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Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras

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Abstract

The notion of bipolar fuzzy subalgebras and bipolar fuzzy ideals of a BCK/BCI-algebra is introduced, and several properties are investigated. Relations between a bipolar fuzzy subalgebra and a bipolar fuzzy ideal are given. A condition for bipolar fuzzy subalgebra to be a bipolar fuzzy ideal is provided, and the characterizations of a bipolar fuzzy ideal are stated. The concept of equivalence relations on the family of all bipolar fuzzy ideals of a BCK/BCI-algebra is considered, and some related properties are discussed.

Key words and phrases: BCK/BCI-algebra, subalgebra, ideal, bipolar fuzzy subalgebra, bipolar fuzzy ideal.

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1. Introduction

In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [1, 5]). In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding

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to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Consider a fuzzy set "*young*" defined on the *age* domain $[0, 100]$ (see Fig. 1 in [3]). Now consider two ages 50 and 95 with membership degree 0. Although both of them do not satisfy the property "*young*", we may say that age 95 is more apart from the property rather than age 50 (see [3]).

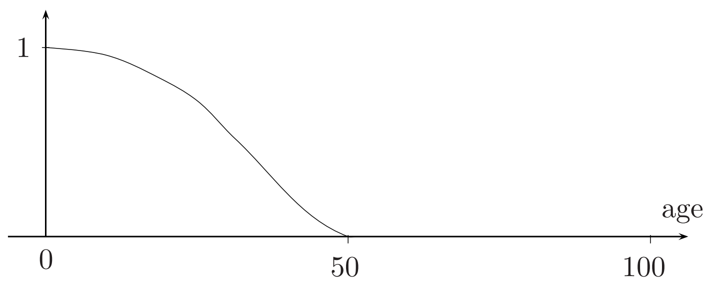

Figure 1. A fuzzy set "*young*"

Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [3] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. He gave two kinds of representations of the notion of bipolar-valued fuzzy sets.

In this paper, we apply the notion of bipolar-valued fuzzy set to BCK/BCI-algebras. We introduce the concept of bipolar fuzzy subalgebras/ideals of a BCK/BCI-algebra, and investigate several properties. We give relations between a bipolar fuzzy subalgebra and a bipolar fuzzy ideal. We provide a condition for a bipolar fuzzy subalgebra to be a bipolar fuzzy ideal. We also give characterizations of a bipolar fuzzy ideal. We consider the concept of equivalence relations on the family of all bipolar fuzzy ideals of a BCK/BCI-algebra, and discuss some related properties.

2. Preliminaries

2.1. **Basic results on BCK/BCI-algebras.** Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BCI-algebra* we mean a system $(X; *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \le y \Rightarrow x * z \le y * z,\ z * y \le z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \le x * y)$

where $x \le y$ if and only if $x * y = 0$. A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. An *ideal* of a BCK/BCI-algebra X is a subset I of X containing 0 such that if $x * y \in I$ and $y \in I$ then $x \in I$. Note that every ideal I of a BCK/BCI-algebra X has the following property:

$$x \le y \text{ and } y \in I \text{ imply } x \in I.$$

A fuzzy set μ in a BCK/BCI-algebra X is said to be a *fuzzy subalgebra* of X if it satisfies:

$$(\forall x, y \in X)\ (\mu(x * y) \ge \min\{\mu(x), \mu(y)\}). \tag{2.1}$$

A fuzzy set μ in a BCK/BCI-algebra X is said to be a *fuzzy ideal* of X if it satisfies:

$$(\forall x \in X)(\mu(0) \ge \mu(x)); \tag{2.2}$$

$$(\forall x, y \in X)(\mu(x) \ge \min\{\mu(x * y),\ \mu(y)\}). \tag{2.3}$$

Note that every fuzzy ideal μ of a BCK/BCI-algebra X is order reversing, i.e., if $x \le y$ then $\mu(x) \ge \mu(y)$.

2.2. **Basic results on bipolar-valued fuzzy set.** Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, etc. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ indicate that elements somewhat satisfy the implicit counter-property (see [3]). Figure 2 shows a bipolar-valued fuzzy set redefined for the fuzzy set "*young*" of Figure 1. The negative membership degrees indicate the satisfaction extent of elements to an implicit counter-property (e.g., old against the property young). This kind of bipolar-valued fuzzy set representation enables the elements with membership degree 0 in traditional fuzzy sets, to be expressed into the elements with membership degree 0 (irrelevant elements) and the elements with negative membership degrees (contrary elements). The age elements 50 and 95, with membership degree 0 in the fuzzy sets of Figure 1, have 0 and a negative membership degree in the bipolar-valued fuzzy set of Figure 2, respectively. Now it is manifested that 50 is an irrelevant age to the property young and 95 is more apart from the property young than 50, i.e., 95 is a contrary age to the property young (see [3]).

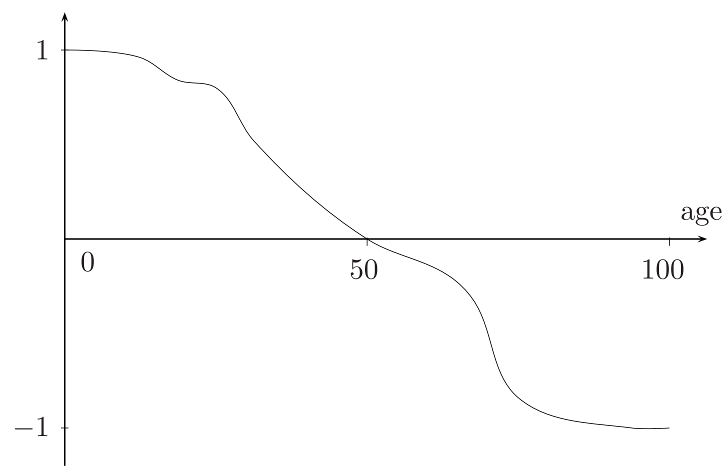

Figure 2. A bipolar fuzzy set "*young*"

In the definition of bipolar-valued fuzzy sets, there are two kinds of representations, so called canonical representation and reduced representation. In this paper, we use the canonical representation of a bipolar-valued fuzzy sets. Let X be the universe of discourse. A *bipolar-valued fuzzy set* Φ in X is an object having the form

$$\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$$

where $\mu_\Phi^P : X \to [0,1]$ and $\mu_\Phi^N : X \to [-1,0]$ are mappings. The positive membership degree $\mu_\Phi^P(x)$ denoted the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$, and the negative membership degree $\mu_\Phi^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$. If $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$. If $\mu_\Phi^P(x) = 0$ and $\mu_\Phi^N(x) \neq 0$, it is the situation that x does not satisfy the property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$ but somewhat satisfies the counter-property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$. It is possible for an element x to be $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [4]). For the sake of simplicity, we shall use the symbol $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ for the bipolar-valued fuzzy set

$\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

3. Bipolar fuzzy subalgebras and bipolar fuzzy ideals

In what follows, let X denotes a BCK/BCI-algebra unless otherwise specified.

Definition 3.1. A bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X is called a *bipolar fuzzy subalgebra* of X if it satisfies:

$$\begin{aligned} &\mu_\Phi^P(x * y) \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}, \\ &\mu_\Phi^N(x * y) \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\} \end{aligned} \tag{3.1}$$

for all $x, y \in X$.

Example 3.2. Consider a BCK-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	a	b	c
μ_Φ^P	0.6	0.6	0.3	0.6
μ_Φ^N	-0.7	-0.7	-0.2	-0.7

Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X.

Example 3.3. Consider a BCI-algebra $X = \{0, a, 1, 2, 3\}$ with the following Cayley table:

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	a	1	2	3
μ_Φ^P	0.8	0.6	0.3	0.3	0.3
μ_Φ^N	-0.7	-0.5	-0.2	-0.2	-0.2

Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X.

Proposition 3.4. *If $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X, then $\mu_\Phi^P(0) \geq \mu_\Phi^P(x)$ and $\mu_\Phi^N(0) \leq \mu_\Phi^N(x)$ for all $x \in X$.*

Proof. Let $x \in X$. Then

$$\mu_\Phi^P(0) = \mu_\Phi^P(x * x) \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(x)\} = \mu_\Phi^P(x)$$

and

$$\mu_\Phi^N(0) = \mu_\Phi^N(x * x) \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(x)\} = \mu_\Phi^N(x).$$

This completes the proof. □

For a bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ and $(s, t) \in [-1, 0] \times [0, 1]$, we define

$$\begin{aligned} \Phi_t^P &:= \{x \in X \mid \mu_\Phi^P(x) \geq t\}, \\ \Phi_s^N &:= \{x \in X \mid \mu_\Phi^N(x) \leq s\} \end{aligned} \tag{3.2}$$

which are called the *positive t-cut* of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ and the *negative s-cut* of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$, respectively. For every $k \in [0, 1]$, the set

$$\Phi_k := \Phi_k^P \cap \Phi_{-k}^N$$

is called the *k-cut* of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$.

Theorem 3.5. *Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy subalgebra of X. Then the following assertions are valid.*

(i) $(\forall t \in [0, 1])$ $(\Phi_t^P \neq \emptyset \Rightarrow \Phi_t^P$ is a subalgebra of $X)$.
(ii) $(\forall s \in [-1, 0])$ $(\Phi_s^N \neq \emptyset \Rightarrow \Phi_s^N$ is a subalgebra of $X)$.

Proof. (i) Let $t \in [0, 1]$ be such that $\Phi_t^P \neq \emptyset$. If $x, y \in \Phi_t^P$, then $\mu_\Phi^P(x) \geq t$ and $\mu_\Phi^P(y) \geq t$. It follows that

$$\mu_\Phi^P(x * y) \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\} \geq t$$

so that $x * y \in \Phi_t^P$. Therefore Φ_t^P is a subalgebra of X. Now let $s \in [-1, 0]$ be such that $\Phi_s^N \neq \emptyset$. If $x, y \in \Phi_s^N$, then

$$\mu_\Phi^N(x * y) \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\} \leq s$$

and so $x * y \in \Phi_s^N$. Hence Φ_s^N is a subalgebra of X. □

Corollary 3.6. *If $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X, then the sets $\Phi^P_{\mu_\Phi^P(0)}$ and $\Phi^N_{\mu_\Phi^N(0)}$ are subalgebras of X.*

Proof. Straightforward. □

Definition 3.7. A bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X is called a *bipolar fuzzy ideal* of X if it satisfies:

$$\mu_\Phi^P(0) \geq \mu_\Phi^P(x) \;\&\; \mu_\Phi^N(0) \leq \mu_\Phi^N(x), \tag{3.3}$$

$$\begin{aligned} &\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(x*y), \mu_\Phi^P(y)\}, \\ &\mu_\Phi^N(x) \leq \max\{\mu_\Phi^N(x*y), \mu_\Phi^N(y)\} \end{aligned} \tag{3.4}$$

for all $x, y \in X$.

Example 3.8. Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	c	d	a	0

Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	a	b	c	d
μ_Φ^P	0.7	0.2	0.7	0.2	0.2
μ_Φ^N	-0.8	-0.7	-0.8	-0.7	-0.7

Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X.

Proposition 3.9. *Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy ideal of X. If the inequality $x * y \leq z$ holds in X, then*

$$\begin{aligned} &\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(y), \mu_\Phi^P(z)\}, \\ &\mu_\Phi^N(x) \leq \max\{\mu_\Phi^N(y), \mu_\Phi^N(z)\}. \end{aligned} \tag{3.5}$$

Proof. Let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and so

$$\begin{aligned} \mu_\Phi^P(x) &\geq \min\{\mu_\Phi^P(x*y), \mu_\Phi^P(y)\} \\ &\geq \min\{\min\{\mu_\Phi^P((x*y)*z), \mu_\Phi^P(z)\}, \mu_\Phi^P(y)\} \\ &= \min\{\min\{\mu_\Phi^P(0), \mu_\Phi^P(z)\}, \mu_\Phi^P(y)\} \\ &= \min\{\mu_\Phi^P(y), \mu_\Phi^P(z)\} \end{aligned}$$

and

$$\begin{aligned} \mu_\Phi^N(x) &\leq \max\{\mu_\Phi^N(x*y), \mu_\Phi^N(y)\} \\ &\leq \max\{\max\{\mu_\Phi^N((x*y)*z), \mu_\Phi^N(z)\}, \mu_\Phi^N(y)\} \\ &= \max\{\max\{\mu_\Phi^N(0), \mu_\Phi^N(z)\}, \mu_\Phi^N(y)\} \\ &= \max\{\mu_\Phi^N(y), \mu_\Phi^N(z)\}. \end{aligned}$$

This completes the proof. □

Proposition 3.10. *Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy ideal of X. If the inequality $x \leq y$ holds in X, then*

$$\mu_\Phi^P(x) \geq \mu_\Phi^P(y) \ \& \ \mu_\Phi^N(x) \leq \mu_\Phi^N(y).$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then

$$\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(x * y), \mu_\Phi^P(y)\} = \min\{\mu_\Phi^P(0), \mu_\Phi^P(y)\} = \mu_\Phi^P(y),$$
$$\mu_\Phi^N(x) \leq \max\{\mu_\Phi^N(x * y), \mu_\Phi^N(y)\} = \max\{\mu_\Phi^N(0), \mu_\Phi^N(y)\} = \mu_\Phi^N(y).$$

This completes the proof. □

Theorem 3.11. *In a BCK-algebra X, every bipolar fuzzy ideal of X is a bipolar fuzzy subalgebra of X.*

Proof. Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy ideal of a BCK-algebra X. Since $x * y \leq x$ for all $x, y \in X$, it follows from Proposition 3.10 that

$$\mu_\Phi^P(x * y) \geq \mu_\Phi^P(x) \text{ and } \mu_\Phi^N(x * y) \leq \mu_\Phi^N(x),$$

so from (3.4) that

$$\mu_\Phi^P(x * y) \geq \mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(x * y), \mu_\Phi^P(y)\} \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}$$

and

$$\mu_\Phi^N(x * y) \leq \mu_\Phi^N(x) \leq \max\{\mu_\Phi^N(x * y), \mu_\Phi^N(y)\} \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\}.$$

Hence $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X. □

The converse of Theorem 3.11 is not true in general. For example, the bipolar fuzzy subalgebra $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in Example 3.2 is not a bipolar fuzzy ideal of X since

$$\mu_\Phi^N(b) = -0.2 > -0.7 = \max\{\mu_\Phi^N(b * a), \mu_\Phi^N(a)\}.$$

We give a condition for a bipolar fuzzy subalgebra to be a bipolar fuzzy ideal in a BCK-algebra.

Theorem 3.12. *Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy subalgebra of a BCK-algebra X such that* (3.5) *holds for all $x, y, z \in X$ satisfying the inequality $x * y \leq z$. Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X.*

Proof. Recall from Proposition 3.4 that $\mu_\Phi^P(0) \geq \mu_\Phi^P(x)$ and $\mu_\Phi^N(0) \leq \mu_\Phi^N(x)$ for all $x \in X$. Since $x * (x * y) \leq y$ for all $x, y \in X$, it follows from (3.5) that

$$\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(x * y), \mu_\Phi^P(y)\} \text{ and } \mu_\Phi^N(x) \leq \max\{\mu_\Phi^N(x * y), \mu_\Phi^N(y)\}.$$

Hence $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X. □

Theorem 3.13. *Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X. Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X if and only if it satisfies the following assertions:*

$$\begin{aligned} &(\forall t \in [0,1])(\Phi_t^P \neq \emptyset \Rightarrow \Phi_t^P \text{ is an ideal of } X), \\ &(\forall s \in [-1,0])(\Phi_s^N \neq \emptyset \Rightarrow \Phi_s^N \text{ is an ideal of } X). \end{aligned} \tag{3.6}$$

Proof. Assume that $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X. Let $(s,t) \in [-1,0] \times [0,1]$ be such that $\Phi_t^P \neq \emptyset$ and $\Phi_s^N \neq \emptyset$. Obviously, $0 \in \Phi_t^P \cap \Phi_s^N$. Let $x, y \in X$ be such that $x * y \in \Phi_t^P$ and $y \in \Phi_t^P$, and let $a, b \in X$ be such that $a * b \in \Phi_s^N$ and $b \in \Phi_s^N$. Then $\mu_\Phi^P(x*y) \geq t$, $\mu_\Phi^P(y) \geq t$, $\mu_\Phi^N(a*b) \leq s$ and $\mu_\Phi^N(b) \leq s$. It follows from (3.4) that

$$\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(x*y), \mu_\Phi^P(y)\} \geq t$$

and

$$\mu_\Phi^N(a) \leq \max\{\mu_\Phi^N(a*b), \mu_\Phi^N(b)\} \leq s$$

so that $x \in \Phi_t^P$ and $a \in \Phi_s^N$. Therefore Φ_t^P and Φ_s^N are ideals of X. Conversely, suppose that the condition (3.6) is valid. For any $x \in X$, let $\mu_\Phi^P(x) = t$ and $\mu_\Phi^N(x) = s$. Then $x \in \Phi_t^P \cap \Phi_s^N$, and so Φ_t^P and Φ_s^N are non-empty. Since Φ_t^P and Φ_s^N are ideals of X, $0 \in \Phi_t^P \cap \Phi_s^N$. Hence $\mu_\Phi^P(0) \geq t = \mu_\Phi^P(x)$ and $\mu_\Phi^N(0) \leq s = \mu_\Phi^N(x)$ for all $x \in X$. If there exists $x', y', a', b' \in X$ such that

$$\mu_\Phi^P(x') < \min\{\mu_\Phi^P(x'*y'), \mu_\Phi^P(y')\}$$

and

$$\mu_\Phi^N(a') > \max\{\mu_\Phi^N(a'*b'), \mu_\Phi^N(b')\},$$

then by taking

$$\begin{aligned} t_0 &= \tfrac{1}{2}(\mu_\Phi^P(x') + \min\{\mu_\Phi^P(x'*y'), \mu_\Phi^P(y')\}), \\ s_0 &= \tfrac{1}{2}(\mu_\Phi^N(a') + \max\{\mu_\Phi^N(a'*b'), \mu_\Phi^N(b')\}), \end{aligned} \tag{3.7}$$

we have

$$\begin{aligned} \mu_\Phi^P(x') &< t_0 < \min\{\mu_\Phi^P(x'*y'), \mu_\Phi^P(y')\}, \\ \mu_\Phi^N(a') &> s_0 > \max\{\mu_\Phi^N(a'*b'), \mu_\Phi^N(b')\}. \end{aligned} \tag{3.8}$$

Hence $x' \notin \Phi_{t_0}^P$, $x'*y' \in \Phi_{t_0}^P$, $y' \in \Phi_{t_0}^P$, $a' \notin \Phi_{s_0}^N$, $a'*b' \in \Phi_{s_0}^N$, and $b' \in \Phi_{s_0}^N$. This is a contradiction, and thus $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X. □

Corollary 3.14. *If $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X, then the intersection of a non-empty positive t-cut and a non-empty negative s-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is an ideal of X for all $(s,t) \in [-1,0] \times [0,1]$. In particular, the non-empty k-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is an ideal of X for all $k \in [0,1]$.*

The following example shows that there exists $(s,t) \in [-1,0] \times [0,1]$ such that if $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X, then the union of a non-empty positive t-cut and a non-empty negative s-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is not an ideal of X in general.

Example 3.15. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	a	b	c
μ_Φ^P	0.7	0.6	0.4	0.4
μ_Φ^N	-0.8	-0.3	-0.7	-0.3

Then

$$\Phi_t^P = \begin{cases} \emptyset & \text{if } 0.7 < t, \\ \{0\} & \text{if } 0.6 < t \leq 0.7, \\ \{0, a\} & \text{if } 0.4 < t \leq 0.6, \\ X & \text{if } 0 \leq t \leq 0.4, \end{cases}$$

and

$$\Phi_s^N = \begin{cases} \emptyset & \text{if } -1 \leq s < -0.8, \\ \{0\} & \text{if } -0.8 \leq s < -0.7, \\ \{0, b\} & \text{if } -0.7 \leq s < -0.3, \\ X & \text{if } -0.3 \leq s < 0. \end{cases}$$

It follows from Theorem 3.13 that $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X. But $\Phi_{0.5}^P \cup \Phi_{-0.6}^N = \{0, a\} \cup \{0, b\} = \{0, a, b\}$ is not an ideal of X since $c * a = b \in \{0, a, b\}$, but $c \notin \{0, a, b\}$.

The following example shows that there exists $k \in [0, 1]$ such that if $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X, then the union of a non-empty positive k-cut and a non-empty negative $(-k)$-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is not an ideal of X in general.

Example 3.16. Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	1	a	b	c
μ_Φ^P	0.8	0.6	0.5	0.3	0.3
μ_Φ^N	-0.7	-0.7	-0.2	-0.5	-0.2

Then

$$\Phi_t^P = \begin{cases} \emptyset & \text{if } 0.8 < t, \\ \{0\} & \text{if } 0.6 < t \leq 0.8, \\ \{0,1\} & \text{if } 0.5 < t \leq 0.6, \\ \{0,1,a\} & \text{if } 0.3 < t \leq 0.5, \\ X & \text{if } 0 \leq t \leq 0.3, \end{cases}$$

and

$$\Phi_s^N = \begin{cases} \emptyset & \text{if } -1 \leq s < -0.7, \\ \{0,1\} & \text{if } -0.7 \leq s < -0.5, \\ \{0,1,b\} & \text{if } -0.5 \leq s < -0.2, \\ X & \text{if } -0.2 \leq s < 0. \end{cases}$$

It follows from Theorem 3.13 that $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X. But $\Phi_{0.4}^P \cup \Phi_{-0.4}^N = \{0,1,a\} \cup \{0,1,b\} = \{0,1,a,b\}$ is not an ideal of X since $c * b = a \in \{0,1,a,b\}$, but $c \notin \{0,1,a,b\}$.

We provide a condition for the union of a non-empty positive k-cut and a non-empty negative $(-k)$-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ to be an ideal of X.

Theorem 3.17. *If $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy ideal of X such that*

$$(\forall x \in X)\,(\mu_\Phi^P(x) + \mu_\Phi^N(x) \geq 0), \tag{3.9}$$

then the union of a non-empty positive k-cut and a non-empty negative $(-k)$-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is an ideal of X for all $k \in [0,1]$.

Proof. Let $k \in [0,1]$. Since $\Phi_k^P \neq \emptyset$ and $\Phi_{-k}^N \neq \emptyset$, they are ideals of X by Theorem 3.13. Hence $0 \in \Phi_k^P \cup \Phi_{-k}^N$. Let $x, y \in X$ be such that $x * y \in \Phi_k^P \cup \Phi_{-k}^N$ and $y \in \Phi_k^P \cup \Phi_{-k}^N$. We can consider the following four cases:

(i) $x * y \in \Phi_k^P$ and $y \in \Phi_k^P$.
(ii) $x * y \in \Phi_k^P$ and $y \in \Phi_{-k}^N$.
(iii) $x * y \in \Phi_{-k}^N$ and $y \in \Phi_k^P$.
(iv) $x * y \in \Phi_{-k}^N$ and $y \in \Phi_{-k}^N$.

Case (i) implies that $x \in \Phi_k^P \subseteq \Phi_k^P \cup \Phi_{-k}^N$. Case (iv) implies that $x \in \Phi_{-k}^N \subseteq \Phi_k^P \cup \Phi_{-k}^N$. For the case (ii), we have $\mu_\Phi^P(x * y) \geq k$ and $\mu_\Phi^N(y) \leq -k$. It follows from (3.4) and (3.9) that

$$\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(x * y), \mu_\Phi^P(y)\} \geq \min\{\mu_\Phi^P(x * y), -\mu_\Phi^N(y)\} \geq k,$$

so that $x \in \Phi_k^P \subseteq \Phi_k^P \cup \Phi_{-k}^N$. Case (iii) implies that $\mu_\Phi^N(x * y) \leq -k$ and $\mu_\Phi^P(y) \geq k$. It follows from (3.4) and (3.9) that

$$\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(x * y), \mu_\Phi^P(y)\} \geq \min\{-\mu_\Phi^N(x * y), \mu_\Phi^P(y)\} \geq k,$$

so that $x \in \Phi_k^P \subseteq \Phi_k^P \cup \Phi_{-k}^N$. Hence $\Phi_k^P \cup \Phi_{-k}^N$ is an ideal of X. □

Let Λ^P and Λ^N be non-empty subsets of $[0,1]$ and $[-1,0]$, respectively.

Theorem 3.18. *Let $\{I_k \mid k \in \Lambda^P \cup \Lambda^N\}$ be a finite collection of ideals of X such that*

(1) $X = (\cup\{I_t \mid t \in \Lambda^P\}) \bigcup (\cup\{I_s \mid s \in \Lambda^N\})$,
(2) $(\forall m, n \in \Lambda^P \cup \Lambda^N)\ (m > n \iff I_m \subset I_n)$.

Then a bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X defined by

$$\begin{aligned} \mu_\Phi^P(x) &= \sup\{t \in \Lambda^P \mid x \in I_t\}, \\ \mu_\Phi^N(x) &= \inf\{s \in \Lambda^N \mid x \in I_s\} \end{aligned} \tag{3.10}$$

for all $x \in X$ is a bipolar fuzzy ideal of X.

Proof. Let $(s,t) \in [-1,0] \times [0,1]$ be such that Φ_t^P and Φ_s^N are non-empty. We claim that Φ_t^P and Φ_s^N are ideals of X. We consider the following two cases:

$$\text{(i)}\ t = \sup\{r \in \Lambda^P \mid r < t\} \ \text{ and }\ \text{(ii)}\ t \neq \sup\{r \in \Lambda^P \mid r < t\}.$$

First case implies that

$$x \in \Phi_t^P \iff x \in I_r \text{ for all } r < t \iff x \in \cap\{I_r \mid r < t\}, \tag{3.11}$$

so that $\Phi_t^P = \cap\{I_r \mid r < t\}$, which is an ideal of X. For the second case, we claim that $\Phi_t^P = \cup\{I_r \mid r \geq t\}$. If $x \in \cup\{I_r \mid r \geq t\}$, then $x \in I_r$ for some $r \geq t$. It follows that $\mu_\Phi^P(x) \geq r \geq t$, so that $x \in \Phi_t^P$. If $x \notin \cup\{I_r \mid r \geq t\}$, then $x \notin I_r$ for all $r \geq t$. Since $t \neq \sup\{r \in \Lambda^P \mid r < t\}$, there exists $\varepsilon > 0$ such that $(t-\varepsilon, t) \cap \Lambda^P = \emptyset$. Hence $x \notin I_r$ for all $r > t-\varepsilon$, which means that if $x \in I_r$ then $r \leq t - \varepsilon$. Thus $\mu_\Phi^P(x) \leq t - \varepsilon < t$, and so $x \notin \Phi_t^P$. Therefore $\Phi_t^P = \cup\{I_r \mid r \geq t\}$ which is an ideal of X since $\{I_k\}$ forms a chain. Next we show that Φ_s^N is an ideal of X. We also consider the following two cases:

$$\text{(iii)}\ s = \inf\{q \in \Lambda^N \mid s < q\} \ \text{ and }\ \text{(iv)}\ s \neq \inf\{q \in \Lambda^N \mid s < q\}.$$

For the case (iii), we get

$$x \in \Phi_s^N \iff x \in I_q \text{ for all } q > s \iff x \in \cap\{I_q \mid s < q\}, \tag{3.12}$$

and so $\Phi_s^N = \cap\{I_q \mid s < q\}$, which is an ideal of X. For the case (iv), we prove that $\Phi_s^N = \cup\{I_q \mid q \leq s\}$. If $x \in \cup\{I_q \mid q \leq s\}$, then $x \in I_q$ for some $q \leq s$. It follows that $\mu_\Phi^N(x) \leq q \leq s$, so that $x \in \Phi_s^N$. Hence $\cup\{I_q \mid q \leq s\} \subseteq \Phi_s^N$. Conversely, if $x \notin \cup\{I_q \mid q \leq s\}$, then $x \notin I_q$ for all $q \leq s$. Since $s \neq \inf\{q \in \Lambda^N \mid s < q\}$, there exists $\varepsilon > 0$ such that $(s, s+\varepsilon) \cap \Lambda^N = \emptyset$, which implies that $x \notin I_q$ for all $q < s+\varepsilon$. This says that if $x \in I_q$ then $q \geq s + \varepsilon$. Thus $\mu_\Phi^N(x) \geq s + \varepsilon > s$, and so $x \notin \Phi_s^N$. Therefore $\Phi_s^N \subseteq \cup\{I_q \mid q \leq s\}$, and consequently $\Phi_s^N = \cup\{I_q \mid q \leq s\}$ which is an ideal of X. This completes the proof. □

Let $BFI(X)$ be the collection of all bipolar fuzzy ideals of X and let $(s,t) \in [-1,0] \times [0,1]$. Define binary relations P^t and N^s on $BFI(X)$ as follows:

$$\begin{aligned} (\Phi,\Psi) \in P^t &\iff \Phi_t^P = \Psi_t^P, \\ (\Phi,\Psi) \in N^s &\iff \Phi_s^N = \Psi_s^N \end{aligned} \tag{3.13}$$

respectively, for all $\Phi = (X;\mu_\Phi^P,\mu_\Phi^N), \Psi = (X;\mu_\Psi^P,\mu_\Psi^N) \in BFI(X)$. Clearly P^t and N^s are equivalence relations on $BFI(X)$. For any $\Phi = (X;\mu_\Phi^P,\mu_\Phi^N) \in BFI(X)$, let $[\Phi]_{P^t}$ (respectively, $[\Phi]_{N^s}$) be the equivalence class of $\Phi = (X;\mu_\Phi^P,\mu_\Phi^N)$ modular P^t (respectively, N^s). Denote by $BFI(X)/P^t$ (respectively, $BFI(X)/N^s$) the system of all equivalence classes modular P^t (respectively, N^s), i.e.,

$$\begin{aligned} BFI(X)/P^t &= \{[\Phi]_{P^t} \mid \Phi = (X;\mu_\Phi^P,\mu_\Phi^N) \in BFI(X)\}, \\ BFI(X)/N^s &= \{[\Phi]_{N^s} \mid \Phi = (X;\mu_\Phi^P,\mu_\Phi^N) \in BFI(X)\}. \end{aligned} \tag{3.14}$$

Now let $Id(X)$ denote the family of all ideals of X. Define maps

$$f_t : BFI(X) \to Id(X) \cup \{\emptyset\},\ \Phi \mapsto \Phi_t^P$$

and

$$g_s : BFI(X) \to Id(X) \cup \{\emptyset\},\ \Phi \mapsto \Phi_s^N$$

for all $\Phi = (X;\mu_\Phi^P,\mu_\Phi^N) \in BFI(X)$. Then f_t and g_s are clearly well defined.

Theorem 3.19. *For any $(s,t) \in [-1,0) \times (0,1]$, the maps f_t and g_s are surjective.*

Proof. Obviously, a bipolar fuzzy set $\mathbf{0} := (X;\mu_{\mathbf{0}}^P,\mu_{\mathbf{0}}^N)$ is a bipolar fuzzy ideal of X where $\mu_{\mathbf{0}}^P(x) = 0 = \mu_{\mathbf{0}}^N(x)$ for all $x \in X$. We have

$$f_t(\mathbf{0}) = \mathbf{0}_t^P = \{x \in X \mid \mu_{\mathbf{0}}^P(x) \geq t\} = \emptyset$$

and

$$g_s(\mathbf{0}) = \mathbf{0}_s^N = \{x \in X \mid \mu_{\mathbf{0}}^N(x) \leq s\} = \emptyset.$$

For any non-empty G in $Id(X)$, consider a bipolar fuzzy set $\Phi(G) = (X;\ \mu_{\Phi(G)}^P,\ \mu_{\Phi(G)}^N)$ in X where

$$\mu_{\Phi(G)}^P : X \to [0,1], \quad x \mapsto \begin{cases} 1 & \text{if } x \in G, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mu_{\Phi(G)}^N : X \to [-1,0], \quad x \mapsto \begin{cases} -1 & \text{if } x \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Phi(G) = (X;\ \mu_{\Phi(G)}^P,\ \mu_{\Phi(G)}^N)$ is a bipolar fuzzy ideal of X. Now we get

$$\begin{aligned} f_t(\Phi(G)) &= \Phi(G)_t^P = \{x \in X \mid \mu_{\Phi(G)}^P(x) \geq t\} \\ &= \{x \in X \mid \mu_{\Phi(G)}^P(x) = 1\} = G \end{aligned}$$

and

$$\begin{aligned} g_s(\Phi(G)) &= \Phi(G)_s^N = \{x \in X \mid \mu_{\Phi(G)}^N(x) \leq s\} \\ &= \{x \in X \mid \mu_{\Phi(G)}^N(x) = -1\} = G. \end{aligned}$$

Hence f_t and g_s are surjective. □

Theorem 3.20. *The quotient sets $BFI(X)/P^t$ and $BFI(X)/N^s$ are equipotent to $Id(X) \cup \{\emptyset\}$ for every $(s,t) \in [-1,0) \times (0,1]$.*

Proof. For $(s,t) \in [-1,0) \times (0,1]$, let

$$f_t^* : BFI(X)/P^t \to Id(X) \cup \{\emptyset\}$$

and

$$g_s^* : BFI(X)/N^s \to Id(X) \cup \{\emptyset\}$$

be defined by

$$f_t^*([\Phi]_{P^t}) = f_t(\Phi) \text{ and } g_s^*([\Phi]_{N^s}) = g_s(\Phi)$$

for all $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N) \in BFI(X)$, respectively. For every $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ and $\Psi = (X; \mu_\Psi^P, \mu_\Psi^N)$ in $BFI(X)$, if $\Phi_t^P = \Psi_t^P$ and $\Phi_s^N = \Psi_s^N$ then $(\Phi, \Psi) \in P^t$ and $(\Phi, \Psi) \in N^s$, and hence $[\Phi]_{P^t} = [\Psi]_{P^t}$ and $[\Phi]_{N^s} = [\Psi]_{N^s}$. Therefore f_t^* and g_s^* are injective. For any $G(\neq \emptyset) \in Id(X)$, consider the bipolar fuzzy ideal $\Phi(G) = (X; \mu_{\Phi(G)}^P, \mu_{\Phi(G)}^N)$ which is given in the proof of Theorem 3.19. Then

$$f_t^*([\Phi(G)]_{P^t}) = f_t(\Phi(G)) = \Phi(G)_t^P = G$$

and

$$g_s^*([\Phi(G)]_{N^s}) = g_s(\Phi(G)) = \Phi(G)_s^N = G.$$

For the bipolar fuzzy ideal $\mathbf{0} := (X; \mu_{\mathbf{0}}^P, \mu_{\mathbf{0}}^N)$ of X, we have

$$f_t^*([\mathbf{0}]_{P^t}) = f_t(\mathbf{0}) = \mathbf{0}_t^P = \emptyset$$

and

$$g_s^*([\mathbf{0}]_{N^s}) = g_s(\mathbf{0}) = \mathbf{0}_s^N = \emptyset.$$

Thus f_t^* and g_s^* are surjective. This completes the proof. □

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