A NEW DEFINITION OF MORPHISM ON PETRI NETS

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0. Introduction.

Petri nets are a fundamental model of concurrent processes and have a wide range of applications. They can be viewed as generalisation of transition systems in which concurrency is not simulated by non-deterministic interleaving. They were invented by C. A. Petri in the 60's. (A reference work is [Br].)

It can be argued that the main effort and success of Petri Net Theory has been in developing techniques for showing properties of arbitrary Petri nets, *e.g.* Kurt Lautenbach has used techniques of linear algebra to discover invariants (properties which hold at all reachable markings). These techniques can be used to prove properties of concurrent programs. First represent the program as one big net and then prove properties about that. The problem is that big nets get out of hand, and more easily out of mind. For this reason chiefly, Hartmann Genrich, Kurt Lautenbach and Kurt Jensen invented predicate transition nets and coloured nets [GL, J] and accompanying techniques to find their invariants. Although they certainly do give a more compact way to model programs and systems they are necessarily more complicated, are more like programs, and need a semantics to relate them to structures which are more simple and universal.

We address another problem, that of constructions on Petri nets and how to prove properties of a compound process by proving properties of its components. The constructions follow from a new notion of morphism on Petri nets—it is not the same as Petri's original notion. The morphisms respect the token game unlike Petri's original. The category of nets with the new morphisms has a product which is closely related to various parallel compositions which have been defined on labelled Petri nets for synchronising processes (see *e.g.* the compositions on nets defined in [LS,...] and section 3). It has a coproduct which is a generalised form of the "sum" operation as used for example in [M].

One can use Petri nets to give semantics to programming languages. But, what is the semantics of nets? In themselves nets are complicated objects whose behaviour is rather intricate. When do Petri nets have the same behaviour? Attempting to answer these questions leads naturally to occurrence nets first introduced in [NPW1, 2]. Occurrence nets form a subcategory which bears a pleasant relation to the larger category of nets; the inclusion functor has a right adjoint which is an operation taking a net to its unfolding to a net of condition and event occurrences. (This construction was introduced in [NPW1, 2, W] but without this abstract characterisation.) It is argued that the meaning, or semantics, of a net is its occurrence net unfolding so that two nets are regarded as having essentially the same behaviour if they have isomorphic unfoldings.

The point of this work is to develop ways to structure (and so prove) properties of behaviour of large, even infinite, Petri nets while still keeping the nets of the straightforward form originally proposed by Petri [P]. I hope the neatness of the constructions and their simple characterisations counter one frequent criticism of Petri nets, that their mathematics is unwieldy.

1. Petri nets.

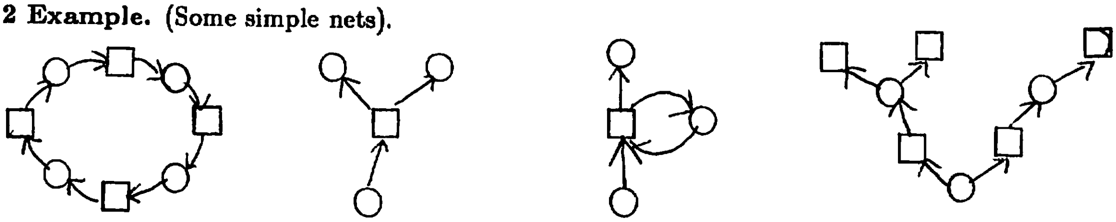
Petri nets have a structural part and a dynamic part. The structural part specifies the causal relation between events and conditions (=local states or propositions that can be made) of a system. The dynamic part specifies how the system evolves in time. Frequently a Petri net is identified with just the structural part, now defined.

1.1 Definition. A *Petri net* is a 3-tuple (B, E, F) where
B is a set of *conditions,*
E is a set of *events,*
$F \subseteq (B \times E) \cup (E \times B)$ is the *flow* (or *causal dependency*) relation
which satisfy the restriction:
$\{ b \in B \mid bFe \}$ is a non-null, finite set for all events $e \in E$.

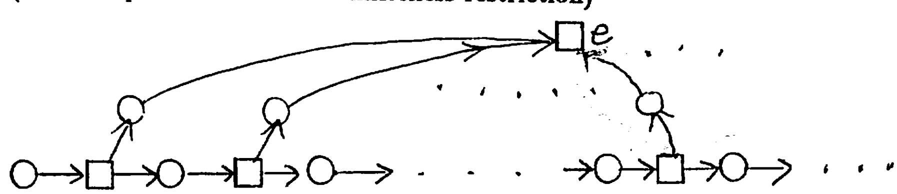
Thus we insist that each event causally depends on at least one condition, but require that the number of conditions on which it depends is finite.

Nets are often drawn as graphs in which events are represented as boxes and conditions as circles with directed arcs between them to represent the flow relation. Here are some examples.

1.2 Example. (Some simple nets).



1.3 Example. (An example which fails the finiteness restriction)



The above structure fails the restriction, $\{ b \in B \mid bFe \} \leq \infty$, which we have imposed on nets. Think of the intuitive behaviour of the net: the infinite chain of events and conditions is imagined to occur and only then does the event e occur—a strange computation! Petri forbids this kind of net by imposing an axiom called K-density (see [P]). However we find that axiom far too restrictive because if one accepts it one cannot model as wide a range of computations as one would wish—see [W1] for arguments against K-density—and so we prefer the weaker axiom we impose. (Later when defining *occurrence nets*—representatives of net behaviour—we shall impose further restrictions.)

1.4 Notation. Let $N = (B, E, F)$ be a net. Let x be an event or a condition so $x \in B \cup E$. Define

$$^\bullet x = F^{-1}\{ x \} = \{ y \in B \cup E \mid yFx \}.$$

When x is an event $e \in E$ we call the set $^\bullet e$ its *preconditions*.Similarly define

$$x^\bullet = F\{ x \} = \{ y \in B \cup E \mid xFy \}.$$

When x is an event e the set $e^\bullet$ is called its *postconditions*. We extend the "dot" notation to sets:

$$^\bullet A = \bigcup_{a \in A} {^\bullet a} \quad \text{and} \quad A^\bullet = \bigcup_{a \in A} a^\bullet .$$

So far, as we have defined them, nets are rather static objects. Their dynamic behaviour is based on these principles which specify how the occurrence of events affect the holding of conditions—a condition is said to *hold* when it is true:

(i) An occurrence of an event e ends the holding of its preconditions $^\bullet e$ and begins the holding of its postconditions $e^\bullet$.

(ii) (a) The holding of a condition b, when it ends, ends because of the occurrence of a unique event in $b^\bullet$.

(ii) (b) The holding of a condition b, when it begins, begins because of the occurrence of a unique event in $^\bullet b$.

Remark. The first principle (i) is often stated. The principles (ii)(a) and (ii)(b) do not seem to be recognised and stated so so widely (they are stated by Winkowski in [Win]). Principles (ii)(a) and (b) are consequences of a more basic principle:

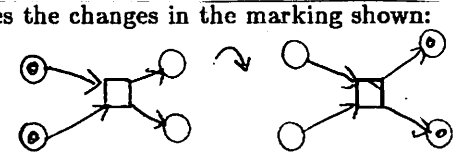
If the occurrences of two events in a net are ever coincident (or synchronised) then the two events are identical.

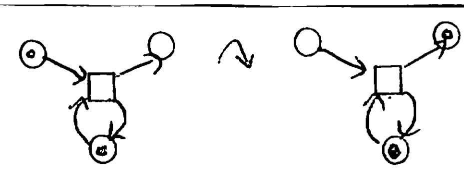
This principle expresses our understanding of the concept of an event; it says if the occurrence of two events is synchronised then they have to be the same event. (This principle does not hold in all applications of nets *e.g.* in [Sif] where two, or more, distinct events in the same net are forced to occur at the same time.)

Of course we need a way to specify what conditions hold. We introduce an idea of global state which just specifies what subset of conditions hold (= are true).

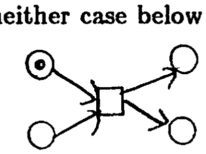
1.5 Definition. Let $N = (B, E, F)$ be a Petri net. A *marking* of N is a subset of conditions $M \subseteq B$.

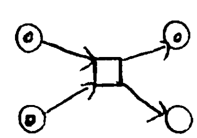
The marking of a net changes over time according to rules, commonly called "the token game" because a marking is often specified by laying tokens on those conditions in the marking; as events occur tokens are picked-up and put-down in accord with the fundamental principles above. From the fundamental principles it follows, only informally, of course, that an event can occur only once all its preconditions hold and none of its postconditions which are not preconditions hold. Here are two cases where the occurrence of an event produces the changes in the marking shown:





In neither case below can the events occur:

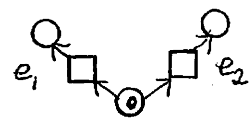
1. 

2. 

In 1 not all the preconditions hold so how could the occurrence of end the holding of the unmarked condition. In 2 a postcondition holds already, so how could the events occurrence begin its holding? The occurrence of the event in either 1 or 2 would contradict the principle (i) above.

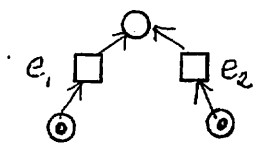
When an event can occur it is said to have *concession* or to be *enabled*.

So far we have looked at the occurrence of one event alone. Petri nets allow more than one event to occur together but there are situations where the occurrence of one event excludes the occurrence of another and vice versa – a phenomenon called *conflict*. Consider two events e_1 and e_2 which are both able to occur but which have a precondition b in common. In a picture we might have, for example



From the principle (ii)(a) it follows that only one of e_1 and e_2 can occur; otherwise they would both end the holding of the condition b. This is an example of *forwards conflict*.

Now consider two events which both have concession but which have a postcondition in common, for example



By (ii)(b) only one of e_1 , and e_2 can occur. This is an example of *backwards conflict*.

Now we can formally define the token game which specifies how the marking changes as events occur.

1.6 Definition. The token game Let $N = (B, E, F)$ be a Petri net. Let M be a marking.
Say an event $e \in E$ has *concession at* M iff

$$^{\bullet}e \subseteq M \ \&\ (e^{\bullet} \setminus {}^{\bullet}e) \cap M = \emptyset.$$

Let e,e' be events with concession at M. Say e and e' are in *forwards conflict* at M iff

$$e \neq e' \ \&\ {}^{\bullet}e \cap {}^{\bullet}e' \neq \emptyset.$$

Say they are in *backwards conflict* at M iff

$$e \neq e' \ \&\ e^{\bullet} \cap e'^{\bullet} \neq \emptyset.$$

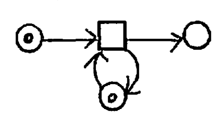
Let M and M' be markings. Let $A \subseteq E$. Define $M \xrightarrow{A} M'$ iff

$$\begin{aligned}&\forall e \in A. e \text{ has concession at } M \ \&\\ &\forall e, e' \in A. e, e' \text{ are not in conflict } \&\\ &M' = (M \setminus {}^{\bullet}A) \cup A^{\bullet}.\end{aligned}$$

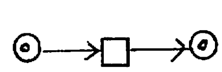
In this situation the events A are said to occur *concurrently*.

A marking M' is said to be reachable from a marking M iff $M = M_0 \xrightarrow{A_0} M_1 \xrightarrow{A_1} \cdots \xrightarrow{A_{n-1}} M_n = M'$ for subsets of events $A_0, A_1, \ldots, A_{n-1}$ and markings $M_0, M_1, \ldots, M_n$.

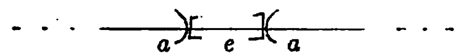
Remark. There are three points to clear up. Firstly we allow the event e to occur in



although we do not allow the event e to occur in



The reason is that in the first, the condition a is ended and then begun by the event occurrence, in time it looks like

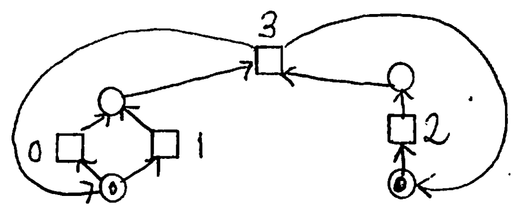


while in the second, the condition b is not first ended by the occurrence of e.

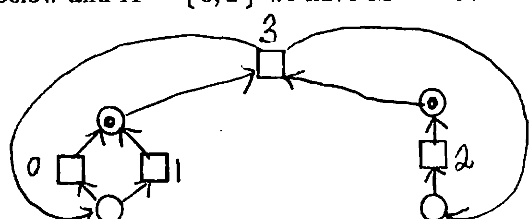
The second point is for those familiar with a token game in which more than one token is allowed or a condition, local states are allowed a certain multiplicity so that they can model, for example, the availability of a number of resources. We shall not allow more than one token on a condition, partly for simplicity and partly because I believe much more complicated nets should ultimately be abbreviations for the simpler nets we consider.

The third point is that in the U.S.A. the token game is often played differently to the way it is played in Europe. In the introductory book by Peterson [Pe], only one event is allowed to occur at a time, while in Europe, generally it is possible for a set of events to occur concurrently, as decribed here.

1.7 Example.

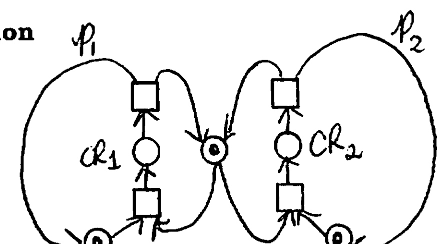


Initially the net is marked as shown. The events 0, 1 are in both forwards and backwards conflict so either 0 or 1, but not both can occur. Certainly the event 2 can occur. It is not in conflict with either 0 or 1 so 2 can occur concurrently with 0 or 1, but not both. For example, taking M to be the marking above, M' to be the marking below and $A = \{0, 2\}$ we have $M \xrightarrow{A} M'$.



Of course from the marking M' the event 3 can occur giving rise to the marking M again, and we can start all over again, perhaps letting event 1 occur this time.

1.8 Example. Mutual exclusion



The two processes P_1 and P_2 cannot both be in their critical regions CR_1 and CR_2 simultaneously.

Generally a process is modelled by a Petri net with an initial marking from which it reaches other markings as events occur.

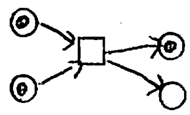
1.9 Definition. A Petri net with initial marking is a structure (B, E, F, M_0) where (B, E, F) is a Petri net and M_0 is a marking called the *initial marking*. Markings reachable from the initial marking are called *reachable markings*.

There is said to be *contact* at a marking M of a net if for some event

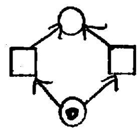
$$^\bullet e \subseteq M \ \& \ (e^\bullet \setminus {}^\bullet e) \cap M \neq \emptyset.$$

A Petri net with initial marking is *contact-free* iff there is not contact at each reachable marking.

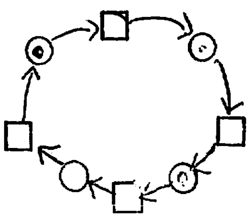
1.10 Example. A simple example of contact

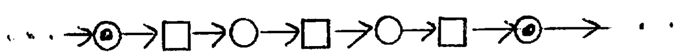


1.11 Example. Here is an example of net with initial marking which is contact-free, but which has backwards conflict at a reachable marking.



1.11 Example. The following nets with initial marking are not contact free.





Contact–free nets have the pleasant property that an event can occur at a reachable marking iff its preconditions are included in the marking. If one accepts the earlier principles, the behaviour of nets with contact is weird; it seems an event is prevented from occurring by the knowledge of what would happen in the future if it did—see the above examples. For this reason it is difficult to understand their behaviour. Later when we come to associate an occurrence net unfolding with the behaviour of a net—thus giving nets a formal semantics in terms of more basic nets—we shall only be able to do this with for nets which are contact–free. One view of nets with contact is that they are improper descriptions. As has been remarked, there are other token games in which conditions can have multiple holdings. For such nets the above principles are invalid. The understanding of such nets is less settled; for example the question of the equivalence of two nets is unsure, though a start has been made in [GR].

When a net is contact-free the token game simplifies as we now describe.

1.12 Proposition. The token game for contact-free nets:
Let $N = (B, E, F, M_0)$ be a contact-free net with initial marking. Let M be a reachable marking.
Let e be an event. Then e has concession at M iff $^\bullet e \subseteq M$.
Let e, e' be events. Then e, e' are in conflict at M iff $^\bullet e \cap {}^\bullet e' \neq \emptyset$.
Let M' be a marking of N. Then

$$\begin{aligned} M \xrightarrow{A} M' \Leftrightarrow & \forall e \in A. {}^\bullet e \subseteq M \\ & \& \forall e, e' \in A. {}^\bullet e \cap {}^\bullet e' = \emptyset \\ & \& M' = (M \setminus {}^\bullet A) \cup A^\bullet. \end{aligned}$$

2. The new definition of morphism on nets.

Our definition of morphism on nets involves binary relations, sometimes specialised to being partial or total functions. Here are the elementary notations, properties and operations on relations we shall use:

2.1 Notation. A *relation* from a set X to a set Y is a subset $R \subseteq X \times Y$. When $(x, y) \in R$ we write xRy. A relation R has an *opposite or (converse)* relation, R^{op}, given by

$$R^{op} = \{ (y, x) \mid xRy \}.$$

Clearly $xRy \Leftrightarrow yR^{op}x$.

When the relation R satisfies the property $\forall y, y' \in Y \forall x \in X . xRy \ \& \ xRy' \Rightarrow y = y'$ the relation R is said to be a *partial function*. A partial function R is said to be *total* when it satisfies the additional property $\forall x \in X \exists y \in Y . xRy$.

The *composition* of relations is defined as follows: Let R be a relation from a set X to a set Y and S a relation from the set Y to a set Z. The *composition* of R with S is the relation $S \circ R$ from X to Z given by

$$S \circ R = \{ (x, z) \in X \times Z \mid \exists y \in Y . xRy \ \& \ ySz \}.$$

Note the order of the composition which follows that generally used for functions but unfortunately not that commonly used for relations—using both functions and relations in the same breath we had to make a choice for one notation and chose to stick with the one for functions. We shall frequently miss-out the composition symbol $\circ$ and write $S \circ R$ as just SR.

When a relation R is a partial function, and we are thinking of it as taking an argument x and giving a value $R(x)$, it is useful to have a symbol to invoke when the value $R(x)$ does not exist. We use $*$ to represent *undefined* and so write

$$R(x) = * \Leftrightarrow \nexists y . xRy$$

when R is a partial function from X to Y.

If R is a relation from X to Y and $A \subseteq X$ we define the *image* of A under R to be the set RA given by

$$RA = \{ y \in Y \mid \exists x \in A . xRy \}.$$

Note the clash with abbreviated relation composition; any ambiguities can be resolved from the context.

Let $N_0 = (B_0, E_0, F_0, M_0)$ and $N_1 = (B_1, E_1, F_1, M_1)$ be two nets. A morphism from N_0 to N_1 is to be a pair of relations (ϵ, β) where ϵ is a relation between events, $\epsilon \subseteq E_0 \times E_1$, and β is a relation between conditions, $\beta \subseteq B_0 \times B_1$. The relation $e_0 \epsilon e_1$ means: when e_0 occurs its occurrence is synchronised with the occurrence of e_1. The relation $b_0 \beta b_1$ means: when b_0 begins to hold its beginning is synchronised with the beginning of the holding of b_1, and when b_0 ends holding its end is synchronised with the end of b_1. (In the following discussion conditions in the initial markings are assumed begun by some starting event.)

An informal argument suggests that ϵ should be a partial function: Assume $e_0 \epsilon e_1$ and $e_0 \epsilon e_1'$ for events e_0 in N_0 and e_1, e_1' in N_1. Then the occurrence of e_0 implies the synchronised occurrence of e_1 and e_1'. This makes the events e_1 and e_1' synchronised together. According to our informal understanding of the behaviour of N_1—as given in the last section—the two events can only be synchronised together if they are the same event so $e_1 = e_1'$.

From our interpretation of β if $b_0 \beta b_1$ and b_0 begins to hold in N_0 then b_1 should begin to hold in N_1. Thus if $e_0 F_0 b_0$ and $b_0 \beta b_1$, so e_0 begins the holding of b_0 which is synchronised with the beginning of the holding of b_1, there should be an event e_1 synchronised with e_0 which begins the holding of b_1 *i.e.* $e_0 \epsilon e_1$ and $e_1 F_1 b_1$. In particular, if $b_0 \in M_0$ and $b_0 \beta b_1$ then as b_0 holds initially so should b_1, making $b_1 \in M_1$. (Recall conditions of the initial markings are imagined started by a starting event.) Similarly if $b_0 F_0 e_0$ and $b_0 \beta b_1$ then there should exist an event e_1 such that $e_0 \epsilon e_1$—consider how the holdings of the conditions end.

In order for the pair (ϵ, β) to be a morphism we insist that some further restrictions are met in the neighbourhood of events. Suppose $e_0 \epsilon e_1$ for an event $e_0 \in E_0$ and event $e_1 \in E_1$. If $b_1 F_1 e_1$, so e_1 ends the holding of b_1, we insist there is a unique condition b_0 so that $b_0 F_0 e_0$ and $b_0 \beta b_1$. Similarly if $e_1 F_1 b_1$ we require there exists a condition b_0 such that $e_0 F_0 b_0$ and $b_0 \beta b_1$. In particular for the initial marking (imagined started by a starting event) we have $\forall b_1 \in M_1 \exists! b_0 \in M_0 . b_0 \beta b_1$.

We define morphisms between general marked Petri nets. Later we shall have reason to specialise to contact-free nets.

2.2 Definition. Let $N = (B_i, E_i, F_i, M_i)$ be nets for $i = 0, 1$. Define a *morphism* of nets from N_0 to N_1 to be a pair of relations (ϵ, β) such that $\epsilon \subseteq E_0 \times E_1$ is a partial function, $\beta \subseteq B_0 \times B_1$ which satisfies the restrictions

$$M_1 = \beta M_0,$$
$$\forall b_1 \in M_1 \exists! b_0 \in M_0 . b_0 \beta b_1,$$

and for all $e_0 \in E_0$, $b_1 \in B_1$

$$\exists e_1 . (e_0 \epsilon e_1 \ \& \ b_1 F_1 e_1) \Rightarrow \exists! b_0 . (b_0 \beta b_1 \ \& \ b_0 F_0 e_0)$$
$$\exists b_0 . (b_0 \beta b_1 \ \& \ b_0 F_0 e_0) \Rightarrow \exists e_1 . (e_0 \epsilon e_1 \ \& \ b_1 F_1 e_1)$$

and

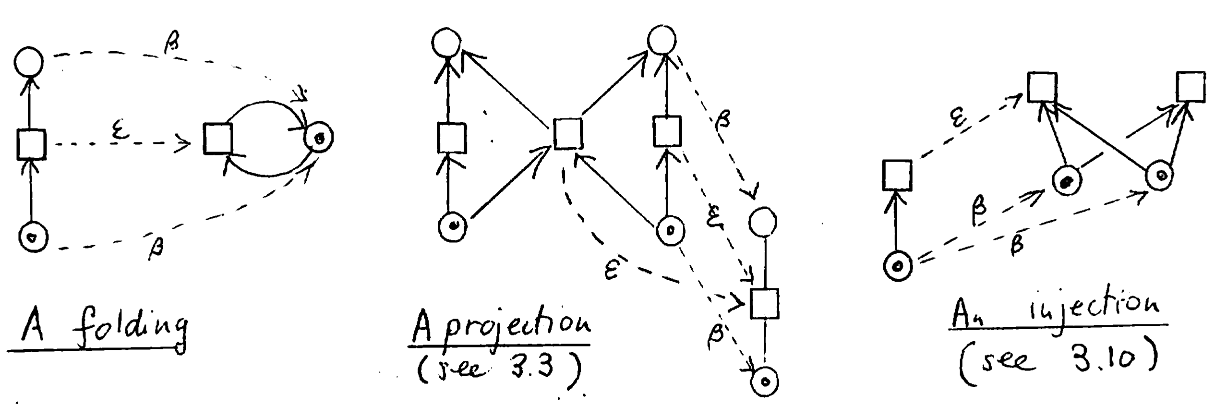
$$\exists e_1 . (e_0 \epsilon e_1 \ \& \ e_1 F_1 b_1) \Rightarrow \exists! b_0 . (b_0 \beta b_1 \ \& \ e_0 F_0 b_0)$$
$$\exists b_0 . (b_0 \beta b_1 \ \& \ e_0 F_0 b_0) \Rightarrow \exists e_1 . (e_0 \epsilon e_1 \ \& \ e_1 F_1 b_1).$$

When the function ϵ is total we say the morphism (ϵ, β) is *synchronous*.
When the relations ϵ and β are total functions we say the morphism (ϵ, β) is a *folding*.
When (ϵ, β) is a morphism, $B_0 \subseteq B_1$ and $E_0 \subseteq E_1$ and the relations ϵ and β are the restrictions of the inclusion relations, *i.e.* $e_0 \epsilon e_1 \Leftrightarrow e_0 = e_1$ and $b_0 \beta b_1 \Leftrightarrow b_0 = b_1$, we say the net N_0 is a *subnet* of N_1.

Recalling our intuition about the F relation, the restrictions above say of a morphism:

An event $\epsilon(e_0)$ ends/begins the holding of a condition b_1 iff e_0 ends/begins the holding of a unique condition b_0 such that $b_0 \beta b_1$.

2.3 Example. Here are three examples of morphisms:



2.3 Proposition. *Let $N_0 = (B_0, E_0, F_0, M_0)$ and $N_1 = (B_1, E_1, F_1, M_1)$ be two nets. Let (ϵ, β) be a pair of relations $\epsilon \subseteq E_0 \times E_1$ and $\beta \subseteq B_0 \times B_1$.*

The pair is a morphism, $(\epsilon, \beta) : N_0 \rightarrow N_1$ *iff* ϵ *is a partial function,* $\beta^{op} \cap M_1 \times M_0 : M_1 \rightarrow M_0$ *is a total function and*

$$\forall e_0, e_1 . e_0 \epsilon e_1 \Rightarrow \beta^\bullet e_0 = {}^\bullet e_1 \ \& \\ \beta^{op} \cap {}^\bullet e_1 \times {}^\bullet e_0 : e_1{}^\bullet \rightarrow e_0{}^\bullet \text{ is a total function, } \& \\ \beta e_0{}^\bullet = e_1{}^\bullet \ \& \\ \beta^{op} \cap e_1{}^\bullet \times e_0{}^\bullet : {}^\bullet e_1 \rightarrow {}^\bullet e_0 \text{ is a total function.}$$

The pair (ϵ, β) *is a folding iff* ϵ *and* β *are total functions,* $\beta \cap M_0 \times M_1$ *is a one–one correspondence between initial markings and* $\beta \cap {}^\bullet e \times {}^\bullet \epsilon(e)$ *(respectively* $\beta \cap e^\bullet \times \epsilon(e)^\bullet$*) is a one–one correspondence between the preconditions (respectively postconditions) of* e *and* $\epsilon(e)$.

Proof. Directly from the definition of morphism. ∎

Thus our definition of folding is not the same as Petri's; his allows, for example, more than one precondition of an event to map to the same condition in the image, a possibility not allowed by our definition of morphism. Still our definition of folding and Petri's appear to agree on all the important examples.

2.4 Lemma. *Let* $(\epsilon, \beta) : N_0 \rightarrow N_1$ *be a morphism between nets* $N_0 = (B_0, E_0, F_0, M_0)$ *and* $N_1 = (B_1, E_1, F_1, M_1)$. *Let* A *be a subset of the events of* N_0. *Then*

$$\beta({}^\bullet A) = {}^\bullet(\epsilon A)$$
$$\beta(A^\bullet) = (\epsilon A)^\bullet.$$

Also, suppose e *and* e' *are two events of* N_0 *such that* $\epsilon(e) \neq *$ *and* $\epsilon(e') \neq *$ *and* ${}^\bullet e, {}^\bullet e' \subseteq M_0$. *Then*

$${}^\bullet\epsilon(e) \cap {}^\bullet\epsilon(e') \neq \emptyset \Rightarrow {}^\bullet e \cap {}^\bullet e' \neq \emptyset.$$

2.5 Theorem. *Let* $N = (B_i, E_i, F_i, M_i)$ *be nets for* $i = 0, 1$. *Let* N_1 *be contact-free. Let* $(\epsilon, \beta) : N_0 \rightarrow N_1$ *be a morphism of nets. Let* C *be a reachable marking of* N_0 *and suppose*

$$C \xrightarrow{A} C' \text{ in } N_0.$$

Then βC *is a reachable marking of* N_1 *and*

$$\beta C \xrightarrow{\epsilon A} \beta C' \text{ in } N_1.$$

Further, for all reachable markings C *of* N_0,

$$\forall b_1 \in \beta C \exists! b_0 \in C . b_0 \beta b_1.$$

Proof. We take the statement of the theorem as inductive hypothesis and prove the theorem by induction on the length of the chain $M_0 \xrightarrow{A_0} \cdots \xrightarrow{A_p} C$ from the initial marking M_0 to a reachable marking C. From the definition of morphism we immediately have that $M_1 = \beta M_0$ and $\forall b_1 \in M_1 \exists! b_0 \in M_0 . b_0 \beta b_1$. Thus the inductive hypothesis holds for the base case when the length of the chain is zero.

To show the inductive step:

Suppose C is a reachable marking and that $C \xrightarrow{A} C'$ in N_0. Then by induction hypothesis βC is a reachable marking of N_1. We require that $\beta C \xrightarrow{\epsilon A} \beta C'$ in N_1—of course it then follows that $\beta C'$ is a reachable marking—and also that $\forall b_1 \in \beta C \exists! b_0 \in C . b_0 \beta b_1$.

Suppose $e \in A$ and that $\epsilon(e)$ is defined. Then e has concession at C so ${}^\bullet e \subseteq C$. However by the previous lemma ${}^\bullet\epsilon(e) = \beta^\bullet e \subseteq \beta C$. Thus each event in ϵA has concession at βC because N_1 is assumed contact–free.

Suppose $\epsilon(e)$ and $\epsilon(e')$ are defined for $e, e' \in A$. Then ${}^\bullet e, {}^\bullet e' \subseteq C$. Suppose $\epsilon(e)$ and $\epsilon(e')$ are in conflict at βC *i.e.* because N_1 is contact–free, $\epsilon(e) \neq \epsilon(e')$ and ${}^\bullet\epsilon(e) \cap {}^\bullet\epsilon(e') \neq \emptyset$. By the previous lemma ${}^\bullet e \cap {}^\bullet e' \neq \emptyset$. As ϵ is a partial function, $e \neq e'$ so e and e' are in conflict at C. This is a contradiction. Consequently $\epsilon(e)$ and $\epsilon(e')$ are not in conflict at βC for $e, e' \in A$.

To complete the proof that $\beta C \xrightarrow{\epsilon A} \beta C'$ in N_1 we show that $\beta C' = (\beta C \setminus {}^\bullet(\epsilon A)) \cup (\epsilon A)^\bullet$. Clearly

$$\begin{aligned} \beta C' &= \beta((C \setminus {}^\bullet A) \cup A^\bullet) \\ &= (\beta(C \setminus {}^\bullet A)) \cup (\beta(A^\bullet)). \end{aligned}$$

Now β^{op} restricted to βC forms a (total) function, f say, such that

$$f = \beta^{op}\lceil \beta C : \beta C \to C.$$

It is easily shown that

$$f^{-1}(X \setminus Y) = (f^{-1}X) \setminus (f^{-1}Y)$$

for such a function f and sets X and Y in the codomain of f. It follows that

$$\beta(C \setminus {}^\bullet A) = f{-}1(C \setminus {}^\bullet A) = (f^{-1}C) \setminus (f^{-1\bullet}A) = (\beta C) \setminus (\beta^\bullet A).$$

By the above lemma we have $\beta^\bullet A = {}^\bullet(\epsilon A)$ and $\beta A^\bullet = (\epsilon A)^\bullet$. Thus

$$\beta C' = (\beta C \setminus {}^\bullet(\epsilon A)) \cup (\epsilon A)^\bullet$$

as required. Therefore $\beta C \xrightarrow{\epsilon A} \beta C'$ and consequently $\beta C'$ is a reachable marking.

Finally, to complete the inductive step we require that

$$\forall b_1 \in \beta C' \exists! b_0 \in C'. b_0 \beta b_1.$$

Clearly it is sufficient to prove

$$\forall b_0, b_0' \in C'. b_0 \beta b_1 \ \&\ b_0' \beta b_1 \Rightarrow b_0 = b_0'.$$

We establish a contradiction by supposing otherwise *i.e.* that there are $b_0, b_0' \in C'$ with $b_0 \neq b_0'$ & $b_0 \beta b_1$ & $b_0' \beta b_1$.

Because of the induction hypothesis on C this could only occur if either $b_0, b_0' \in A^\bullet$ or $b_0 \in (C \setminus {}^\bullet A)$ & $b_0' \in A^\bullet$—or essentially the same case with b_0 and b_0' interchanged. Fortunately the first case can be reduced to the second: Take $e \in A$ such that $b_0 \in e^\bullet$ and $C^+ = (C \setminus {}^\bullet e) \cup e^\bullet$ and $A^+ = A \setminus \{e\}$; clearly then $b_0 \in (C^+ \setminus {}^\bullet A^+)$ & $b_0' \in A^{+\bullet}$.

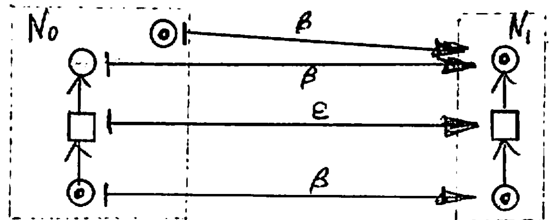
Thus we need only consider the case $b_0 \in (C \setminus {}^\bullet A)$ & $b_0' \in A^\bullet$. Then $e_0 F_0 b_0'$ for some $e_0 \in A$. Consequently for some event $e_1 \in E_1$ we have $e_0 \epsilon e_1$ & $e_1 F_1 b_1$. We show there is contact at βC. We have $b_1 \notin {}^\bullet e_1$ as $b_0 \notin {}^\bullet e_0$. Also ${}^\bullet e_1 \subseteq \beta C$ and $b_1 \in \beta C$. Thus ${}^\bullet e_1 \subseteq \beta C$ and $\beta C \setminus e^\bullet \neq \emptyset$ so there is contact at βC—a contradiction as N_1 is contact free. Therefore

$$\forall b_1 \in \beta C' \exists! b_0 \in C'. b_0 \beta b_1$$

as required to complete the induction. ∎

The next example shows that the restriction, that N_1 be contact-free, is necessary in theorem 2.5.

2.6 Example. Let $(\epsilon, \beta) : N_0 \rightarrow N_1$ be a morphism between nets with initial markings as shown:



It is easily checked that (ϵ, β) is a morphism. However because there is contact in N_1 the image of an event in N_0 with concession does not have concession in N_1.

2.7 Definition. Let $N_i = (B_i, E_i, F_i, M_i)$ be Petri nets for $i = 0, 1, 2$. Let $(\epsilon_0, \beta_0) : N_0 \rightarrow N_1$ and $(\epsilon_1, \beta_1) : N_1 \rightarrow N_2$ be morphisms. Define their composition $(\epsilon_1, \beta_1) \circ (\epsilon_0, \beta_0)$ to be $(\epsilon_1 \circ \epsilon_0, \beta_1 \circ \beta_0)$—where $\epsilon_1 \circ \epsilon_0$ and $\beta_1 \circ \beta_0$ are the compositions of relations given above.

2.8 Proposition. *Contact-free Petri nets with morphisms and composition as above form a category, i.e. each net $N = (B, E, F, M)$ has an identity morphism $(1_E, 1_B)$ with respect to composition and composition is associative. When morphisms are restricted to being synchronous or foldings we obtain respective subcategories.*

2.9 Definition. Define **Net** to be the category of contact-free nets with morphisms on nets as defined above. Define $\mathbf{Net}_{syn}$ to be the subcategory with synchronous morphisms on nets. Define $\mathbf{Net}_{fol}$ to be the subcategory with morphisms which are foldings.

In the next section we explore further the consequences of our definition of morphism.

3. Categorical constructions.

In this section we shall see that our choice of morphism throws out several interesting and useful categorical constructions. One important consequence of the constructions being categorical is that each comes accompanied by a characterisation to within isomorphism. Such characterisations are useful when reasoning about processes modelled by nets built-up from the constructions. It is not just a hope that that the constructions will eventually be found a use. The product is related to many forms of parallel composition defined on nets (see for example the work of Lauer and Shields ....[ ]). The synchronous product (in the category with synchronous morphisms), itself a somewhat stricter form of parallel composition, provides a natural interleaving, or serialising operator, on nets, by setting them in synchronous product with a "clock process", while the coproduct construction connects well with "sum" operations used by for example Robin Milner et al [ ].

The categorical constructions we shall introduce will depend on the properties of two more basic categories. One is well-known; it is the category of sets with partial functions. It corresponds to that part of morphisms on nets which act between sets of events. The other is new, at least to me; it is called the category of marked sets and corresponds to that part of morphisms on nets which act between sets of conditions while respecting the initial marking.

3.1 Lemma. *Product and coproduct for the category of sets with partial functions.*

Let $\mathbf{Set}_*$ *be the category of sets and partial functions given in definition 2.1.* $\mathbf{Set}_*$ *has products and coproducts of the following form:*

Let E_0 and E_1 be sets.

Their product, to within isomorphism, is $E_0 \times_* E_1$ with projections π_0, π_1 where

$$E_0 \times_* E_1 = \{ (e_0, *) \mid e_0 \in E_0 \} \cup \{ (*, e_1) \mid e_1 \in E_1 \} \cup \{ (e_0, e_1) \mid e_0 \in E_0 \ \&\ e_1 \in E_1 \},$$

and $\pi_0(x, y) = x$, $\pi_1(x, y) = y$.

Their coproduct, to within isomorphism, is $E_0 + E_1 =_{def} \{0\} \times E_0 \cup \{1\} \times E_1$ with injections $in_0(e_0) = (0, e_0)$ and $in_1(e_1) = (1, e_1)$ for $e_0 \in E_0$ and $e_1 \in E_1$.

Proof. The proof is left to the reader. These facts are well known see *e.g.* [Mac] or [Arb] but note our sets are not their pointed sets. ∎

3.2 Lemma. *Product and coproduct of marked sets.*

Define a marked set to be a pair of sets (B, M) where $M \subseteq B$. Define a morphism of marked sets from (B_0, M_0) to (B_1, M_1) to be a relation $R \subseteq B_0 \times B_1$ such that $RM_0 = M_1$ and

$\forall b_0, b'_0 \in M_0 \forall b_1 \in M_1 . b_0 R b_1 \ \&\ b'_0 R b_1 \Rightarrow b_0 = b'_0$.

Define composition to be the usual composition of relations given in 2.1. Then marked sets with the morphisms above form a category with identity morphisms the identity relations. It has products and coproducts of the following form:

Let (B_0, M_0) and (B_1, M_1) be marked sets.

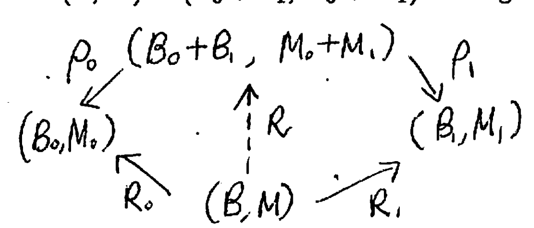
Their product, to within isomorphism, is $(B_0 + B_1, M_0 + M_1)$ with projections the relations ρ_0 and ρ_1 given by $(b, 0)\rho_0 b$ for $b \in B_0$ and $(b, 1)\rho_1 b$ for $b \in B_1$.(The projection relations ρ_i are the opposite relations to the injection functions from the set B_i into the disjoint union $B_0 + B_1$.)

Their coproduct, to within isomorphism, is (B, M) with injections ι_0 and ι_1 where

$$\begin{aligned} B &= \{ (b_0, *) \mid b_0 \in B_0 \setminus M_0 \} \cup \{ (*, b_1) \mid b_1 \in B_1 \setminus M_1 \} \cup \{ (b_0, b_1) \mid b_0 \in B_0 \ \&\ b_1 \in B_1 \}, \\ M &= M_0 \times M_1, \\ b_0 \iota_0 b &\Leftrightarrow \exists b_1 \in B_1 \cup \{ * \} . b = (b_0, b_1), \\ b_1 \iota_1 b &\Leftrightarrow \exists b_0 \in B_0 \cup \{ * \} . b = (b_0, b_1). \end{aligned}$$

(Thus the injection relations are opposite to the obvious partial functions taking a condition in B to its first or second component.)

Proof. The product in marked sets. We verify that the construction above does indeed give a product. Firstly it is easily checked that the relations ρ_0 and ρ_1 above are morphisms of marked sets $\rho_0 : (B_0 + B_1, M_0 + M_1) \to (B_0, M_0)$ and $\rho_1 : (B_0 + B_1, M_0 + M_1) \to (B_1, M_1)$. Let $R_0 : (B, M) \to (B_0, M_0)$ and $R_1 : (B, M) \to (B_1, M_1)$ be morphisms of marked sets from a marked set (B, M). We require that there exists a unique morphism $R : (B, M) \to (B_0 + B_1, M_0 + M_1)$ making the following diagram commute:

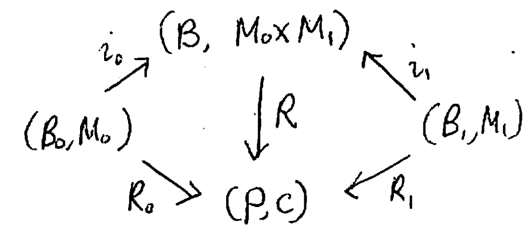


We take $R = \{ (b, (0, b_0)) \mid bR_0 b_0 \} \cup \{ (b, (1, b_1)) \mid bR_1 b_1 \}$. Clearly $RM = M_0 + M_1$ and supposing bRc & $b'Rc$ implies c has the form $(0, b_0)$ or $(1, b_1)$. Without loss of generality assume $c = (0, b_0)$ for some $b_0 \in B_0$. Then from the definition of R we know $bR_0 b_0$ and $b'R_0 b_0$. As R_0 is a morphism we obtain $b = b'$. Thus R is a morphism of marked sets.

From the definition of R it follows directly that the diagram commutes. Suppose $S : (B, M) \to (B_0 + B_1, M_0 + M_1)$ is a morphism making the diagram commute. Then as $\rho_j S = R_j$ for $j = 0, 1$ we get

$bS(j, b_j) \Leftrightarrow bR_jb_j \Leftrightarrow bR(j, b_j)$ which makes $S = R$. Thus R is the unique morphism such that the diagram commutes. Therefore the construction really is the product in marked sets as required.

The coproduct in marked sets. We verify that the construction above does indeed give a coproduct. Firstly it is easily checked that that the relations ι_0 and ι_1 above are morphisms of marked sets $\iota_0 : (B_0, M_0) \to (B, M)$ and $\iota_1 : (B_1, M_1) \to (B, M)$. Let $R_0 : (B_0, M_0) \to (P, C)$ and $R_1 : (B_1, M_1) \to (P, C)$ be morphisms of marked sets for a marked set (P, C). We require that there exists a unique morphism of marked sets $R : (B, M_0 \times M_1) \to (P, C)$ making the following diagram commute:



Define

$$\begin{aligned} R = &\{((b_0, *), p) \mid b_0 \in B_0 \setminus M_0 \ \&\ b_0R_0p\} \\ &\cup \{((*, b_1), p) \mid b_1 \in B_1 \setminus M_1 \ \&\ b_1R_1p\} \\ &\cup \{((b_0, b_1), p) \mid b_0 \in M_0 \ \&\ b_1 \in M_1 \ \&\ b_0R_0p \ \&\ b_1R_1p\}. \end{aligned}$$

Clearly as R_0 and R_1 are morphisms

$$\begin{aligned} RM &= \{p \mid b_0 \in M_0 \ \&\ b_0R_0p \ \&\ b_1 \in M_1 \ \&\ b_1R_1p\} \\ &= R_0M_0 \cup R_1M_1 = C \cup C = C. \end{aligned}$$

Also, suppose $b, b' \in M$ and bRp and $b'Rp$. Then for some $b_0, b_0' \in B_0$ and $b_1, b_1' \in B_1$ we have

$$\begin{aligned} b &= (b_0, b_1) \ \&\ b_0R_0p \ \&\ b_1R_1p \quad \text{and} \\ b &= (b_0', b_1') \ \&\ b_0'R_0p \ \&\ b_1'R_1p. \end{aligned}$$

But, as R_0 and R_1 are morphisms $b_0 = b_0'$ and $b_1 = b_1'$ so $b = b'$. Thus $R : (B, M) \to (P, C)$ is a morphism of marked sets.

Now we show R makes the above diagram commute *i.e.* $R_0 = R\iota_0$ and $R_1 = R\iota_1$. (Recall our composition of relations follows the same order as the usual one for functions!) Clearly directly from the definition of R we obtain $R\iota_0 \subseteq R_0$ and $R\iota_1 \subseteq R_1$. Now suppose b_0R_0p. Either $b_0 \notin M_0$ or $b_0 \in M_0$. If $b_0 \notin M_0$ this gives $(b_0, *)Rp$. Otherwise, $b_0 \in M_0$ making $p \in C = R_0M_0$. But then there is some $b_1 \in M_1$ so that b_1R_1p. This gives $(b_0, b_1)Rp$. In either case this yields $b_0(R\iota_0)p$. Thus $R_0 \subseteq R\iota_0$ which combined with the converse inclusion proved earlier gives $R_0 = R\iota_0$. Similarly $R_1 = R\iota_1$. Thus R does make the above diagram commute.

In addition we need that R is the unique morphism making the diagram commute. Suppose $S : (B, M) \to (P, C)$ made the above diagram commute *i.e.* $S\iota_0 = R_0$ and $S\iota_1 = R_1$. Considering the three different kinds of element of B we have:

$$\begin{aligned} (b_0, *)Sp &\Leftrightarrow b_0R_0p, \quad \text{for} \quad b_0 \in B_0 \setminus M_0, \\ (*, b_1)Sp &\Leftrightarrow b_1R_1p, \quad \text{for} \quad b_1 \in B_1 \setminus M_1, \\ (b_0, b_1)Sp &\Leftrightarrow b_0R_0p \ \&\ b_1R_1p, \quad \text{for} \quad b_0 \in M_0 \ \&\ b_1 \in M_1. \end{aligned}$$

Thus $S = R$.

And so finally we have proved that the construction above is a coproduct. ∎

Now we give a construction of the product of two nets. In view of the two lemmas on the more basic categories above it will follow that the construction really is a categorical product in **Net**.

3.3 Definition. The product of nets.

Let $N_0 = (B_0, E_0, F_0, M_0)$ and $N_1 = (B_1, E_1, F_1, M_1)$ be contact–free nets.

Let $\pi_0 : E_0 \times_* E_1 \to E_0$ and $\pi_1 : E_0 \times_* E_1 \to E_1$ be the projections from the product of sets in **Set**$_*$ given in 3.1. Let $\rho_0 : (B_0 + B_1, M_0 + M_1) \to (B_0, M_0)$ and $\rho_1 : (B_0 + B_1, M_0 + M_1) \to (B_1, M_1)$ be the projections from the product of marked sets given in 3.2.

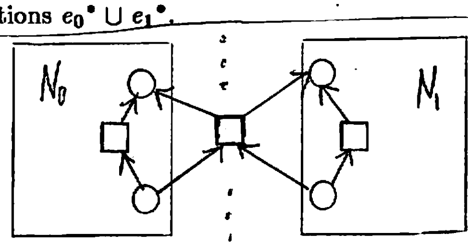
Define the *product* of the nets, $N_0 \times N_1$, to be the net (B, E, F, M) where $B = B_0 + B_1$, $M = M_0 + M_1$, $E = E_0 \times_* E_1$ and

$$
\begin{aligned}
eFb \Leftrightarrow & (\exists e_0 \in E_0, b_0 \in B_0 . e\pi_0 e_0 \ \&\ b\rho_0 b_0 \ \&\ e_0 F_0 b_0) \\
& \text{or } (\exists e_1 \in E_1, b_1 \in B_1 . e\pi_1 e_1 \ \&\ b\rho_1 b_1 \ \&\ e_1 F_1 b_1) \\
bFe \Leftrightarrow & (\exists e_0 \in E_0, b_0 \in B_0 . e\pi_0 e_0 \ \&\ b\rho_0 b_0 \ \&\ b_0 F_0 e_0) \\
& \text{or } (\exists e_1 \in E_1, b_1 \in B_1 . e\pi_1 e_1 \ \&\ b\rho_1 b_1 \ \&\ b_1 F_1 e_1).
\end{aligned}
$$

Define *projection* morphisms of nets:

$$
\begin{aligned}
\Pi_0 &= (\pi_0, \rho_0) : N_0 \times N_1 \to N_0 \\
\Pi_1 &= (\pi_1, \rho_1) : N_0 \times N_1 \to N_1.
\end{aligned}
$$

The product construction can be summarised in a simple picture. Disjoint copies of the two nets N_0 and N_1 are juxtaposed and extra events of synchronisation of the form (e_0, e_1) are adjoined, for e_0 an event of N_0 and e_1 an event of N_1; an extra event (e_0, e_1) has as preconditions those of its components ${}^\bullet e_0 \cup {}^\bullet e_1$ and similarly postconditions $e_0{}^\bullet \cup e_1{}^\bullet$.



The product on nets is closely related to various forms of parallel composition which have been defined on nets to model synchronised communication—see[ ]. For the moment imagine that the events of nets are labelled in order to specify how they can or cannot synchronise with events in the environment—the synchronisation algebras of [W2, W3] are a way of formalising this idea. Then the parallel composition of two labelled nets will be modelled as a restriction of the product to those synchronised events—of the form (e_0, e_1)—and those unsynchronised events—of the form $(e_0, *)$ and $(*, e_1)$—allowed by the discipline of synchronisation.

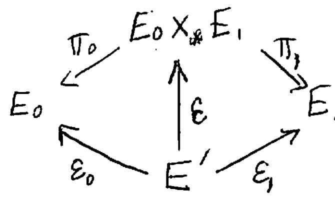
3.4 Theorem. *The above construction $N_0 \times N_1, \Pi_0, \Pi_1$ is a product in* **Net**, *the category of nets.*

Proof. It follows straightforwardly from the definitions that $\Pi_0 = (\pi_0, \rho_0)$ and $\Pi_1 = (\pi_1, \rho_1)$ are morphisms of nets.

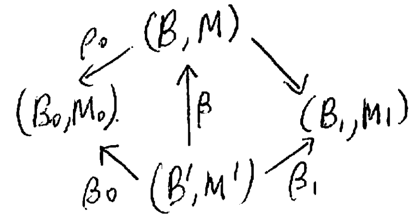
We need that the construction $N_0 \times N_1$ gives an object in **Net**and so that $N_0 \times N_1$ is contact–free. Suppose there is contact at a reachable marking of the product *i.e.* there is a reachable marking C, a condition b and an event e of $N_0 \times N_1$ such that ${}^\bullet e \subseteq C$ and $b \in (e^\bullet \setminus {}^\bullet e) \cap C$. Either $b = (0, b_0)$ for some $b_0 \in B_0$ or $b = (1, b_1)$ for some $b_1 \in B_1$. Without loss of generality suppose $b = (0, b_0)$ for some $b_0 \in B_0$. Then $\pi_0(e) = e_0$ for some $e_0 \in E_0$. Thus ${}^\bullet e_0 \subseteq \pi_0 C$ and $b_0 \in (e_0{}^\bullet \setminus {}^\bullet e_0) \cap \pi_0 C$. However as N_0 is contact–free, by theorem 2.5, $\pi_0 C$ is a reachable marking of N_0 at which e_0 has concession—a contradiction. Therefore $N_0 \times N_1$ is contact–free.

Now suppose there are morphisms $\Phi_0 = (\epsilon_0, \beta_0) : N' \to N_0$ and $\Phi_1 = (\epsilon_1, \beta_1) : N' \to N_1$ from a contact–free net $N' = (B', E', F', M')$.

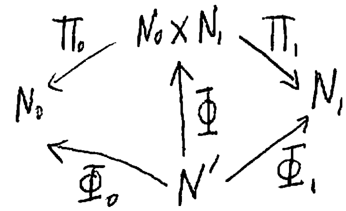
As $\epsilon_0 : E' \to E_0$ and $\epsilon_1 : E' \to E_1$ are morphisms in $\mathbf{Set}_*$ and E, π_0, π_1 is a product in $\mathbf{Set}_*$ there is a unique partial function $\epsilon : E' \to E$ such that the following diagram commutes in $\mathbf{Set}_*$:



Similarly, as $\beta_0 : (B', M') \to (B_0, M_0)$ and $\beta_1 : (B', M') \to (B_1, M_1)$ are morphisms of marked sets and (B, M), ρ_0, ρ_1 is a product in the category of marked sets—by lemma 3.2—there is a unique relation β so that the following diagram commutes in the category of marked sets:



Define $\Phi = (\epsilon, \beta)$. Clearly provided Φ is a morphism of nets $\Phi : N' \to N$ it will be the unique morphism of nets such that the following diagram commutes:



So finally we check that $\Phi : N' \to N$ is indeed a morphism of nets. Because of the properties of marked sets Φ behaves well on initial markings.

Suppose $e'\epsilon e$ & eFb for $e' \in E'$, $e \in E$ and $b \in B$. Either $b\rho_0 b_0$ for some $b_0 \in B_0$ or $b\rho_1 b_1$ for some $b \in B_1$. Without loss of generality assume $b\rho_0 b_0$ for some $b_0 \in B_0$. Then $e\pi_0 e_0$ and $e_0 F_0 b_0$ for some $e_0 \in E_0$ as Π_0 is a morphism. Because $\epsilon_0 = \pi_0\epsilon$ we get $e\epsilon_0 e_0$. As Φ_0 is a morphism, there is some unique b' such that $b'\beta_0 b_0$ and $e'F'b'$. Then because $\beta_0 = \rho_0\beta$, the condition b' is unique so that $b'\beta b$ and $e'F'b'$, as required. The proof that $e'\epsilon e$ & bFe implies there is a unique b' such that $b'\beta b$ and $b'F'e'$ is virtually the same.

Suppose $b'\beta b$ & $b'F'e'$ for $b' \in B'$, $b \in B$ and $e' \in E'$. Without loss of generality assume $b\rho_0 b_0$. By commutativity $b'\beta_0 b_0$. As $b'F'e'$ there is some e_0 such that $e'\epsilon_0 e_0$ & $b_0 F_0 e_0$. But then as Π_0 is a morphism there is an event $e \in E$ such that $e\pi_0 e_0$ and bFe. As ϵ is a partial function making $\epsilon_0 = \pi_0\epsilon$ we must have $e'\epsilon e$ as well as bFe, that which was required. The remaining case is virtually the same.

Thus we conclude that Φ is the unique morphism making the diagram commute. Consequently the above construction really is a product. ∎

Of course the token game tells us how we can view a net as giving rise to a transition system in which the arrows between states are associated with sets of events imagined to occur concurrently. Let us see how the product construction looks from this point of view.

3.5 Theorem. *Let $N_0 \times N_1$, $\Pi_0 = (\pi_0, \rho_0)$ and $\Pi_1 = (\pi_1, \rho_1)$ be a product of nets. Then M is a reachable* marking of $N_0 \times N_1$ and $M \xrightarrow{A} M'$ iff

$\rho_0 M$ is a reachable marking of N_0 and
$\rho_0 M \xrightarrow{\pi_0 A} \rho_0 M'$ and
$\forall e, e' \in A \forall e_0 \in E_0 . e \pi_0 e_0 \;\&\; e' \pi_0 e_0 \Rightarrow e = e'$ and
$\rho_1 M$ is a reachable marking of N_1 and
$\rho_1 M \xrightarrow{\pi_1 A} \rho_1 M'$ and
$\forall e, e' \in A \forall e_1 \in E_1 . e \pi_1 e_1 \;\&\; e' \pi_1 e_1 \Rightarrow e = e'$.

Proof. Omitted. ∎

3.6 Definition. Synchronous product. Let $N_0 = (B_0, E_0, F_0, M_0)$ and $N_1 = (B_1, E_1, F_1, M_1)$ be contact-free nets. Define their *synchronous product* $N_0 \otimes N_1$ to be the restriction $N_0 \times N_1 \lceil (E_0 \times E_1)$ with synchronous projections $\Pi'_0 = (\pi'_0, \rho_0)$ and $\Pi'_1 = (\pi'_1, \rho_1)$ where $\pi'_0(e_0, e_1) = e_0$ and $\pi'_1(e_0, e_1) = e_1$.

3.7 Theorem. *The above construction $N_0 \otimes N_1$, Π'_0, Π'_1 is a product in* $\mathbf{Net}_{syn}$*, the category of nets with synchronous morphisms.*

Proof. Use the previous result that $N_0 \times N_1$, Π_0, Π_1 is the product in **Net** and just check that this time the mediating morphism stays inside the category $\mathbf{Net}_{syn}$. ∎

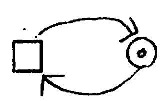
Again we can view this new construction as an operation on transition systems.

3.8 Theorem. *Let $N_0 \otimes N_1$, $\Pi'_0 = (\pi'_0, \rho_0)$ and $\Pi'_1 = (\pi'_1, \rho_1)$ be the synchronous product of nets. Then M is a reachable marking of $N_0 \otimes N_1$ and $M \xrightarrow{A} M'$ iff*

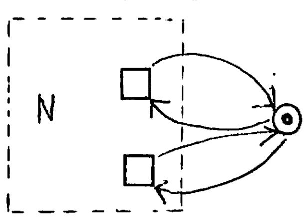
$\rho_0 M$ is a reachable marking of N_0 and
$\rho_0 M \xrightarrow{\pi'_0 A} \rho_0 M'$ and
$\forall e, e' \in A \forall e_0 \in E_0 . e \pi'_0 e_0 \;\&\; e' \pi'_0 e_0 \Rightarrow e = e'$ and
$\rho_1 M$ is a reachable marking of N_1 and
$\rho_1 M \xrightarrow{\pi'_1 A} \rho_1 M'$ and
$\forall e, e' \in A \forall e_1 \in E_1 . e \pi'_1 e_1 \;\&\; e' \pi'_1 e_1 \Rightarrow e = e'$.

Proof. Omitted. ∎

3.9 Example. One can represent a ticking clock as the following simple net, call it Ω:



Given an arbitrary contact-free net N it is a simple matter to serialise, or interleave, its event occurrences; just synchronise them one at a time with the ticks of the clock. This amounts to forming the synchronous product $N \otimes \Omega$ of N with Ω, in a picture:



Of course one would like to check, in a formal way, that this construction really does interleave event occurrences. The techniques for doing this are presented in section 5 on occurrence nets.

Now we give the form of coproducts in **Net** and **Net**$_{syn}$.

3.10 Definition. The coproduct of nets.

Let $N_0 = (B_0, E_0, F_0, M_0)$ and $N_1 = (B_1, E_1, F_1, M_1)$ be contact-free nets.

Let $in_0 : E_0 \to E_0 + E_1$ and $in_1 : E_1 \to E_0 + E_1$ be the injections into the coproduct of sets in **Set**$_*$given in 3.2. Let $\iota_0 : (B_0, M_0) \to (B, M)$ and $\iota_1 : (B_1, M_1) \to (B, M)$ be the injections into the coproduct of marked sets given in 3.3.

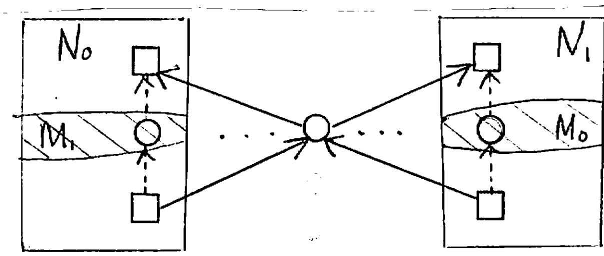
Define the *coproduct* of the nets, $N_0 + N_1$, to be the net (B, E, F, M) where

$$\begin{aligned}
&(B, M) \text{ is the coproduct of marked sets}\\
&E = E_0 + E_1\\
&eFb \Leftrightarrow (\exists e_0 \in E_0, b_0 \in B_0 . e_0 in_0 e \ \&\ b_0 \iota_0 b \ \&\ e_0 F_0 b_0)\\
&\quad \text{or } (\exists e_1 \in E_1, b_1 \in B_1 . e_0 in_1 e \ \&\ b_1 \iota_1 b \ \&\ e_1 F_1 b_1)\\
&bFe \Leftrightarrow (\exists e_0 \in E_0, b_0 \in B_0 . e_0 in_0 e \ \&\ b_0 \iota_0 b \ \&\ b_0 F_0 e_0)\\
&\quad \text{or } (\exists e_1 \in E_1, b_1 \in B_1 . e_0 in_1 e \ \&\ b_1 \iota_1 b \ \&\ b_1 F_1 e_1).
\end{aligned}$$

Define *injection* morphisms of nets:

$$\begin{aligned}
I_0 &= (in_0, \iota_0) : N_0 \to N_0 + N_1\\
I_1 &= (in_1, \iota_1) : N_1 \to N_0 + N_1.
\end{aligned}$$

The coproduct construction can be summarised in a simple picture. The two nets N_0 and N_1 are laid side by side and then a little surgery is performed on their initial markings. For each pair of conditions b_0 in the initial marking of N_0 and b_1 in the initial marking of N_1 a new condition (b_0, b_1) is created and made to have the same pre and post events as b_0 and b_1 together. The conditions in the original initial markings are removed and replaced by a new initial marking consisting of these newly created conditions. Here is the picture:



3.11 Theorem. *The above construction $N_0 + N_1$, I_0, I_1 is a coproduct in the categories* **Net** *and* **Net**$_{syn}$.

Proof. Omitted. ∎

Again the construction translates over to a natural construction on transition systems.

3.12 Theorem. *Let $N_0 + N_1$, $I_0 = (in_0, \iota_0)$ and $I_1 = (in_1, \iota_1)$ be the coproduct of nets. Then*

M is a reachable marking of $N_0 + N_1$ and $M \xrightarrow{A} M'$

iff

$$\begin{aligned}
&\exists M_0, A_0, M'_0 .\\
&\quad M_0 \xrightarrow{A_0} M'_0 \ \&\ A = in_0 A_0 \ \&\ M = \iota_0 M_0 \ \&\ M' = \iota_0 M'_0\\
&\qquad\qquad \text{or}\\
&\exists M_1, A_1, M'_1 .\\
&\quad M_1 \xrightarrow{A_1} M'_1 \ \&\ A = in_1 A_1 \ \&\ M = \iota_1 M_1 \ \&\ M' = \iota_1 M'_1 .
\end{aligned}$$

Proof. Omitted. ∎

Equalisers do not exist for arbitrary nets because they do not exist for sets with relations as morphisms. I do not yet know whether or not coequalisers exist.

4. The subnet ordering, restriction and a "cpo" of nets.

We consider two natural partial orders on nets. One is the relation of one net being a subnet of another. The other is that of net inclusion induced by componentwise inclusion of nets. Both will have least upper bounds of ω-chains but only net inclusion has a least element making it a complete partial order (cpo) for the purposes of giving and solving recursive definitions of nets—of course nets form a class and not a set so solely for this reason, it is not strictly speaking a cpo. Our operations on nets will be continuous with respect to both orders so we shall be able to define nets recursively following now standard lines—see *e.g.* [S]—by taking least fixed-points in the cpo. Recall the definition of subnet.

4.1 Lemma. *Let $N_0 = (B_0, E_0, F_0, M_0)$ and $N_1 = (B_1, E_1, F_1, M_1)$ be nets. Then N_0 is a subnet of N_1 iff $B_0 \subseteq B_1$, $E_0 \subseteq E_1$, $M_0 = M_1$ and*

$$\forall e_0 \in E_0 \forall b \in B_1 . e_0 F_1 b \Leftrightarrow e_0 F_0 b,$$
$$\forall e_0 \in E_0 \forall b \in B_1 . b F_1 e_0 \Leftrightarrow b F_0 e_0 .$$

Proof. Directly from the definition of subnet. ∎

4.2 Definition. Restriction. Let $N = (B, E, F, M)$ be a net. Let $E' \subseteq E$. Define the *restriction* of N to E', written $N\lceil E'$, to be (B, E', F', M) where $F' = F \cap ((B \times E') \cup (E' \times B))$.

In other words the restriction of a net to a subset of events is just the net with all the events not in the subset deleted. Obviously the restriction of a net is a subnet.

4.3 Proposition. *The restriction of a net N, in* **Net**, *to a subset of events E' gives a subnet $N\lceil E'$ which is contact-free and so in* **Net**.

4.4 Example. Obviously the synchronous product of two nets is a restriction of the product of two nets.

Clearly $\leq$ is a partial order on nets. Another obvious partial order is induced by coordinatewise inclusion of nets.

4.5 Definition. Let $N_0 = (B_0, E_0, F_0, M_0)$ and $N_1 = (B_1, E_1, F_1, M_1)$ be nets. Write $N_0 \leq N_1$ iff N_0 is a subnet of N_1. Write $N_0 \subseteq N_1$ iff $B_0 \subseteq B_1$, $E_0 \subseteq E_1$, $F_0 \subseteq F_1$ and $M_0 \subseteq M_1$.

This inclusion order makes a complete partial order of nets, apart from the the fact that nets form a class and not a set. All the operations we have and shall introduce on nets will be continuous with respect to this cpo structure. Unfortunately the subnet order $\leq$, though it does have lubs of ω-chains, does not have a least net so it is not a cpo—this may indicate that my choice of morphism on nets could usefully be made a little more general.

4.6 Proposition. *(i) The null net, $(\emptyset, emptyset, emptyset, emptyset)$ is the $\subseteq$-least net i.e. for all nets N, $(\emptyset,\emptyset,\emptyset,\emptyset) \subseteq N$. Let $N_0 N_1 \cdots N_n \subseteq \cdots$ be an ω-chain of nets of the form $N_n = (B_n, E_n, F_n, M_n)$. Then it has a least upper bound $\bigcup_{n\in\omega} N_n = (\bigcup_{n\in\omega} B_n, \bigcup_{n\in\omega} E_n, \bigcup_{n\in\omega} F_n, \bigcup_{n\in\omega} M_n)$. Similarly if $N_0 \leq N_1 \cdots N_n \leq \cdots$ is an ω-chain of nets it has a least upper bound $\bigcup_{n\in\omega} N_n$.*

4.7 Definition. Say a unary operation operation *op* on nets is $\leq$-($\subseteq$-)*continuous* iff it preserves least upper bounds of ω-chains of nets ordered by $\leq$ ($\subseteq$). If *op* is an n-ary operation on nets, say it is $\leq$-($\subseteq$)*continuous* iff it is continuous in each argument separately.

4.8 Theorem. *The constructions $\times$, $\otimes$ and $+$ and restriction are continuous operations on nets ordered by $\subseteq$ and the subnet ordering $\leq$.*

Proof. Omitted. ∎

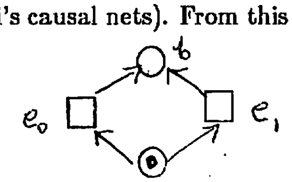
Thus each of the operations $\times$, $\otimes$ and $+$ and restriction can be used to define nets recursively because they are all continuous with respect to the cpo of nets.

5. The semantics of Petri nets.

Nets are rather complex objects with an intricate behaviour. Clearly we would like to know when two nets have essentially the same behaviour. In this section we put forward the view that the behaviour of a net is captured naturally by its unfolding to a net of occurrences, an operation very like that of unfolding a transition system to a tree [W4] or Dana Scott's operation of unravelling a flow diagram to a possibly infinite element in his lattice of flow diagrams [S1]. Naturally we would like the operations we perform on nets to "commute" with the represention of their behaviour.

Here we show how an *occurrence net*, in which conditions and events stand for occurrences, can be associated with a contact-free net. The occurrence net we associate with a contact-free net will be built up essentially by unfolding the net to its occurrences. This unfolding is a canonical representative of the behaviour of the original net. Of course we assume the behaviour of isomorphic nets is the same. Occurrence nets and the operation of unfolding a net to an occurrence net were first introduced in [NPW1, 2 and W].

In general because of the presence of forwards and backwards conflict that part of a net "caused by" or "causing" an event or condition need not be unique. In an occurrence net we wish the elements to represent occurrences (as is the case with Petri's causal nets). From this point of view backwards conflict is undesirable. For instance in



the condition b can be caused to hold in two different ways, either through the occurrence of e_0 or e_1. In occurrence nets we choose to allow only forwards conflict arising through events sharing a common precondition. This explains axioms (i) and (iv).

Because we do not want repeated occurrences represented by an occurrence net we ban nets like by insisting there be no loops in the F^+ relation. This explains half of axiom (iv).

We identify the initial marking with those conditions b for which $^\bullet b = \emptyset$—axiom (ii). Because we imagine the process to have a definite start, to have not gone on forever in the past, we assume that there are no infinitely descending F–chains—axiom (iii).

For occurrence nets there is an especially simple definition of a *concurrency relation* and *conflict relation* which was previously only defined with respect to a marking.

5.1 Definition. An *occurrence net* is a net (B, E, F, M) for which the following restrictions are satisfied:

(i) $\forall b \in B. |^\bullet b| \leq 1$,
(ii) $b \in M \Leftrightarrow {}^\bullet b = \emptyset$,
(iii) F^+ is irreflexive and $\forall x \in B \cup E. \{ z \mid zF^* x \}$ is finite,
(iv) $\#$ is irreflexive where

$$e \#_1 e' \Leftrightarrow_{def} e \in E \ \&\ e' \in E \ \&\ {}^\bullet e \cap {}^\bullet e' \neq \emptyset \text{ and}$$
$$x \# x' \Leftrightarrow_{def} \exists e, e' \in E. e \#_1 e' \ \&\ eF^* x \ \&\ e'F^* x'.$$

Suppose $N = (B, E, F, M)$ is an occurrence net. We call the relation $\#_1$ defined above the *immediate conflict relation* and $\#$ the *conflict relation*. We define the *concurrency relation*, *co*, between pairs $x, y \in B \cup E$ by:

$$x \ co \ y \Leftrightarrow_{def} \neg(xF^+ y \text{ or } yF^+ x \text{ or } x \# y).$$

5.2 Definition. Write **Occ** for the category of occurrence nets with net morphisms. Write $\mathbf{Occ}_{syn}$ for the subcategory of occurrence nets with synchronous morphisms. Write $\mathbf{Occ}_{fol}$ for the subcategory of occurrence nets with foldings as morphisms.

There is a natural idea of depth of an element of an occurrence net, useful to prove properties of occurrence nets by induction.

5.3 Definition. Let $N = (B, E, F, M)$ be an occurrence net. Inductively define the *depth* of an element $x \in B \cup E$ as follows:
For $b \in M$ take $depth(b) = 0$;
For $e \in E$ take $depth(e) = \max\{ depth(b) \mid bFe \} + 1$;
For $b \in B \setminus M$ take $depth(b) = depth(e)$ for that unique e such that eFb.

As expected every condition and event of an occurrence net can occur in a play of the token game of 1.6. We show that the concurrency and conflict relations on occurrence nets agree with the earlier notions. By insisting that events and conditions in an occurrence net correspond to occurrences we do not need to specify at which marking we assume its conditions to hold and its events to have concession.

5.4 Proposition. *Let $N = (B, E, F, M)$ be an occurrence net. Then every event of N has concession at some reachable marking and every condition of N holds at some reachable marking.*

Let e, e' be two events of N. Let b, b' be two conditions of N.

The relations $\#_1 \subseteq E^2$ and $\# \subseteq (B \cup E)^2$ are binary, symmetric, irreflexive relations. The relation of immediate conflict $e \#_1 e'$ holds iff there is a reachable marking of N at which the events e and e' are in conflict.

The relation co is a binary, symmetric, reflexive relation between conditions and events of N. We have $b \ co \ b'$ iff there is a reachable marking of N at which b and b' both hold. We have $e \ co \ e'$ iff there is a reachable marking at which e and e' can occur concurrently.

Let $(\epsilon, \beta) : N_0 \to N_1$ *be a morphism between occurrence nets. Then* $e_0 \epsilon e_1$ & $e'_0 \epsilon e_1 \Rightarrow e_0 = e'_0$ *or* $e_0 \# e'_0$.
and $b_0 \beta b_1$ & $b'_0 \beta b_1 \Rightarrow b_0 = b'_0$ or $b_0 \# b'_0$.

Proof. Omitted. ∎

5.5 Proposition. *An occurrence net* $N = (B, E, F, M)$ *is the lub of its subnets* $N^{(n)}$ *of depth* n *i.e. Define* $N^{(n)} =_{def} (B^{(n)}, E^{(n)}, F^{(n)}, M)$ *where*

$$B^{(n)} = \{ b \in B \mid depth(b) \leq n \}$$
$$E^{(n)} = \{ e \in e \mid depth(e) \leq n \}$$
$$xF^{(n)}y \Leftrightarrow x, y \in B^{(n)} \cup E^{(n)} \;\&\; xFy.$$

Then $N^{(n)} \leq N$ *and* $N = \bigcup_{n \in \omega} N^{(n)}$.

Proof. Left to the reader. ∎

5.6 Proposition. *Let* $N = (B, E, F, M)$ *be a contact-free net. There is a* $\leq$*-least occurrence net* $N_O \doteq (B_O, E_O, F_O, M_O)$ *with a folding* $f = (\epsilon_O, \beta_O) : N_O \to N$ *which satisfies:*

$$B_O = \{ (\emptyset, b) \mid b \in M \} \cup \{ (\{ e_0 \}, b) \mid e_0 \in E_O \;\&\; b \in B \;\&\; \epsilon_O(e_0)Fb \},$$
$$E_O = \{ (S, e) \mid S \subseteq B_O \;\&\; e \in E \;\&\; \beta_O S = {}^\bullet e \;\&\; \forall b_0, b'_0 \in S . b_0 \; co \; b'_0 \},$$
$$xF_O y \Leftrightarrow \exists w, z . y = (w, z) \;\&\; x \in z,$$
$$M_O = \{ (\emptyset, b) \mid b \in M \},$$
and
$$e_0 \epsilon_O e \Leftrightarrow \exists S \subseteq B_O . e_0 = (S, e),$$
$$b_0 \beta_O b \Leftrightarrow b \in M \;\&\; b_0 = (\emptyset, b) \text{ or } \exists e_0 \in E_O . b_0 = (\{ e_0 \}, b).$$

Proof. We define N_O as a lub of subnets, so $N_O = \bigvee_{n \in \omega} N^n_O$ and $f = (\bigcup_{n \in \omega} \epsilon^n, \bigcup_{n \in \omega} \beta^n)$, for an increasing chain of subnets N^n_O and foldings $f^n = (\epsilon^n, \beta^n) : N^n_O \to N$ for $n \in \omega$.

For $n \in \omega$, take the *occurrence net unfolding of* N *to depth* n to be $N^n_O = (B^n_O, E^n_O, F^n_O, M^n_O)$,and the *folding to depth* n to be $f^n = (\epsilon^n, \beta^n) : N^n_O \to N$ where both N^n_O and f^n are defined inductively as follows:

For the base case take .

$$E^0_O = \emptyset,$$
$$B^0_O = \{ \emptyset \} \times M,$$
$$F^0_O = \emptyset,$$
$$M^0_O = \{ \emptyset \} \times M.$$
and
$$\epsilon^0 = \emptyset,$$
$$b_0 \beta^0 b \Leftrightarrow b \in M \;\&\; b_0 = (\emptyset, b).$$

For the $n + 1$st case take

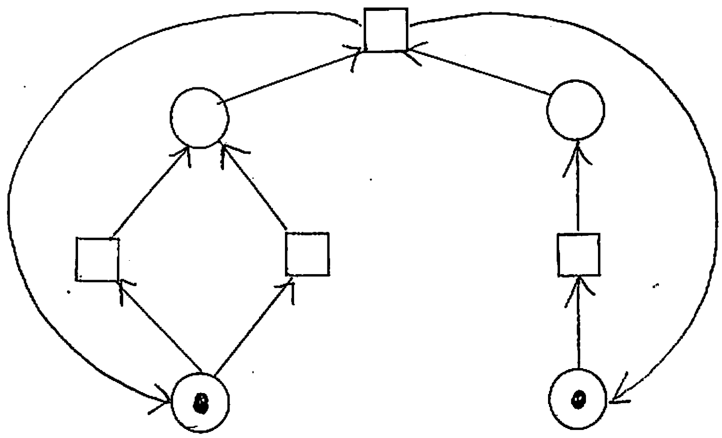
$$
\begin{aligned}
e_0 \in E_O^{n+1} &\Leftrightarrow \exists S \subseteq B_O^n, e \in E.\beta^n = {}^\bullet e \;\&\; (\forall b_0, b_0' \in S.b_0 \; co^n \; b_0') \;\&\; e_0 = (S, e),\\
b_0 \in B_O^{n+1} &\Leftrightarrow b_0 \in B_O^0\\
&\quad \text{or } \exists e_0 \in E_O^{n+1}.\epsilon^{n+1}(e_0) = e \;\&\; eFb \;\&\; b_0 = (\{e_0\}, b),\\
xF_O^{n+1}y &\Leftrightarrow \exists w, z.y = (w, z) \;\&\; x \in z,\\
M_O^{n+1} &= \{\emptyset\} \times M,\\
&\quad \text{where}\\
x\#^n y &\Leftrightarrow \exists e, e' \in E.e \neq e' \;\&\; {}^\bullet e \cap {}^\bullet e' \neq \emptyset \;\&\; eF^{n^*}x \;\&\; e'F^{n^*}y,\\
x \; co^n \; y &\Leftrightarrow \text{neither } xF^{n+}y \text{ nor } yF^{n+}x \text{ nor } x\#^n y,\\
&\quad \text{and}\\
e_0\epsilon^{n+1}e &\Leftrightarrow \exists S \subseteq B_O^n.e_0 = (S, e),\\
b_0\beta^{n+1}b &\Leftrightarrow b_0\beta^0 b\\
&\quad \text{or } \exists e_0 \in E_O^{n+1}.b_0 = (\{e_0\}, b).
\end{aligned}
$$

It is easy to check, by induction, that each N_O^n is an occurrence net, each $f^n : N_O^n \to N$ is a folding and that $N_O^n \leq N_O^{n+1}$ for $n \in \omega$. Thus taking $N_O = (B_O, E_O, F_O, M_O) = \bigcup_{n\in\omega} N_O^n$ and $f = (\bigcup_{n\in\omega} \epsilon^n, \bigcup_{n\in\omega} \beta^n)$ ensures N_O is an occurrence net and that f is a folding. As each event occurrence depends on only a finite set of occurrences of conditions and each condition occurrence depends on only one event occurrence, the sets satisfy the recursive conditions stated above. That the unfolding is the least follows from the construction. ∎

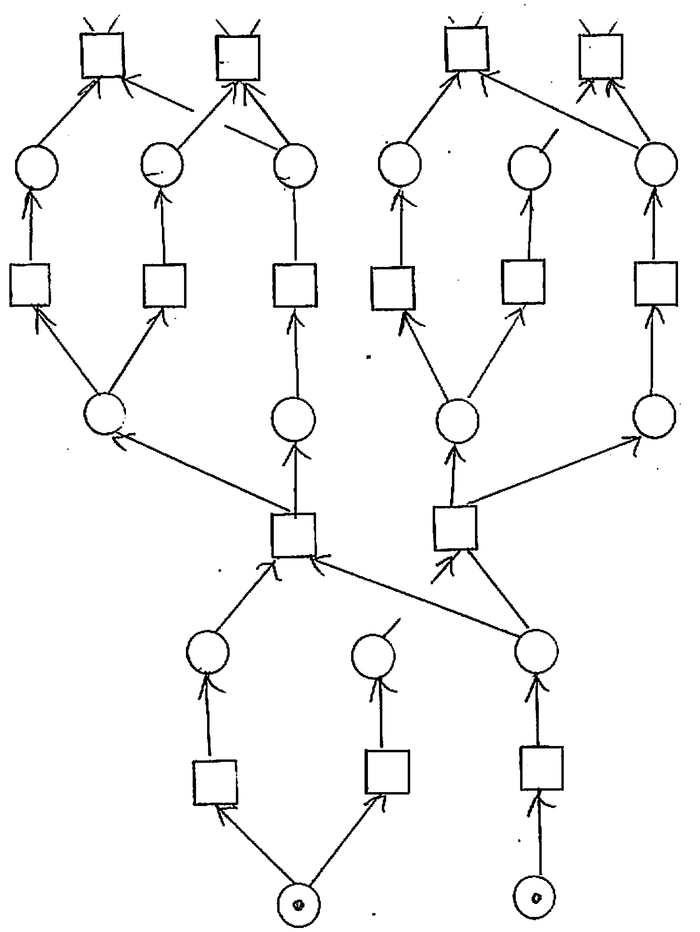
5.7 Definition. Let N be a contact-free net. Define its *occurrence net unfolding*, $\mathcal{U}N$, to be the unique net and the *folding* morphism that folding satisfying the requirements of the proposition above.

5.8 Example. This example illustrates a contact-free net together with its occurrence net unfolding.

5.8 Example. This example illustrates a contact-free net together with its occurrence net unfolding.



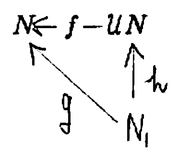
N



UN.

A few minutes thought should convince the reader that the unfolding construction is quite natural, at least provided it is accepted that occurrence nets do capture the essence of net behaviour. Still the construction alone would be quite unwieldy when used as a method for comparing the behaviours of nets. Fortunately there is an abstract characterisation of the occurrence net unfolding of a contact-free net. In a sense it was there all the time, because the unfolding operation acts on nets as the right adjoint to the inclusion functor **Occ** $\rightarrow$ **Net** so it was determined by the categorical set-up. Another way to say the same thing is to say the occurrence net unfolding $\mathcal{U}N$ of a net together with the folding morphism $f : \mathcal{U}N \rightarrow N$ is *cofree* over N. And another way is to say that **Occ** is a *coreflective* subcategory of **Net**. (See [Mac] for further details.) The latter terminology is apt because as we shall see the subcategory **Occ** of occurrence nets, which can be thought of as the meanings of nets, really does reflect the category **Net**. The proof of the cofreeness of the occurrence net unfolding is long. But the theorem enables us to sweep all the unpleasant details of the construction under the carpet; they're there if you want to look but you don't have to, just use the theorem.

5.9 Theorem. *Let N be a contact-free Petri net. Then the occurrence net unfolding $\mathcal{U}N$ and folding f are cofree over N i.e. for any morphism $g : N_1 \rightarrow N$ with N_1 an occurrence net there is a unique morphism $h : N_1 \rightarrow \mathcal{U}N$ such that the following diagram commutes:*



Proof. Assume $N = (B, E, F, M)$ is a contact-free net which has an occurrence net unfolding $\mathcal{U}N = (B_O, E_O, F_O, M_O)$ and folding $f = (\epsilon_0, \beta_0) : \mathcal{U}N \rightarrow N$. Assume N_1 is an occurrence net of the form $N_1 = (B_1, E_1, F_1, M_1)$ and that $g = (\epsilon_1, \beta_1) : N_1 \rightarrow N$ is a morphism.

It is convenient to first establish necessary and sufficient conditions for there to be a morphism making the above diagram commute, and then later to construct a pair of relations which is clearly unique so the conditions are satisfied.

Let $h = (\epsilon, \beta)$ be a pair of relations $\epsilon \subseteq E_1 \times E_O$ and $\beta \subseteq B_1 \times B_O$. We show that h is a morphism, so $h : N_1 \rightarrow \mathcal{U}N$, such that $g = f \circ h$ iff the following two conditions are satisfied:

(i) $e_1 \epsilon e_0 \Leftrightarrow \exists e \in E. e_0 = (\beta^\bullet e_1, e) \ \&\ e_1 \epsilon_1 e,$

(ii) $b_1 \beta b_0 \Leftrightarrow \exists b \in B. b_0 = (\epsilon^\bullet b_1, b) \ \&\ b_1 \beta_1 b.$

Firstly suppose h is a morphism such that $g = f \circ h$. We show that the conditions (i) and (ii) must then be satisfied.

"(i)$\Rightarrow$." Let $e_1 \epsilon e_0$. Then because $g = fh$ we have $e_1 \epsilon_1 e$ for some e and S such that $e_0 = (S, e)$. However because h is a morphism we must have $S = \beta^\bullet e_1$, as required.

"(i)$\Leftarrow$." Suppose $e_0 = (\beta^\bullet e_1, e)$ and $e_1 \epsilon_1 e$ for some $e \in E$. We first show $e_0 \in E_0$. Because h is a morphism $\beta^\bullet e_1$ is a pairwise *co* set of conditions. Also as $g = fh$ and g is a morphism, we have $\beta_0 \beta^\bullet e_1 = \beta_1{}^\bullet e_1 = {}^\bullet e$. Thus $e_0 = (\beta^\bullet e_1, e) \in E_O$ so $e_0 \epsilon_0 e$. Take $b_1 \in {}^\bullet e_1$. As h is a morphism $b_1 \beta b_0$ for some $b_0 F_O e_0$. But then, again as h is a morphism, we obtain some e'_0 such that $e_1 \epsilon e'_0$ and $b_0 F_O e'_0$. By the commutativity $g = fh$ we get $\epsilon_0(e'_0) = \epsilon_1(e_1) = e$. Because h is a morphism ${}^\bullet e'_0 = \beta^\bullet e_1$. Thus $e'_0 = (\beta^\bullet e_1, e) = e_0$, so $e_1 \epsilon e_0$ as required.

"(ii)$\Rightarrow$." Suppose $b_1\beta b_0$. Then by the commutativity, $b_1\beta_1 b$ and $b_0 = (A, b)$ for some $b \in B$ where either $A = \emptyset$ or $A = \{e_0\}$ for some $e_0 \in E_O$. Assume $A = \emptyset$. In this case ${}^\bullet b_0 = \emptyset$. Now if ${}^\bullet b_1 \neq \emptyset$ then as h is a morphism ${}^\bullet b_0 \neq \emptyset$. Thus ${}^\bullet b_1 = \emptyset$ so $b_0 = (\epsilon^\bullet b_1, b)$ as required.

"(ii)$\Leftarrow$." Suppose $b_0 = (\epsilon^\bullet b_1, b)$ and $b_1\beta_1 b$ for some $b \in B$. Either $b_1 \in M_1$ or ${}^\bullet b_1 \neq \emptyset$. Assume $b_1 \in M_1$. Then $b_0 = (\emptyset, b) \in M_O$. As h is a morphism there is some $b'_1 \in M_1$ such that $b'_1\beta b_0$. As g is a morphism $b_1 = b'_1$ so $b_1\beta b_0$ as required. Now assume the other case, that ${}^\bullet b_1 \neq \emptyset$ and let e_1 be the unique event such that $e_1 F_1 b_1$. As g is a morphism $\epsilon_1(e_1) \neq *$ and $\epsilon_1(e_1)Fb$. By the commutativity $\epsilon(e_1) \neq *$. Thus $b_0 = (\{\epsilon(e_1)\}, b)$ so $\epsilon(e_1)F_O b_0$. As h is a morphism there is some b_1 so that $b'_1\beta b_0$ and $e_1 F_1 b'_1$. Therefore by the commutativity $b'_1\beta_1 b$. Thus

$$b'_1\beta_1 b \;\&\; e_1 F_1 b'_1 \quad \text{and}$$
$$b_1\beta_1 b \;\&\; e_1 F_1 b_1.$$

But g is a morphism so $\exists! b_1.(b_1\beta_1 b \;\&\; e_1 F_1 b_1)$, making $b_1 = b'_1$. Therefore $b_1\beta b_0$ as required.

Thus we have shown that if $h : N_1 \to \mathcal{U}N$ is a morphism such that $g = fh$ then the conditions (i) and (ii) are satisfied. Now we show the converse, that the conditions (i) and (ii) ensure that h is a morphism such that $g = fh$.

Suppose the conditions (i) and (ii) are satisfied. First we show h is a morphism $h : N_1 \to \mathcal{U}N$.

Clearly

$$b_1\beta b_0 \;\&\; b_1 \in M_1 \Rightarrow b_0 = (\emptyset, b) \in M_O.$$

Also

$$\begin{aligned} b_1, b'_1 \in M_1 \;\&\; b_1\beta b_0 \;\&\; b'_1\beta b_0 & \\ &\Rightarrow b_1\beta_1 b \;\&\; b'_1\beta_1 b \quad \text{where } b_0 = (\emptyset, b) \\ &\Rightarrow b_1 = b'_1. \end{aligned}$$

Suppose $e_1\epsilon e_0 \;\&\; e_0 F_O b_0$. Then by (i), $e_0 = (\beta^\bullet e_1, e) \;\&\; e_1\epsilon_1 e$ for some $e \in E$. From the definition of the unfolding, $eFb \;\&\; b_0 = (\{e_0\}, b)$ for some $b \in B$. As g is a morphism $\exists! b_1 \in B_1.e_1 F_1 b_1 \;\&\; b_1\beta_1 b$. Therefore b_1 is the unique condition such that $b_1\beta b_0 \;\&\; e_1 F_1 b_1$, as required.

Suppose $b_1\beta b_0 \;\&\; e_1 F_1 b_1$. Then by (ii), $b_0 = (\{\epsilon(e_1)\}, b) \;\&\; b_1\beta_1 b$ for some $b \in B$. As g is a morphism $e_1\epsilon_1 e \;\&\; eFb$ for some e so $\epsilon(e_1) = (\beta^\bullet e_1, e) \neq *$. Take $e_0 = \epsilon(e_1)$. Then $e_1\epsilon e_0 \;\&\; e_0 F_O b_0$, as required.

Suppose $e_1\epsilon e_0 \;\&\; b_0 F_0 e_0$. Then, by (i) $e_0 = (\beta^\bullet e_1, e) \;\&\; e_1\epsilon_1 e$ for some $e \in E$. By the properties of the folding morphism, $b_0 \in \beta^\bullet e_1$. Thus $b_1\beta b_0 \;\&\; b_1 F_1 e_1$ for some $b_1 \in B_1$. We also need the uniqueness of b_1. Let $\beta_O(b_0) = b$. Assume $b'_1\beta b_0 \;\&\; b'_1 F_1 e_1$ for some $b'_1 \in B_1$. Then by (ii) $b'_1\beta_1 b$, which combined with $b'_1 F_1 e_1$ implies $b'_1 = b_1$ as g is a morphism. So, as required b_1 is unique so that $b_1\beta b_0 \;\&\; b_1 F_1 e_1$.

Suppose $b_1\beta b_0 \;\&\; b_1 F_1 e_1$ for $e_1 \in E_1$. Then by (ii), $b_0 = (\epsilon^\bullet b_1, b) \;\&\; b_1\beta_1 b$ for some $b \in B$. As g is a morphism $bFe \;\&\; e_1\epsilon_1 e$ for some $e \in E$. Take $e_0 = (\beta^\bullet e_1, e)$. Then $e_1\epsilon e_0 \;\&\; b_0 F_O e_0$, as required.

We require that $g = f \circ h$ *i.e.* $(\epsilon_1, \beta_1) = (\epsilon_0, \beta_0)\circ(\epsilon, \beta)$. Clearly it follows from (i) and (ii) that $\epsilon_0 \circ \epsilon \subseteq \epsilon_1$ and $\beta_0 \circ \beta \subseteq \beta_1$. It remains to prove the converse inclusions:

Suppose $e_1\epsilon_1 e$. Take $e_0 = (\beta^\bullet e_1, e)$. Then by "(i)$\Leftarrow$" $e_0 \in E_0$ and so $e_0\epsilon_0 e$. Therefore $e_1(\epsilon_0 \circ \epsilon)e$ as needed.

Suppose $b_1\beta_1 b$. Take $b_0 = (\epsilon^\bullet b_1, b)$. Then by "(ii)$\Leftarrow$" $b_0 \in B_0$ and so $b_0\beta_0 b$. Therefore $b_1(\beta_0 \circ \beta)b$, as needed to complete the proof that $g = f \circ h$.

Thus we have completed that part of the proof showing that $h : N_1 \to \mathcal{U}N$ is a morphism and $g = fh$ iff h satisfies (i) and (ii). Of course it remains to show that such a morphism h exists and moreover is unique.

Now we show the existence of such an h. Define $h = (\epsilon, \beta) = (\bigcup_{n\in\omega} \epsilon^n, \bigcup_{n\in\omega} \beta^n)$ where $\epsilon^n \subseteq E_1 \times E_0$ and $\beta^n \subseteq B_1 \times B_0$ are given inductively as follows:

For the basis of the construction take

$$\epsilon^0 = \emptyset$$
$$b_1\beta^0 b_0 \Leftrightarrow \exists b \in B.b_0 = (\emptyset, b) \;\&\; b_1\beta_1 b.$$

For the inductive step in the construction take

$$e_1\epsilon^{n+1}e_0 \Leftrightarrow \exists e \in E.e_0 = (\beta^{n\bullet}e_1, e) \;\&\; e_1\epsilon_1 e$$
$$b_1\beta^{n+1}b_0 \Leftrightarrow \exists b \in B.b_0 = (\epsilon^{n+1\bullet}b_1, b) \;\&\; b_1\beta_1 b.$$

This inductive definition provides an $h = (\epsilon, \beta)$ which satisfies (i) and (ii). (We leave the verification of this to the reader; note the inductive definition has closure ordinal ω because we assume an event has only a finite number of preconditions.) Thus by our previous work $h : N_1 \to \mathcal{U}N$ is a morphism for which $g = fh$.

The ultimate step in the proof is to show that the h defined inductively above is the unique morphism $h : N_1 \to \mathcal{U}N$ for which $g = fh$. Suppose $h' = (\epsilon', \beta')$ were another morphism such that $g = fh'$. Then it too would satisfy (i) and (ii). Consequently by induction on n, $\epsilon \subseteq \epsilon'$ and $\beta \subseteq \beta'$. The converse inclusions are established by induction on the depth of the conditions and events of N_1:

Zero Depth. Clearly if $b_1 \in M_1$ and $b_1\beta' b_0$ then, as β' satisfies (ii), $b_1\beta b_0$ too.

Nonzero Depth. Assume $e_1\epsilon' e_0$ where $depth(e_1) = n + 1$. As ϵ' satisfies (i) we have $e_0 = (\beta'^{\bullet}e_1, e)$ and $e_1\epsilon_1 e$ for some $e \in E$. Each condition in $\beta'^{\bullet}e_1$ has strictly less depth than $n + 1$. Thus $\beta'^{\bullet}e_1 = \beta^{\bullet}e_1$ so as ϵ satisfies (i) we obtain $e_1\epsilon e_0$.

Assume $b_1\beta' b_0$ where $depth(b_1) = n + 1$. As β' satisfies (ii), $b_0 = (\epsilon'^{\bullet}b_1, b)$ and $b_1\beta_1 b$. Here the unique event e_1 such that $e_1 F_1 b_1$ has depth $n + 1$. By the argument just given $e_1\epsilon' e_0 \Leftrightarrow e_1\epsilon e_0$. Because ϵ satisfies (ii) we obtain $b_1\beta b_0$.

This induction shows that $\epsilon' \subseteq \epsilon$ and $\beta' \subseteq \beta$ which together with the previously shown converse inclusions yields $h = h'$. We have established the existence and uniqueness of a morphism $h : N_1 \to \mathcal{U}N$ making $g = fh$.

Finally we conclude that $\mathcal{U}N, f$ is cofree over N, completing the proof of the theorem. ∎

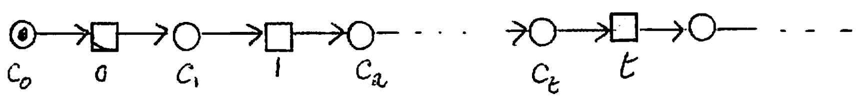
5.10 Corollary. *The unfolding operation on contact-free nets preserves limits; in particular it preserves products. Thus the unfolding of the product (in* **Net***) of two nets $\mathcal{U}(N_0 \times N_1)$ is isomorphic to the product (in* **Occ** *) of the unfoldings $(\mathcal{U}N_0) \times_{occ} (\mathcal{U}N_1)$. To within isomorphism, the product of two occurrence nets $N_0 \times_{occ} N_1$ in* **Occ** *is the net $\mathcal{U}(N_0 \times N_1)$.*

Proof. See [Arb] or [Mac] for the proof that right adjoints preserve limits. To prove the result characterising product in **Occ** note that the unfolding of an occurrence net yields an occurrence net isomorphic to the original. ∎

In the same way the occurrence net unfolding $\mathcal{U}N$ and folding f are also cofree over N in the category **Net**$_{syn}$—just check that the mediating morphism h in theorem 5.6 is synchronous provided g is. It follows that the (synchronous) product in $\mathbf{Occ}_{syn}$ is just the unfolding of the synchronous product in **Net.** Let us look again at example 3.9 and prove our claim that forming the synchronous product of a net with the "clock" Ω serialises or interleaves its event occurrences, *i.e.*no two distinct event occurrences can occur concurrently (be in the *co*–relation).

5.8 Proposition. *Let N be a contact-free net and Ω the "clock" of example 3.9. If e, e' are events of $\mathcal{U}(N \bigotimes \Omega)$ then eF^*e' or $e'F^*e$ or $e\#e'$.*

Proof. Clearly Ω unfolds to the net:



where for simplicity we name the tick occurrences $0, 1, 2, \cdots$ and their preceding conditions $c_0, c_1, c_2 \ldots$. Let $\Pi = (\epsilon, \beta) : \mathcal{U}(N \bigotimes \Omega) \cong (\mathcal{U}N \bigotimes_{occ} \mathcal{U}\Omega) \to \mathcal{U}\Omega$ be the projection morphism in $\mathbf{Occ}_{syn}$, taking an event occurrence synchronised with a tick (occurrence) to that tick. To avoid clutter we shall overload the symbol F allowing it to represent the flow relation in several nets.

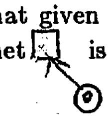
Let e, e' be event occurrences of $N \bigotimes \Omega$, so they are events of $\mathcal{U}(N \bigotimes \Omega)$. As Π is synchronous there are ticks t and t' so that $\epsilon(e) = t$ and $\epsilon(e') = t'$. Without loss of generality assume $t'F^*t$.

Because Π is a morphism and c_tFt there is a condition bFe in $\mathcal{U}(N \bigotimes \Omega)$ such that $b\beta c_t$. Because $\mathcal{U}(N \bigotimes \Omega)$ is an occurrence net, either b is in the initial marking (so c_t is) or there is some unique event so eFb. Thus continuing inductively we obtain a chain $e_0F \ldots FbFe$ where $\epsilon(e_0) = t'$. If $e_0 = e'$ then $e'F^*e$. Otherwise, because Π is a morphism between occurrence nets $e_0\#e'$ so $e\#e'$, as required to prove the proposition. ▮

Now we consider coproducts.

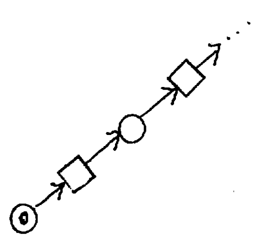
5.9 Proposition. *The categories* **Occ** *and* $\mathbf{Occ}_{syn}$ *have coproducts which coincide with those in* **Net.**

The next example shows that the unfolding need not preserve coproducts however.

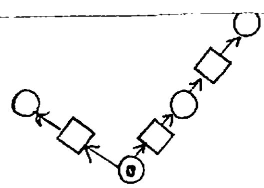
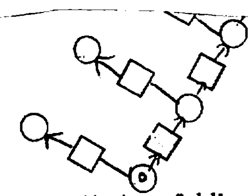
5.8 Example. This example is essentially the same as that given in [W3] for a category of transition systems where unfolding yields a tree. The unfolding of the net  is of course itself.

The unfolding of the net 

is



The coproduct of their unfoldings in **Occ** and the unfolding of their coproduct in **Net**are:

Of course we can restrict to subcategories of nets so that unfolding does preserve coproducts. A subcategory for which this is true is that for which nets satisfy: every condition in the initial marking has no pre–events.

We have proposed one subcategory of nets, **Occ** the category of occurrence nets, as that category which captures the idea of net behaviour. There may be larger subcategories which capture a more refined notion of net behaviour while still capturing a suitably abstract idea of net behaviour. There may well for example be some way of unfolding nets to the subcategory of nets in which conditions can hold once and only once in a play of the token game. (Is there a right adjoint to the inclusion functor associated with this subcategory? If so it should correspond to some form of unfolding.) Certainly there are cruder subclasses of nets which reflect certain aspects of net behaviour while forgetting others, and some of them are used all the time. For example trees can be regarded as special kinds of nets and they are basic to so much work on concurrency in which concurrency is simulated by non–deterministic interleaving. Inside **Net**there is a subcategory naturally equivalent to the category of trees introduced in [W3] and that is inside a slightly larger subcategory of transition systems, where events occur one at a time. And then the category of event structures sits inside **Net**as a subcategory. All these subcategories have right adjoints to the associated inclusion functors, so there are analogues of the unfolding operation taking a net to a canonical representative in each of these classes. Moreover these representatives are natural in themselves; for example the product in the subcategory of trees is closely related to parallel compositions that have been defined on labelled trees by Milner [M].

6. Conclusion.

Petri nets are a very natural model of of concurrent computation. However they have two major drawbacks. For one , they often describe a computation in too much detail; they are not abstract enough. For another, they are generally presented in an unstructured way making it difficult to reason about their behaviour; net descriptions often get too big, out of hand and out of mind. It was for these reasons that Petri introduced morphisms on nets—see [Br] for the definition. It was intended that the resulting category would provide a formal framework for operations on nets. In my view, Petri's choice of definition falls far short of its goal and this is because, in general, his definition fails to respect the dynamic behaviour of nets. This paper gives a new definition of morphism on nets, significantly different from Petri's, which, while probably not the final story, has several points in its favour:

• The new morphisms preserve the dynamic behaviour of nets; there is a forgetful functor from the new category of nets to a category of transition systems where states correspond to markings and transitions to concurrently firing sets of events.

• The new category of nets gives useful categorical constructions, accompanied by abstract characterisations. For example the product is closely related to many parallel compositions that have been defined on nets and the coproduct is an operation which "fuses" nets together at their initial markings. There is a systematic way of labelling events (using the synchronisation algebras of [W1–4]) to give net semantics to parallel programming languages like CCS and CSP.

• The category has a pleasant relation with subcategories based on familiar objects such as trees, transition systems (both the many–events–to–a–transition and the one–event–to–a–transition variety), event structures and occurrence nets (unfolded or unravelled nets). In each case the inclusion functor has a right adjoint; for trees it is an interleaving operation and for occurrence nets it is an unfolding operation which can be viewed as associating with a net a canonical representative of its behaviour. So the category of nets is reflected in the subcategories (and for example the results of [W3] follow).

My hope is that the highly structured view of Petri nets presented here will not only make nets more managable but also be a great help in giving net semantics to concurrent programs and proving their properties. I hope to demonstrate this in the future and provide proof rules to accompany the constructions; there should be proof rules for constructions like product, relating properties in the product to properties in the components, and a form of induction rule associated with the operation of unfolding.

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