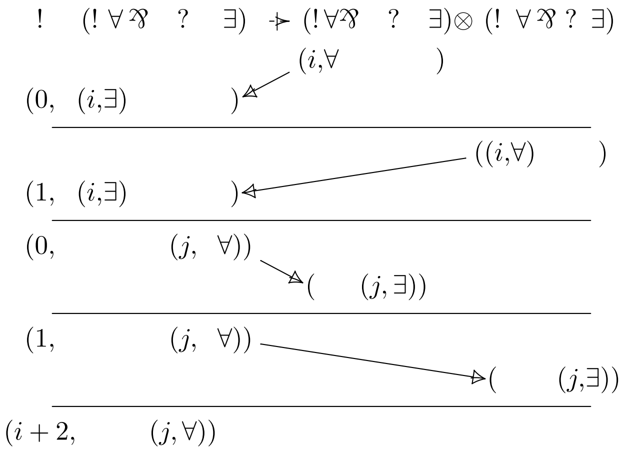



We do not detail the construction of this Σ-strategy, but it is easy to get from the definitions. This Σ-strategy $co_{\forall\vee\exists}$ obviously performs an infinitary duplication, however it does not show by itself that the interpretation is infinitary, as $co_{\forall\vee\exists}$ is just an auxiliary device in the definition of the interpretation, rather than itself the interpretation of a proof.

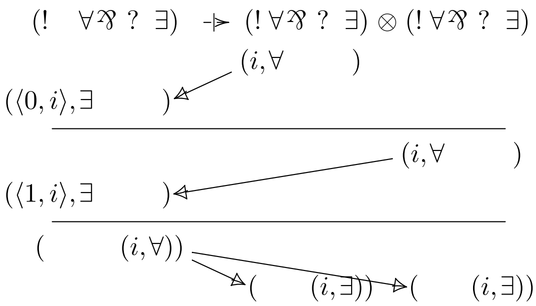
To get contraction on $\varphi$ from $co_{\forall\vee\exists}$, we compose it with the derelicted version of contraction on $!\varphi$:

$$!(!\forall \parr ?\exists) \rightarrow\!\!\!\!\!+\!\!\!> !(!\forall \parr ?\exists) \otimes !(!\forall \parr ?\exists) \rightarrow\!\!\!\!\!+\!\!\!> (!\forall \parr ?\exists) \otimes (!\forall \parr ?\exists)$$

which we display here:

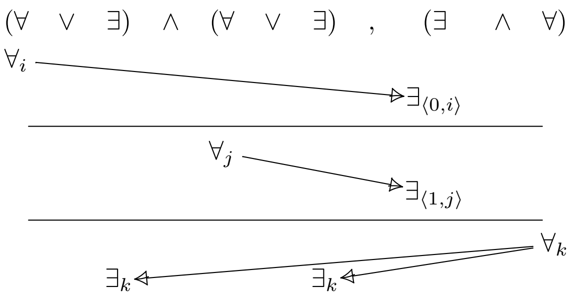



where the final case is just closure under receptivity. Performing the composition, we get the contraction $\Sigma$-strategy $\delta_{\forall\vee\exists}$:

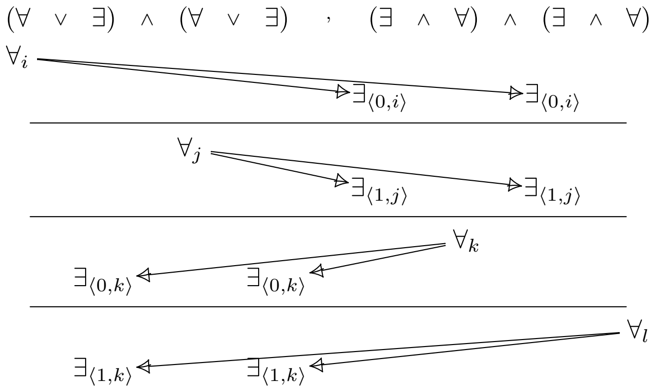



With that in place, we can finally obtain by composition (where we adopt again the simplified annotation for copy indices, since in this games ! and ? are again always attached to quantifiers – we still omit the trivial term annotations):

$$\left[\!\left[ \dfrac{\dfrac{\dfrac{}{\vdash \varphi, \varphi^\perp}\,\text{Ax} \quad \dfrac{}{\vdash \varphi, \varphi^\perp}\,\text{Ax}}{\vdash \varphi \wedge \varphi, \varphi^\perp, \varphi^\perp}\,\wedge\text{I}}{\vdash \varphi \wedge \varphi, \varphi^\perp}\,\text{C} \right]\!\right] =$$




The second branch of $\varpi_1$ is symmetric, so we do not make it explicit. Now, we interpret the Cut rule and the composition yields $[\![\varpi_1]\!]$ below (again, we omit term annotations which coincide with the unique predecessor for ∃loïse's moves).



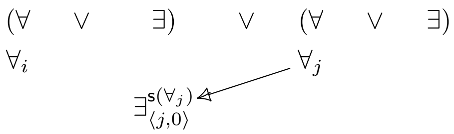

It is interesting to note that although $\varpi_1$ has arbitrarily large cut-free forms, the corresponding strategy only plays finitely many ∃loïse moves for every ∀bélard move. However, we are on the right path to finding a truly infinitary Σ-strategy.

### D.2.3.2 An infinitary proof.

The next step is to set (with $\mathsf{s}$ some unary function symbol):

$$\varpi_2 = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^{\mathsf{x}} \top[\mathsf{s}(\mathsf{x})/\mathsf{y}], \bot}\,\text{Ax}}{\vdash^{\mathsf{x}} \exists \mathsf{y}.\, \top, \bot}\,\exists\text{I}}{\vdash \exists \mathsf{y}.\, \top, \forall \mathsf{x}.\, \bot}\,\forall\text{I}}{\dfrac{\vdash \forall \mathsf{x}.\, \bot, \exists \mathsf{y}.\, \top, \forall \mathsf{x}.\, \bot, \exists \mathsf{x}.\, \top}{\vdash (\forall \mathsf{x}.\, \bot \vee \exists \mathsf{y}.\, \top) \vee (\forall \mathsf{x}.\, \bot \vee \exists \mathsf{x}.\, \top)}\,\vee\text{I}}\,\text{W}$$

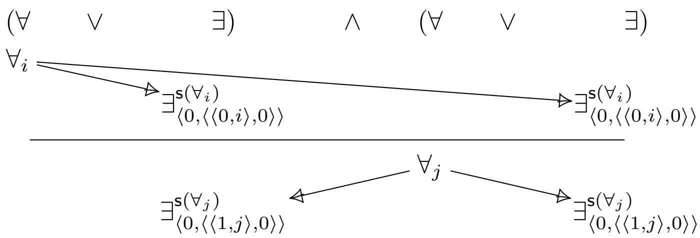
Leaving to the reader the details of the interpretation, we have by design that $[\![\varpi_2]\!]$ is:




We now use these to compute the interpretation of:

$$\varpi_3 = \dfrac{\dfrac{\varpi_1}{\vdash \varphi \wedge \varphi, \varphi^{\perp} \wedge \varphi^{\perp}} \qquad \dfrac{\varpi_2}{\vdash (\forall \vee \exists) \vee (\forall \vee \exists)}}{\vdash \varphi \wedge \varphi}\,\textsc{Cut}$$

The associated composition reveals $[\![\varpi_3]\!]$ to be:



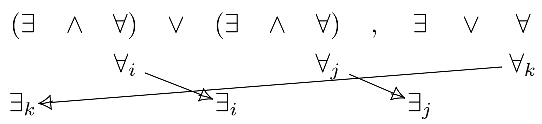
We are almost there. It suffices now to note that $\varpi_3$ provides a proof of

$$(\exists \mathsf{x}.\top \implies \exists \mathsf{x}.\top) \wedge (\exists \mathsf{x}.\top \implies \exists \mathsf{x}.\top)$$

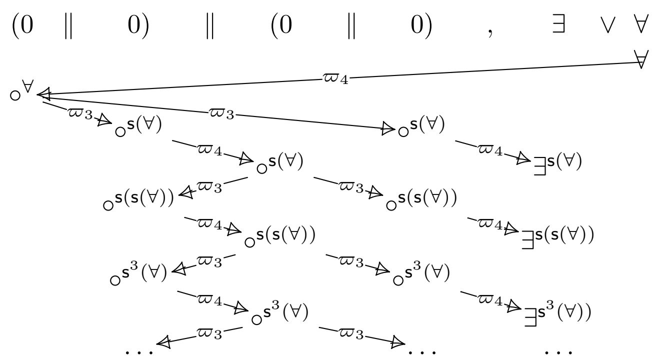
These two implications can be *composed* by cutting $\varpi_3$ against the proof $\varpi_4$ or $(\exists \Rightarrow \exists) \wedge (\exists \Rightarrow \exists) \Rightarrow (\exists \Rightarrow \exists)$ performing the composition:

$$\varpi_4 = \vee\mathrm{I}\,\dfrac{\mathrm{Ex}\,\dfrac{\wedge\mathrm{I}\,\dfrac{\wedge\mathrm{I}\,\dfrac{\mathrm{Ax}\,\dfrac{}{\vdash \forall, \exists} \quad \mathrm{Ax}\,\dfrac{}{\vdash \forall, \exists}}{\vdash \forall, \exists \wedge \forall, \exists} \quad \mathrm{Ax}\,\dfrac{}{\vdash \forall, \exists}}{\vdash \forall, \exists \wedge \forall, \exists \wedge \forall, \exists}}{\vdash \exists \wedge \forall, \exists \wedge \forall, \exists, \forall}}{\vdash (\exists \wedge \forall) \vee (\exists \wedge \forall), \exists \vee \forall}$$

with interpretation:



Write $\varpi_5$ for the proof of $\exists \mathsf{x}.\top \vee \forall \mathsf{y}.\bot$ obtained by cutting $\varpi_3$ and $\varpi_4$ in the obvious way. The interpretation of $\varpi_5$ is the composition of $[\![\varpi_3]\!]$ and $[\![\varpi_4]\!]$, which triggers the feedback loop causing the infiniteness phenomenon. We display below the corresponding interaction. For the "synchronised" part of formulas, we will use 0 for components resulting from matching dual quantifiers, and $\parallel$ for components resulting for matching dual propositional connectives. We write $\circ$ for synchronized events (*i.e.* of neutral polarity), and omit copy indices, which get very unwieldy. For readability, we also annotate the immediate causal links with the sub-proof that they originate from, *i.e.* $\varpi_3$ or $\varpi_4$.



Therefore, after hiding, $\exists$loïse responds to an initial $\forall$bélard move $\forall$ by playing simultaneously all $\exists^{\mathsf{s}^n(\forall)}$, for $n \geq 1$. Finally, cutting $\varpi_5$ against a proof of $\exists \mathsf{x}.\top$ playing a constant symbol 0, we get a proof $\varpi_6$ of $\vdash \exists \mathsf{x}.\top$ whose interpretation plays simultaneously all $\exists^{\mathsf{s}^n(0)}$ for $n \geq 1$.

## E Compactness

Restricting any winning $\Sigma$-strategy $\sigma : [\![\varphi]\!]$ to $[\![\varphi]\!]^\exists$ (ignoring $\forall$bélard's replications) yields $\sigma^\exists : [\![\varphi]\!]^\exists$, not necessarily finite. Yet, we will show that it has a finite *top-winning* sub-strategy.

A game $\mathcal{A}$ is **a prefix** of $\mathcal{B}$ if $|A| \subseteq |B|$, and all the structure coincides on $|A|$. Notice that $[\![\varphi]\!]^\exists$ embeds (subject to renaming) as a prefix of $[\![\varphi]\!]$. Keeping the renaming silent, we have:

▶ **Lemma 203.** *For any winning $\sigma : [\![\varphi]\!]$, setting*

$$|\sigma^\exists| = \{a \in |\sigma| \mid [a]_\sigma \subseteq |[\![\varphi]\!]^\exists|\}$$

*and inheriting the order, polarity and labelling from $\sigma$, we obtain $\sigma^\exists : [\![\varphi]\!]^\exists$ a winning $\Sigma$-strategy.*

**Proof.** Most conditions are direct. For $\sigma^\exists : [\![\varphi]\!]^\exists$ winning we use that for any $\exists$-maximal $x \in \mathscr{C}^\infty(\sigma^\exists)$, $x \in \mathscr{C}^\infty(\sigma)$ $\exists$-maximal as well: this follows from $[\![\varphi]\!]^\exists$ being itself $\exists$-maximal in $[\![\varphi]\!]$. ◀

As mentioned above, the extracted $\sigma^\exists$ may not be finite! Indeed there are classical proofs for which our interpretation yields infinite strategies, even after removing ∀bélard's replications (see Appendix D.2.3). This reflects the usual issues one has in getting strong normalization in a proof system for classical logic [8] without enforcing too much sequentiality as with a negative translation.

Despite this, the compactness theorem for propositional logic entails that we can always extract a finite top-winning sub-strategy. For $\sigma : [\![\varphi]\!]^\exists$ any $\Sigma$-strategy, we denote $\mathscr{C}^\forall(\sigma)$ the set of **∀-maximal** configurations of $\sigma$, *i.e.* they can only be extended in $\sigma$ by ∃loïse moves – inheriting all structure from $\sigma$ they correspond to its *sub-strategies*, as they are automatically receptive. The proof relies on:

▶ **Lemma 204.** *Let $X$ be a directed set of ∀-maximal configurations. Then, $\mathcal{W}_{[\![\varphi]\!]^\exists}(\bigcup X)$ is logically equivalent to $\bigvee_{x \in X} \mathcal{W}_{[\![\varphi]\!]^\exists}(x)$.*

**Proof.** By induction on $\varphi$, using simple logical equivalences and that if $x_1 \subseteq x_2$ are ∀-maximal, then $\mathcal{W}_{[\![\varphi]\!]^\exists}(x_1)$ implies $\mathcal{W}_{[\![\varphi]\!]^\exists}(x_2)$. ◀

We complete the proof. For $\sigma : [\![\varphi]\!]^\exists$ winning, by the lemma above the (potentially infinite) disjunction of finite formulas

$$\bigvee_{x \in \mathscr{C}^\forall(\sigma)} \mathcal{W}_{[\![\varphi]\!]^\exists}(x)[\lambda_\sigma]$$

is a tautology. By the compactness theorem there is a finite $X = \{x_1, \dots, x_n\} \subseteq \mathscr{C}^\forall(\sigma)$ such that $\bigvee_{x \in X} \mathcal{W}_{[\![\varphi]\!]^\exists}(x)[\lambda_\sigma]$ is a tautology – *w.l.o.g.* $X$ is directed as $\mathscr{C}^\forall(\sigma)$ is closed under union. By Lemma 204 again, $\mathcal{W}_{[\![\varphi]\!]^\exists}(\bigcup X)[\lambda_\sigma]$ is a tautology. So, restricting $\sigma$ to events $\bigcup X$ gives a top-winning finite sub-strategy of $\sigma$.

Although this argument is non-constructive, the extraction of a finite sub-strategy can still be performed effectively: $\Sigma$-strategies and their operations can be effectively presented, and the finite top-winning sub-strategy can be computed by Markov's principle.