Distributed strategies made easy

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Abstract

Distributed/concurrent strategies have been introduced as special maps of event structures. As such they factor through their “rigid images,” themselves strategies. By concentrating on such “rigid image” strategies we are able to give an elementary account of distributed strategies and their composition, resulting in a category of games and strategies. This is in contrast to the usual development where composition involves the pullback of event structures explicitly and results in a bicategory. It is shown how, in this simpler setting, to extend strategies to probabilistic strategies; and indicated how through probability we can track nondeterministic branching behaviour, that one might otherwise think lost irrevocably in restricting attention to “rigid image” strategies.

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1 Introduction

Traditionally in understanding and analysing a large system, whether it be in computer science, physics, biology or economics, the system’s behaviour is thought of as going through a sequence of actions as time progresses. This is bound up with our experience of the world as individuals; in our conscious understanding of the world we experience and narrate our individual history as a sequence, or total order, of events, one after the other. However, a complex system is often much more than an individual agent. It is better thought of as several or many agents interacting together and distributed over various locations. In which case it can be fruitful to abandon the view of its behaviour as caught by a total order of events and instead think of the events of the systems system as comprising a partial order. The partial order expresses the causal dependency between events, how an event depends on possibly several previous events. The view that causal dependency should be paramount over an often incidental temporal order has been discovered and rediscovered in many disciplines: in physics in the understanding of the causal structure of space time; in biology and chemistry in the description of biochemical pathways; in computer science, originally in the work of Petri on Petri nets, and later in the often more mathematically amenable event structures.

Interacting systems are often represented mathematically via games. A system operates in an unknown environment, so often a prescription for its intended behaviour can be expressed as a strategy in which the system is Player against (an unpredictable) Opponent, standing for the environment. Games and their strategies are ubiquitous. They appear in logic (proof theory, set theory, ...), computer science (semantics, algorithmics, ...), biology, economics, *etc.*. They codify the mathematics of interacting systems. But they almost always follow the traditional line of representing the history of a play of the game as a sequence of moves, most often alternating between Player and Opponent. Until recently there was no mathematical theory of games based on partial orders of causal dependency between move occurrences. This handicapped their use in modelling and analysing a system of distributed agents.

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What was lacking was a mathematical theory of distributed games in which Player and Opponent are more accurately thought of as teams of players, distributed over different locations, able to move and communicate with each other. Although there are glimpses of such a mathematical theory of distributed games in earlier work of Abramsky, Mellies and Mimram [1, 13], Faggian and Piccolo [8], and others, a breakthrough occurred with the systematic use of event structures to formalise distributed games and strategies [14]. This meant that we could harness the mathematical techniques developed around event structures in an early mathematical foundation for work on synchronising processes [18]; the move from total to partial orders brings in its wake a lot of technical difficulty and potential for undue complexity unless it's done artfully.

But here we meet an obstacle for many people. Distributed/concurrent strategies have been based on maps of event structures and composition on pullback, which in the case of event structures has to be defined rather indirectly. Then, one obtains not a category but a bicategory of games and strategies. At what seems like an increasingly slight cost, a more elementary treatment can be given. Its presentation is the purpose of this article. The maps and pullbacks are still there of course, but pushed into the background.

The realisation that a more elementary presentation will often suffice has been a gradual one. It is based on the fact that a strategy, presented as a map of event structures, has a "rigid image" in the game and that in many cases this image can stand as a proxy for the original strategy [25]. True some branching behaviour is lost, just as it, and possible deadlock and divergence, can be lost in the composition of strategies. But extra structure on strategies generally remedies this. For example, the introduction of probability to strategies allows the detection of divergence in composition, or hidden branching, through leaks of probability. One can go far with rigid images of strategies. They permit the elementary development presented here.

In their CONCUR'16 paper [2] Castellan and Clairambault used the simple presentation of "rigid image" strategies here. Meanwhile rigid images of strategies had come to play an increasing role in Winskel's ECSYM notes [25]. Before this, Nathan Bowler recognised essentially the same subcategory of games and "rigid image" strategies, within the bicategory of concurrent games and strategies. (At the time, Winskel thought that too much of the nondeterministic branching behaviour would be lost irrecoverably to be very enthusiastic.)

Finally, an apology: we obtained the results here by specialising more general results on strategies to their rigid-images [25]; elementary proofs of the results would be desirable for a fully self-contained presentation, and should be written up shortly.

2 Event structures

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency* relation Con consisting of finite subsets of E. The relation $e' \leq e$ expresses that event e causally depends on the previous occurrence of event e'. That a finite subset of events is consistent conveys that its events can occur together by some stage in the evolution of the process. Together the relations satisfy several axioms:

$$
\begin{aligned}
&[e] =_{\mathrm{def}} \{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \mathrm{Con} \text{ for all } e \in E,\\
&Y \subseteq X \in \mathrm{Con} \text{ implies } Y \in \mathrm{Con}, \quad \text{and}\\
&X \in \mathrm{Con} \ \&\ e \leq e' \in X \text{ implies } X \cup \{e\} \in \mathrm{Con}.
\end{aligned}
$$

There is an accompanying notion of state, or history, those events that may occur up to some stage in the behaviour of the process described. A *configuration* is a, possibly infinite, set of events $x \subseteq E$ which is: *consistent,* $X \subseteq x$ and X is finite implies $X \in \mathrm{Con}$; and *down-closed,* $e' \leq e \in x$ implies $e' \in x$.

Two events e, e' are considered to be causally independent, and called *concurrent* if the set $\{e, e'\}$ is in Con and neither event is causally dependent on the other; then we write $e \mathrel{co} e'$. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, plays a very important role. We write $[X]$ for the down-closure of a subset of events X. Write $\mathcal{C}^\infty(E)$ for the configurations of E and $\mathcal{C}(E)$ for its finite configurations. (Sometimes we shall need to distinguish the precise event structure to which a relation is associated and write, for instance, $\leq_E$, $\rightarrowtriangle_E$ or co_E.)

We can describe a computation path by an *elementary* event structure, which is a partial order $p = (|p|, \leq_p)$ for which the set $\{e' \in |p| \mid e' \leq_p e\}$ is finite for all $e \in |p|$. We can regard an elementary event structure as an event structure in which the consistency relation consists of all finite subsets of events. There is a useful subpath order of rigid inclusion of one elementary event structure in another. Let $p = (|p|, \leq_p)$ and $q = (|q|, \leq_q)$ be elementary event structures. Write

$$p \hookrightarrow q \text{ iff } |p| \subseteq |q| \;\&\; \forall e \in |p|, e' \in |q|.\ e' \leq_p e \iff e' \leq_q e\,.$$

We shall often view a configuration x of E as an elementary event structure, *viz.* a partial order with underlying set x and partial order the causal dependency of E restricted to x.

In an interactive context a configuration x may be subject to causal dependencies beyond those of E. It will become an elementary event structure $p = (|p|, \leq_p)$ comprising an underlying set $|p| = x$ with a partial order $\leq_p$ which augments that from E:

$$\forall e \in |p|, e' \in E.\ e' \leq_E e \implies e' \leq_p e\,.$$

Write $\mathrm{Aug}(E)$ for the set of such *augmentations* associated with E. The order of rigid inclusion of one augmentation in another expresses when one augmentation is a sub-behaviour of another.

It will be useful to combine augmentations, in effect subjecting a configuration simultaneously to the causal dependencies of the two augmentations—provided this does not lead to causal loops. Define a key partial operation

$$\wedge : \mathrm{Aug}(E) \times \mathrm{Aug}(E) \rightharpoonup \mathrm{Aug}(E)$$

by taking

$$p \wedge q = \begin{cases} (|p|, (\leq_p \cup \leq q)^*) & \text{if } |p| = |q| \;\&\; (\leq_p \cup \leq q)^* \text{ is antisymmetric,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

▸ **Lemma 1.** *Letting $p, q \in \mathrm{Aug}(E)$ for which $p \wedge q$ is defined, $e' \rightarrowtriangle_{p \wedge q} e$ implies*

$$[e' \rightarrowtriangle_p e \;\&\; (e' \rightarrowtriangle_q e \text{ or } e' \mathrel{co_q} e)] \text{ or } [e' \rightarrowtriangle_q e \;\&\; (e' \rightarrowtriangle_p e \text{ or } e' \mathrel{co_p} e)]\,.$$

In fact we can see $\mathrm{Aug}(E)$ as an event structure in its own right. Its events are those augmentations with a top event, their causal dependency and consistency induced given by rigid inclusion [20]. The remark is an instance of a general fact:

▸ **Proposition 2.** *A* rigid family $\mathcal{R}$ *comprises a non-empty subset of finite elementary event structures which is down-closed w.r.t. rigid inclusion, i.e. $p \hookrightarrow q \in \mathcal{R}$ implies $p \in \mathcal{R}$. A rigid family determines an event structure* $\mathrm{Pr}(\mathcal{R})$ *whose order of finite configurations is* *isomorphic to* $(\mathcal{R}, \hookrightarrow)$. *The event structure* $\mathrm{Pr}(\mathcal{R})$ *has events those elements of* $\mathcal{R}$ *with a top event; its causal dependency is given by rigid inclusion; and its consistency by compatibilty w.r.t. rigid inclusion. The order isomorphism* $\theta_{\mathcal{R}} : \mathcal{C}(\mathrm{Pr}(\mathcal{R})) \cong \mathcal{R}$ *is given by* $\theta_{\mathcal{R}}(x) = \bigcup x$ *for* $x \in \mathcal{C}(\mathrm{Pr}(\mathcal{R}))$.

3 Event structures with polarity

An event structure with polarity comprises (A, pol) where A is an event structure with a polarity function $pol_A : A \to \{+, -, 0\}$ ascribing a polarity $+$ (Player), $-$ (Opponent) or 0 (neutral) to its events. The events correspond to (occurrences of) moves. It will be technically useful to allow events of neutral polarity; they arise, for example, in a play between a strategy and a counterstrategy. A *game* shall be represented by an event structure with polarity in which no moves are neutral.

▸ **Notation** 3. In an event structure with polarity (A, pol), with configurations x and y, write $x \subseteq^- y$ to mean inclusion in which all the intervening events are moves of Opponent. Write $x \subseteq^+ y$ for inclusion in which the intervening events are neutral or moves of Player.

3.1 Operations

We introduce two fundamental operations on event structures with polarity. We shall adopt the same operations for elementary event structures, and also for configurations, regarding a configuration as an elementary event structure with the order of the ambient event structure.

3.1.1 Dual

The *dual*, $A^\perp$, of A, an event structure with polarity, comprises the same underlying event structure A but with a reversal of polarities, events of neutral polarity remaining neutral.

We shall implicitly adopt the view of Player and understand a strategy in a game A as strategy for Player. A counterstrategy in a game A is a strategy for Opponent in the game A, *i.e.* a strategy (for Player) in the game $A^\perp$.

3.1.2 Simple parallel composition

This operation simply juxtaposes two event structures with polarity. Let $(A, \leq_A, \mathrm{Con}_A, pol_A)$ and $(B, \leq_B, \mathrm{Con}_B, pol_B)$ be event structures with polarity. The events of $A \| B$ are $(\{1\} \times A) \cup (\{2\} \times B)$, their polarities unchanged, with the only relations of causal dependency given by $(1, a) \leq (1, a')$ iff $a \leq_A a'$ and $(2, b) \leq (2, b')$ iff $b \leq_B b'$; a subset of events C is consistent in $A \| B$ iff $\{a \mid (1, a) \in C\} \in \mathrm{Con}_A$ and $\{b \mid (2, b) \in C\} \in \mathrm{Con}_B$. The empty event structure with polarity, written $\varnothing$, is the unit w.r.t. $\|$.

4 Strategies

A strategy in a game will be a (special) subset of plays in the game.

▸ **Definition 4.** A *play* in A, an event structure with polarity, comprises an augmentation, a finite elementary event structure $p = (|p|, \leq_p)$ with underlying set $|p| \in \mathcal{C}(A)$, which may augment with extra causal dependencies provided it does so *courteously*:

$$\forall a, a' \in |p|.\ a' \rightarrowtriangle_p a \ \&\ pol_A(a') = + \text{ or } pol_A(a) = - \implies a' \rightarrowtriangle_A a\,.$$

Note A, and so p, may involve neutral moves. Write $\mathrm{Plays}(A)$ for the set of plays in A.

If A is a game, so with no neutral moves, the only augmentations allowed of a play p to the immediate causal dependency of A are those of the form $\ominus \rightarrowtriangle \oplus$.

The order of rigid inclusion between plays, $p \hookrightarrow q$, expresses that p is a subplay of q. We shall write

$$p \hookrightarrow^{+} q \text{ iff } p \hookrightarrow q \ \&\ |p| \subseteq^{+} |q|\,,$$

when the extension only involves neutral or Player moves, and similarly $p \hookrightarrow^{-} q$ when only Opponent moves are involved.

▸ **Definition 5.** A *bare strategy* in A, an event structure with polarity, is a rigid family of plays, so a nonempty subset $\sigma \subseteq \mathrm{Plays}(A)$ satisfying $p \hookrightarrow q \in \sigma \implies p \in \sigma$, which is also

receptive, $p \in \sigma \ \&\ |p| \subseteq^{-} x \in \mathcal{C}(A) \implies \exists q \in \sigma.\ p \hookrightarrow q \in \sigma \ \&\ |q| = x\,.$

(Note that q is unique by courtesy.)

Write $\sigma : A$ when σ is a bare strategy of A. When A is a game, so an event structure with polarity without neutral moves, we say σ is a *strategy*.

One simple example of a strategy $\sigma : A$ in a game A is got by taking σ to consist of all the finite configurations of A regarded as elementary event structures in which their order of causal dependency is inherited from A. (Bare strategies, with neutral events, have been called “partial strategies” in [25] and “uncovered strategies” in [16].)

We shall regard a strategy in the compound game $A^{\perp} \| B$, where A and B are games, as a strategy *from* the game A *to* the game B [7, 12].

4.1 Copycat

We shall shortly define the composition of strategies. Identities w.r.t. composition are given by copycat strategies. Let A be a game. The copycat strategy $\mathrm{c\!c}_A : A^{\perp} \| A$ is an instance of a strategy. We obtain copycat from the finite configurations of an event structure $\mathbb{C}\mathbb{C}_A$ based on the idea that Player moves, of +ve polarity, in one component of the game $A^{\perp} \| A$ always copy previous corresponding moves of Opponent, of −ve polarity in the other component.

For $c \in A^{\perp} \| A$ we use $\bar{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component, *i.e.* $\overline{(1,a)} = (2,a)$ and $\overline{(2,a)} = (1,a)$. Define $\mathbb{C}\mathbb{C}_A$ to comprise the event structure with polarity $A^{\perp} \| A$ together with extra causal dependencies $\bar{c} \leq_{\mathbb{C}\mathbb{C}_A} c$ for all events c with $pol_{A^{\perp} \| A}(c) = +$. Take a finite subset to be consistent in $\mathbb{C}\mathbb{C}_A$ iff its down-closure w.r.t. the relation $\leq_{\mathbb{C}\mathbb{C}_A}$ is consistent in $A^{\perp} \| A$.

▸ **Example 6.** We illustrate the construction of $\mathbb{C}\mathbb{C}_A$ for the event structure A comprising the single immediate dependency $a_1 \rightarrowtriangle a_2$ from an Opponent move a_1 to a Player move a_2. The event structure $\mathbb{C}\mathbb{C}_A$ is obtained from $A^{\perp} \| A$ by adjoining the additional immediate dependencies shown:

$$\begin{tikzcd}
A^{\perp} & \bar{a}_2\ \ominus \arrow[r, dashed] & \oplus\ a_2 & A \\
& \bar{a}_1\ \oplus \arrow[u] & \ominus\ a_1 \arrow[u] \arrow[l, dashed] &
\end{tikzcd}$$

▸ **Lemma 7.** *Let A be an event structure with polarity. Then, $\mathbb{C}\mathbb{C}_A$ is an event structure with polarity. Moreover,*

$$x \in \mathcal{C}(\mathbb{C}\mathbb{C}_A) \textit{ iff } x \in \mathcal{C}(A^{\perp} \| A) \ \&\ \forall c \in x.\ pol_{A^{\perp} \| A}(c) = + \implies \bar{c} \in x\,.$$

The *copycat* strategy $\mathrm{c\!c}_A : A^{\perp} \| A$ is defined by taking

$$\mathrm{c\!c}_A = \{(x, \leq_{\mathbb{C}\mathbb{C}_A} \restriction x) \mid x \in \mathcal{C}(\mathbb{C}\mathbb{C}_A)\}\,.$$

In other words, $\mathfrak{C}_A$ consists of all the finite configurations of $\mathbb{C}_A$, each understood as a finite partial order through inheriting the causal dependency of $\mathbb{C}_A$.

5 Composition of strategies

A play of a strategy σ in a game $A^\perp \| B$ and a play of a strategy τ in a game $B^\perp \| C$ can interact at the common game B, where the two strategies adopt complementary views, in which one sees a move of Player the other sees a move of Opponent, and *vice versa*. In effect, the two plays synchronise at common moves in B, one strategy being receptive to the Player moves of the other. Together they produce a play in the event structure with polarity $A^\perp \| B^0 \| C$ —the event structure with polarity B^0 has the same underlying event structure as B but where all events now carry neutral polarity. This is because the interaction over the game B produces moves which are no longer open to Player or Opponent.

We can express this interaction through a partial operation

$$\circledast : \mathrm{Plays}(B^\perp \| C) \times \mathrm{Plays}(A^\perp \| B) \rightharpoonup \mathrm{Plays}(A^\perp \| B^0 \| C)$$

defined as follows. Let $p \in \mathrm{Plays}(A^\perp \| B)$, $q \in \mathrm{Plays}(A^\perp \| B)$ with $|p| = x_{A^\perp} \| x_B$ and $|q| = y_{B^\perp} \| y_C$. Take

$$q \circledast p =_{\mathrm{def}} (p \| y_C) \wedge (x_{A^\perp} \| q)\,,$$

where we understand the configurations y_C and $x_{A^\perp}$ to inherit the partial order of their ambient event structures. Notice that $q \circledast p$ is defined only if $x_B = y_{B^\perp}$, and then only if no causal loops are introduced.

▸ **Lemma 8.** *Let $p \in \mathrm{Plays}(A^\perp \| B)$ and $q \in \mathrm{Plays}(B^\perp \| C)$. Then, if defined, $q \circledast p \in \mathrm{Plays}(A^\perp \| B^0 \| C)$.*

Define the *projection*

$$(\_)\!\downarrow : \mathrm{Plays}(A^\perp \| B^0 \| C) \to \mathrm{Plays}(A^\perp \| C)\,,$$

of a play p in $A^\perp \| B^0 \| C$,with $|p| = x_{A^\perp} \| x_B \| x_C$, to a play $p\!\downarrow$ in $A^\perp \| C$, to be the restriction of the order on p to the set $x_{A^\perp} \| x_C$. There may be several p and q which share the same image $q \odot p$. However, amongst these there are minimum p, q w.r.t. rigid inclusion.

Define a partial operation

$$\odot : \mathrm{Plays}(B^\perp \| C) \times \mathrm{Plays}(A^\perp \| B) \rightharpoonup \mathrm{Plays}(A^\perp \| C)$$

by

$$q \odot p = (q \circledast p)\!\downarrow$$

for $p \in \mathrm{Plays}(A^\perp \| B)$ and $q \in \mathrm{Plays}(B^\perp \| C)$.

▸ **Lemma 9.** *Let $p \in \mathrm{Plays}(A^\perp \| B)$ and $q \in \mathrm{Plays}(B^\perp \| C)$. Then, if defined, $q \odot p \in \mathrm{Plays}(A^\perp \| C)$. Moreover, w.r.t. rigid inclusion there are minimum $p' \hookrightarrow p$ and $q' \hookrightarrow q$ such that $q' \odot p' = q \odot p$.*

Let $\sigma : A^\perp \| B$ and $\tau : B^\perp \| C$ be strategies. Define their composition

$$\tau \odot \sigma = \{ q \odot p \mid p \in \sigma \ \&\ q \in \tau \ \&\ q \odot p \text{ is defined} \}\,.$$

It is sometimes useful to consider their composition *without hiding*, the *interaction*

$$\tau \circledast \sigma = \{ q \circledast p \mid p \in \sigma \ \&\ q \in \tau \ \&\ q \circledast p \text{ is defined} \}\,,$$

which is like the strategy $\tau \odot \sigma$, but before hiding the neutral moves over the game B.

▸ **Lemma 10.** *The interaction of strategies $\sigma : A^\perp \| B$ and $\tau : B^\perp \| C$ yields a bare strategy $\tau \circledast \sigma : A^\perp \| B^0 \| C$.*

▸ **Theorem 11.** *The composition of strategies $\sigma : A^\perp \| B$ and $\tau : B^\perp \| C$ yields a strategy $\tau \odot \sigma : A^\perp \| C$. Taking objects to be games and arrows from a game A to a game B to be strategies in the game $A^\perp \| B$, with composition as above, yields a category in which copycat is identity. (This is in contrast to the* bicategory *of [14].)*

5.1 Deterministic strategies

Let A be an event structure with polarity. A bare strategy $\sigma : A$ is *deterministic* iff

$$p \hookrightarrow^+ q \ \&\ p \hookrightarrow r \text{ in } \sigma \implies \exists s \in \sigma.\ q \hookrightarrow s \ \&\ r \hookrightarrow s\,.$$

The interaction of deterministic bare strategies is deterministic. Similarly, the composition of deterministic strategies is deterministic. However, for general games A, the copycat strategy need not be deterministic. It will be deterministic iff A is *race-free*, *i.e.*,

$$x \subseteq^+ y \ \&\ x \subseteq^- z \implies y \cup z \in \mathcal{C}(A)\,.$$

Restricting to race-free games as objects and deterministic strategies as arrows we obtain a category. Deterministic strategies coincide with the *receptive* ingenuous strategies of Melliès and Mimram [13] and are closely related to the strategies of Faggian and Piccolo [8], and Abramsky and Melliès' strategies as closure operators [1].

The subcategory of deterministic strategies on games which countable and purely positive, *i.e.* for which there are no Opponent moves, is isomorphic to that of Berry's dI-domains and stable functions. If we restrict the subcategory further to objects in which causal dependency is simply the identity relation we obtain Girard's qualitative domains with linear maps and if yet further insist that consistency Con is determined in a binary fashion, *i.e.*

$$X \in \text{Con} \iff \forall a_1, a_2 \in X.\ \{a_1, a_2\} \in Con\,,$$

his coherence spaces. In this sense we can see strategies as extending the world of stable domain theory. The relationship with the broader world of traditional domain theory, following in the footsteps of Scott, is more subtle. In [23], it is shown how a strategy determines a presheaf and a strategy between games a profunctor, giving a relationship with a form of generalised domain theory [10, 4].

6 Strategies as maps of event structures

A strategy σ in a game A is a rigid family and so, by Proposition 2, determines an event structure S whose events are those plays in σ which have a top element. Each top element is an event of the game A so there is a function from the events of S to those of A; this function is a total map of event structures and indeed a *concurrent strategy* in the sense of [14]. Not all the concurrent strategies of [14] are obtained this way. But any concurrent strategy of [14] has a rigid image [25] which corresponds to a strategy as presented here. Though not essential to the rest of the paper, we now explain this summary of the relation with the concurrent strategies of [14] in more detail.

Recall a *(total) map* of event structures $f : E \to E'$ is a function f from E to E' such that the image of a configuration x is a configuration fx and any event of fx arises as the image of a unique event of x. Maps compose as functions. Write $\mathcal{E}$ for the ensuing category.

A map $f : E \to E'$ reflects causal dependency locally, in the sense that if e, e' are events in a configuration x of E for which $f(e') \leq f(e)$ in E', then $e' \leq e$ also in E; the event structure E inherits causal dependencies from the event structure E' via the map f. Consequently, a map $f : E \to E'$ preserves concurrency: if two events are concurrent, $e_1 \; co_E \; e_2$, then their images are also concurrent, $f(e_1) \; co_{E'} \; f(e_2)$. In general a map of event structures need not preserve causal dependency; when it does we say it is *rigid*. Write $\mathcal{E}_r$ for the subcategory of rigid maps.

The inclusion functor $\mathcal{E}_r \hookrightarrow \mathcal{E}$ has a right adjoint ([20], Proposition 2.3): There is an obvious map of event structures $\epsilon_B : \mathrm{Pr}(\mathrm{Aug}(B)) \to B$ taking an event of $\mathrm{Pr}(\mathrm{Aug}(B))$ to its top element. Post-composition by ϵ_B yields a bijection

$$\epsilon_B \circ \_ : \mathcal{E}_r(A, \mathrm{Pr}(\mathrm{Aug}(B))) \cong \mathcal{E}(A, B)\,,$$

furnishing the data required for an adjunction. Hence $\mathrm{Pr}(\mathrm{Aug}(\_))$ extends to a right adjoint to the inclusion $\mathcal{E}_r \hookrightarrow \mathcal{E}$. From the bijection of the adjunction, we have a correspondence between maps $f : A \to B$ and rigid maps $\bar{f} : A \to \mathrm{Pr}(\mathrm{Aug}(B))$. The adjunction is unchanged by the addition of polarity to event structures; maps are assumed to preserve polarity.

A strategy determines a map and indeed a "concurrent strategy"as in [14]:

▸ **Proposition 12.** *Let $\sigma : A$ be a strategy in a game A. The function $f_\sigma : \mathrm{Pr}(\sigma) \to A$, taking an event of $\mathrm{Pr}(\sigma)$ to its top element, is a map of event structures with polarity. It is a concurrent strategy in the sense of [14], viz. a map which is*

- courteous*, $s' \rightarrowtriangle s$ and $pol(s') = +$ or $pol(s) = -$ in $\mathrm{Pr}(\sigma)$ implies $f_\sigma(s') \rightarrowtriangle_A f_\sigma(s)$ in A, (called* "innocent" *in [14]), and*
- receptive*, $f_\sigma x \subseteq^- y$ in $\mathcal{C}(A)$, for $x \in \mathcal{C}(\mathrm{Pr}(\sigma))$, implies there is a unique $x' \in \mathcal{C}(\mathrm{Pr}(\sigma))$ such that $f_\sigma x' = y$.*

Not all the concurrent strategies of [14] are obtained in the manner of Proposition 12. However, from any concurrent strategy $f : S \to A$ in a game A there is $\sigma : A$ obtained as the image

$$\sigma =_{\mathrm{def}} \{\theta(\bar{f}x) \in \mathrm{Aug}(A) \mid x \in \mathcal{C}(S)\}$$

of the finite configurations of S as augmentations of A; recall from Proposition 2, the order isomorphism $\theta : \mathcal{C}(\mathrm{Pr}(\mathrm{Aug}(A))) \cong \mathrm{Aug}(A)$. From the definition of σ, the rigid map $\bar{f} : S \to \mathrm{Pr}(\mathrm{Aug}(A))$ cuts down to a rigid map $\bar{f} : S \to \mathrm{Pr}(\sigma)$. The concurrent strategy f factors through its "rigid image" $f_\sigma : \mathrm{Pr}(\sigma) \to A$ in that

$$f : S \xrightarrow{\bar{f}} \mathrm{Pr}(\sigma) \xrightarrow{f_\sigma} A\,,$$

where the rigid image f_σ is itself a concurrent strategy. The simple strategies of this article correspond to such rigid image strategies.

The determination of a strategy, call it σ_f, from a concurrent strategy f is functorial: identity, copycat, strategies are preserved and if concurrent strategies f and g are composable then $\sigma_{g \odot f} = \sigma_g \odot \sigma_f$. Often extra structure on a concurrent strategy f can be pushed forward along the rigid map $\bar{f}$ from to its rigid image, so to a simple strategy of this article. For example, probabilistic structure (in the form of a valuation—see the next section) making a concurrent strategy probabilistic can be pushed forward along the rigid map $\bar{f}$ from S to $\mathrm{Pr}(\sigma_f)$, and so to σ_f [25]. As a consequence, in the next section, we are able to develop probabilistic strategies in the simpler framework of this paper.

A major result of [14] is that receptivity and courtesy (called innocence there) are necessary and sufficient conditions in order for copycat to behave as identity w.r.t. composition; this motivated the definition of concurrent strategy there. That article directly spawned work on games with winning conditions and payoff [5, 6], imperfect information [21], probabilistic strategies [24], "stopping configurations" [3] and "essential events" [16]—the latter two concerned with capturing the liveness behaviour of concurrent strategies viewed as processes. (Concurrent strategies are currently being extended to cope with quantum computation of the kind addressed in the quantum lambda calculus [15].) As an indication of how much of the work ensuing from [14] could be reformulated in terms of the simple strategies on which this article concentrates we next address the issue of how to make strategies probabilistic. Probabilistic strategies developed in this simpler framework, instead of that of concurrent strategies [14], do not suffer from any loss of information *e.g.* with regard to expected payoff.

7 Probabilistic strategies

As a first step we describe how to make event structures probabilistic, in itself an issue, as event structures lie outside the models of probabilistic processes most commonly considered.

7.1 Probabilistic event structures

A probabilistic event structure essentially comprises an event structure together with a continuous valuation on the Scott-open sets of its domain of configurations.[1] The continuous valuation assigns a probability to each open set and can then be extended to a probability measure on the Borel sets [11]. However open sets are several levels removed from the events of an event structure, and an equivalent but more workable definition is obtained by considering the probabilities of sub-basic open sets, generated by single finite configurations; for each finite configuration x this specifies $\mathrm{Prob}(x)$ the probability of obtaining events x, so as result a configuration which extends the finite configuration x. Such valuations on configuration determine the continuous valuations from which they arise, and can be characterised through the device of "drop functions" which measure the drop in probability across certain generalised intervals. The characterisation yields a workable general definition of probabilistic event structure as event structures with *configuration-valuations*, *viz.* functions from finite configurations to the unit interval for which the drop functions are always nonnegative [22].

In detail, a *probabilistic event structure* comprises an event structure E with a *configuration-valuation*, a function v from the finite configurations of E to the unit interval which is

- *(normalized)* $v(\varnothing) = 1$ and satisfies the
- *(drop condition)* $d_v[y; x_1, \cdots, x_n] \geq 0$ when $y \subseteq x_1, \cdots, x_n$ for finite configurations $y, x_1, \cdots, x_n$;

where the "drop" across the generalized interval starting at y and ending at one of the $x_1, \cdots, x_n$ is given by

$$d_v[y; x_1, \cdots, x_n] =_{\mathrm{def}} v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i)$$

—the index I ranges over nonempty $I \subseteq \{1, \cdots, n\}$ such that the union $\bigcup_{i \in I} x_i$ is a configuration. The "drop" $d_v[y; x_1, \cdots, x_n]$ gives the probability of the result being a configuration which includes the configuration y and does not include any of the configurations $x_1, \cdots, x_n$.

[1] A *Scott-open* subset of configurations is upwards-closed w.r.t. inclusion and such that if it contains the union of a directed subset S of configurations then it contains an element of S. A *continuous valuation* is a function w from the Scott-open subsets of $\mathcal{C}^\infty(E)$ to $[0, 1]$ which is *((normalized)* $w(\mathcal{C}^\infty(E)) = 1$; *(strict)* $w(\varnothing) = 0$; *(monotone)* $U \subseteq V \implies w(U) \leq w(V)$; *(modular)* $w(U \cup V) + w(U \cap V) = w(U) + w(V)$; and *(continuous)* $w(\bigcup_{i \in I} U_i) = \sup_{i \in I} w(U_i)$, for *directed* unions.

If $x \subseteq y$ in $\mathcal{C}(E)$, then, provided $v(x) \neq 0$, the conditional probability $\mathrm{Prob}(y \mid x)$ is $v(y)/v(x)$; this is the probability that the resulting configuration includes the events y conditional on it including the events x.

7.2 Probability with an Opponent

This prepares the ground for a definition of probabilistic distributed strategies. Firstly though, we should restrict to race-free games, in particular because without copycat being deterministic there would be no probabilistic identity strategies. A probabilistic strategy in a game A, is a strategy $\sigma : A$ in which we endow σ with probability, while taking account of the fact that in the strategy Player can't be aware of the probabilities assigned by Opponent. To this end we notice that σ, being a rigid family, has the form of a family of configurations. We can't just regard σ as a probabilistic event structure however. This is because Player is oblivious to the probabilities of Opponent moves beyond those determined by causal dependencies of σ. An appropriate *valuation* for σ needs to take account of Opponent moves. It turns out to be useful to extend the concept of valuation to *bare* strategies, which may also have neutral moves.

Let $\sigma : A$ be a bare strategy in A, an event structure with polarity; so both A and σ may involve neutral moves. A *valuation* on σ is a function v, from σ to the unit interval, which is

- *(normalized)* $v(\varnothing) = 1$,
- *(oblivious)* $v(p) = v(q)$ when $p \hookrightarrow^- q$ for $p, q \in \sigma$, and satisfies the
- *(drop condition)* $d_v[q; p_1, \cdots, p_n] \geq 0$ when $q \hookrightarrow^+ p_1, \cdots, p_n$ for elements of σ.

When $p \hookrightarrow^+ q$ in σ, we can still express $\mathrm{Prob}(q \mid p)$, the conditional probability of the additional neutral or Player moves making the play q given p, as $v(q)/v(p)$, provided $v(p) \neq 0$. The game being race-free and the valuation being oblivious ensure the probabilistic independence of Player or neutral moves with Opponent moves with which they are concurrent.

For a race-free game A, the copycat strategy is deterministic and we obtain a valuation on æ_A by taking $v_{\text{æ}_A}$ to be the function which is constantly 1.

7.3 Composing probabilistic strategies

Let A, B and C be race-free games. Assume $\sigma : A^\perp \| B$, with valuation v_σ, and $\tau : B^\perp \| C$, with valuation v_τ, are probabilistic strategies. To define their interaction and composition we must define the valuations $v_\tau \circledast v_\sigma$ on $\tau \circledast \sigma$ and $v_\tau \odot v_\sigma$ on $\tau \odot \sigma$, respectively.

▸ **Lemma 13.** *For $r \in \tau \circledast \sigma$, defining*

$$(v_\tau \circledast v_\sigma)(r) =_{\mathrm{def}} \sum \{v_\tau(q).v_\sigma(p) \mid q \circledast p = r\},$$

yields a valuation on $\tau \circledast \sigma$.

▸ **Lemma 14.** *For $r \in \tau \odot \sigma$, defining*

$$(v_\tau \odot v_\sigma)(r) =_{\mathrm{def}} \sum \{v_\tau(q).v_\sigma(p) \mid p, q \textit{ minimum s.t. } q \odot p = r\},$$

yields a valuation on $\tau \odot \sigma$.

In the above lemma it is important to restrict to minimum p, q such that $q \odot p = r$ in the sense of Lemma 9; otherwise we over-count contributions to the probability.

▸ **Theorem 15.** *For race-free games A, B and C, we define the composition of probabilistic strategies σ from A to B, with valuation v_σ, and τ from B to C, with valuation v_τ, to be $\tau \odot \sigma$, with valuation $v_\tau \odot v_\sigma$. Taking objects to be games and arrows from a game A to a game B to be probabilistic strategies in the game $A^\perp \| B$, with composition as above, yields a category in which copycat, with the constantly-1 valuation, is identity.*

The next example illustrates how through probability leaks we can track deadlocks, or divergences, that can arise in the composition of strategies. (Such branching behaviour might otherwise be lost in the composition of strategies and through concentrating on rigid images.)

▸ **Example 16.** Let B be the game consisting of two concurrent Player events b_1 and b_2, and C the game with a single Player event c. We illustrate the composition of two probabilistic strategies σ from the empty game $\varnothing$ to B and τ from B to C. The strategy $\sigma : \varnothing^\perp \| B$ plays b_1 with probability $2/3$ and b_2 with probability $1/3$ (and plays both with probability 0). The strategy $\tau : B^\perp \| C$ does nothing if just b_1 is played and plays the single Player event c of C with certainty, probabilty 1, if b_2 is played. Their composition yields the strategy $\tau \odot \sigma : \varnothing^\perp \| C$ which plays c with probability $1/3$, so has a $2/3$ chance of doing nothing.

One way in which the probabilistic interaction of strategies is important is in calculating the expected outcome of the competition between a probabilistic strategy and a counterstrategy, the subject of the following example.

▸ **Example 17.** Given a probabilistic strategy $\sigma : A$, with valuation v_σ, and a counterstrategy $\tau : A^\perp$, with valuation v_τ, we obtain a valuation $v_\tau \circledast v_\sigma$ on their interaction $\tau \circledast \sigma : A^0$, where now all the events of the interaction are neutral. Via the order isomorphism $\theta : \mathcal{C}(\mathrm{Pr}(\tau \circledast \sigma)) \cong \tau \circledast \sigma$ we obtain a configuration-valuation $(v_\tau \circledast v_\sigma) \circ \theta$, making $\mathrm{Pr}(\tau \circledast \sigma)$ a probabilistic event structure. As such we get a probability measure $\mu_{\sigma,\tau}$ on the Borel sets of its configurations. Assuming a *payoff* given as a Borel measurable function X from $\mathcal{C}^\infty(A)$ to the real numbers, the *expected payoff* is obtained as the Lebesgue integral

$$\mathbf{E}_{\sigma,\tau}(X) =_{\mathrm{def}} \int_{x \in \mathcal{C}^\infty(\mathrm{Pr}(\tau \circledast \sigma))} X(|x|)\ d\mu_{\sigma,\tau}(x)\,,$$

where $|x| \in \mathcal{C}^\infty(A)$ is the configuration of A over which $x \in \mathcal{C}^\infty(\mathrm{Pr}(\tau \circledast \sigma))$ lies.

8 Conclusion

We have provided an elementary account of a form of distributed strategies by choosing only to represent the rigid images of concurrent strategies. Is anything irredeemably lost through this simplification? (In the sense that it can't be regained through adding extra structure, in the way that probabilistic structure recovers hidden branching.) Not obviously. Though, for instance, we couldn't exactly reproduce the result of [3], establishing a bijection between events of a strategy and derivations in an operational semantics. Though an elementary account is more accessible, a more abstract, categorical account can be helpful too. As often, there are pros and cons. To some extent, one pays for the elementary treatment in not seeing the abstract picture, the wood for the trees.

On another tack, the account of strategies here reveals an alternative way to develop strategies while capturing noneterministic branching explicitly, *viz.* as (pre)sheaves over plays rather than subsets, in the form of rigid families. For instance, we can recover the concurrent strategies of [14] as certain separated presheaves in the manner of [19]; this brings us close to the developments of Hirschowitz and Pous [9] and Ong and Tsukada [17].

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