Computing the Extremal Possible Ranks with Incomplete Preferences*

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ABSTRACT

Various voting rules are based on ranking the candidates by scores induced by aggregating voter preferences. A winner (respectively, unique winner) is a candidate who receives a score not smaller than (respectively, strictly greater than) the remaining candidates. Examples of such rules include the positional scoring rules and the Bucklin, Copeland, and Maximin rules. When voter preferences are known in an incomplete manner as partial orders, a candidate can be a possible/necessary winner based on the possibilities of completing the partial votes. Past research has studied in depth the computational problems of determining the possible and necessary winners and unique winners. These problems are all special cases of reasoning about the range of possible positions of a candidate under different tiebreakers. We investigate the complexity of determining this range, and particularly the extremal positions. Among our results, we establish that finding each of the minimal and maximal positions is NP-hard for each of the above rules, including all positional scoring rules, pure or not. Hence, none of the tractable variants of necessary/possible winner determination remain tractable for extremal position determination. Tractability can be retained when reasoning about the top-k positions for a fixed k. Yet, exceptional is Maximin where it is tractable to decide whether the maximal rank is k for $k = 1$ (necessary winning) but it becomes intractable for all $k > 1$.

KEYWORDS

Social Choice; Elections; Voting Rules; Incomplete Preferences; Partial Profile; Possible Winner; Necessary Winner

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1 INTRODUCTION

A central task in social choice is that of winner determination—how to aggregate voter preferences to decide who wins. Relevant scenarios may be political elections, document rankings in search engines, hiring dynamics in the job market, decision making in multiagent systems, determination of outcomes in sports tournaments, and so on [6]. Different voting rules can be adopted for this task, and the computational social-choice community has investigated the algorithmic aspects of various specific instances of rules. We focus here on rules that are based on ranking the candidates by scores induced by aggregating voter preferences. A prominent example is the family of the *positional scoring rules*: each voter assigns to each candidate a score based on the candidate's position in the voter's ranking, and a winning candidate is one who receives the maximal sum of scores. Famous instantiations include the plurality rule (where a winner is most frequently ranked first), the veto rule (where a winner is least frequently ranked last), their generalizations to t-approval and t-veto, respectively, and the Borda rule (where the score is the position in the reverse order). There are also non-positional voting rules that are based on candidate scoring, such as the Bucklin, Copeland, and Maximin rules.

The seminal work of Konczak and Lang [18] has addressed the situation where voter preferences are expressed or known in just a partial manner. More precisely, a partial voting profile consists of a partial order for each voter, and a completion consists of a linear extension for each of the partial orders. The framework gives rise to the computational problems of determining the *necessary winners* who win in every completion, and the *possible winners* who win in at least one completion. Each of these problems has two variants that correspond to two forms of winning: having a score not smaller than any other candidate (i.e., being a *co-winner*) and a having a score strictly greater than all other candidates (i.e., being the *unique winner*). These computational problems are challenging since, conceptually, they involve reasoning about the entire (exponential-size) space of completions. The complexity of these problems has been thoroughly studied in a series of publications that established the tractability of the necessary winners for positional scoring rules [28], and a full classification of a general class of positional scoring rules (the "pure" scoring rules) into tractable and intractable for the problem of the possible winners [3, 4, 28].

Yet, the outcome of an election often goes beyond just reasoning about the maximal score. For example, the ranking among the other candidates might determine who will be the elected parliament members, the entries of the first page of the search engine, the job candidates to recruit, and the finalists of a sports competition. Studies on *social welfare*, for instance, concern the aggregation of voter preferences into a full ranking of the candidates [5, 26]. In the case of a positional scoring rule, the ranking order is determined by the sum of scores from voters and some tie-breaking mechanism [22]. When voter preferences are partial, a candidate can be ranked in different positions for every completion, and we can then reason about the range of these positions. In fact, the aforementioned computational problems can all be phrased as reasoning about the minimal and maximal ranks under different tiebreakers. A candidate c is a *possible co-winner* if the minimal rank is one

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when the tiebreaker favors c most, a *possible unique winner* if the minimal rank is one when the tiebreaker favors c least, a *necessary co-winner* if the maximal rank is one when the tiebreaker favors c most, and a *necessary unique winner* if the maximal rank is one when the tiebreaker favors c least.

We study the computational problems $\mathsf{Min}\{\theta\}$ and $\mathsf{Max}\{\theta\}$, where θ is one of $<$ and $>$. The input consists of a partial profile, a candidate, a tie-breaking (total) order, and a number k, and the goal is to determine whether $x\theta k$ where x is the minimal rank and the maximal rank, respectively, of the candidate. Our results are summarized in Table 1 and Table 2 for positional scoring rules and for other rules, respectively. (We exclude famous rules that are not naturally expressed as candidate scoring, e.g., Condorcet.)

As Table 1 shows for positional scoring rules, determining the extremal ranks of a candidate is fundamentally harder than the $k = 1$ counterparts (necessary/possible winners). For example, it is known that detecting the possible winners is NP-hard for every pure rule, with the exception of plurality and veto where the problem is solvable in polynomial time [3, 4, 28]. In contrast, we show that determining each of the minimum and maximum ranks is NP-hard for *every* positional scoring rule, pure or not, including plurality and veto. In particular, the tractability of the necessary winners *does not* extend to reasoning about the maximal rank. The same goes for the Bucklin and Maximin rules, as can be seen in Table 2.

We also study the impact of fixing k and consider the problems $\mathsf{Min}\{\theta k\}$ and $\mathsf{Max}\{\theta k\}$ where the goal is to determine whether $x\theta k$ where, again, x is the minimal/maximal rank. As shown in Table 1, we establish a more positive picture in the case of positional scoring rules: tractability for the maximum (assuming that the scores are polynomial in the number of candidates), and tractability of the minimum under plurality and veto. The degree of the polynomials depend on k, and we show that this is necessary (under standard assumptions of parameterized complexity) at least for the case of minimum, where the problem is W[2]-hard for plurality, and for the case of maximum, where the problem is W[1]-hard for every positional scoring rule. Tractability for the maximum is also retained for the non-positional Bucklin rule, as shown in Table 2. Interestingly, Maximin behaves differently: while it is tractable to decide whether the maximal rank is at least k for $k = 1$ (i.e., the necessary-winner problem), it is NP-complete for all $k > 1$.

The study of the range of possible ranks, beyond the very top, is related to the problem of *multi-winner election* that has been studied mostly in the context of *committee selection*. In that respect, our work can be viewed as reasoning about (necessary/possible) membership in the committee that consists of the k highest ranked candidates. Yet, common studies consider richer notions of committee selection that look beyond the individual achievements of candidates. Indeed, various utilities have been studied for qualifying the selected committee, such as maximizing the number of voters with approved candidates [1] and, in that spirit, the Condorcet committees [10, 12], aiming at a proportional representation via frameworks such as Chamberlin and Courant's [9] and Monroe's [23], and the satisfaction of fairness and diversity constraints [7, 8]. Moreover, for some of the famous committee selection rules, determining the elected committee can be intractable even if voter preferences are complete [10, 24, 25, 27], in contrast to rank determination (which is always in polynomial time in the framework we adopt).

The problem of multi-winner determination for incomplete votes has been studied by Lu and Boutilier [19] in a perspective different from pure ranking: find a committee that minimizes the maximum objection (or "regret") over all possible completions.

Due to lack of space, some of the proofs are excluded from the paper and are presented in the full version of this work [16].

2 PRELIMINARIES

We begin with some notation and terminology. We focus on positional scoring rules, and we extend the definitions to other voting rules in Section 4.

Voting Profiles and Positional Scoring. Let $C = \{c_1, \dots, c_m\}$ be the set of *candidates* (or *alternatives*) and let $V = \{v_1, \dots, v_n\}$ be the set of *voters*. A *voting profile* $\mathbf{T} = (T_1, \dots, T_n)$ consists of n linear orders on C, where each T_i represents the ranking of C by v_i.

A *positional scoring rule* r is a series $\{\vec{s}_m\}_{m\in\mathbb{N}^+}$ of m-dimensional score vectors $\vec{s}_m = (\vec{s}_m(1), \dots, \vec{s}_m(m))$ of natural numbers where $\vec{s}_m(1) \geq \cdots \geq \vec{s}_m(m)$ and $\vec{s}_m(1) > \vec{s}_m(m)$. We denote $\vec{s}_m(j)$ by $r(m, j)$. Some examples of positional scoring rules include the *plurality* rule $(1, 0, \dots, 0)$, the t*-approval* rule $(1, \dots, 1, 0, \dots, 0)$ that begins with t ones, the *veto* rule $(1, \dots, 1, 0)$, the t*-veto* rule that ends with t zeros, and the *Borda* rule $(m-1, m-2, \dots, 0)$.

Given a voting profile $\mathbf{T} = (T_1, \dots, T_n)$, the score $s(T_i, c, r)$ that the voter v_i contributes to the candidate c is $r(m, j)$ where j is the position of c in T_i. The score of c in $\mathbf{T}$ is $s(\mathbf{T}, c, r) = \sum_{i=1}^{n} s(T_i, c, r)$ or simply $s(\mathbf{T}, c)$ if r is clear from context. A candidate c is a *winner* (or *co-winner*) if $s(\mathbf{T}, c) \geq s(\mathbf{T}, c')$ for all candidates c', and a *unique winner* if $s(\mathbf{T}, c) > s(\mathbf{T}, c')$ for for all candidates $c' \neq c$.

We make some conventional assumptions about the positional scoring rule r. We assume that $r(m, i)$ is computable in polynomial time in m, and the scores in each $\vec{s}_m$ are co-prime (i.e., their greatest common divisor is one). A positional scoring rule is *pure* if every $\vec{s}_{m+1}$ is obtained from $\vec{s}_m$ by inserting a score at some position.

Partial Profiles. A *partial voting profile* $\mathbf{P} = (P_1, \dots, P_n)$ consists of n partial orders (i.e., reflexive, anti-symmetric and transitive relations) on the set C of candidates, where each P_i represents the incomplete preference of the voter v_i. A *completion* of $\mathbf{P} = (P_1, \dots, P_n)$ is a complete voting profile $\mathbf{T} = (T_1, \dots, T_n)$ where each T_i is a completion (i.e., a linear extension) of the partial order P_i. The computational problems of determining the *necessary winners* and *possible winners* for partial voting preferences were introduced by Konczak and Lang [18].

Given a partial voting profile $\mathbf{P}$, a candidate $c \in C$ is a *necessary winner* if c is a winner in every completion $\mathbf{T}$ of $\mathbf{P}$, and c is a *possible winner* if there exists a completion $\mathbf{T}$ of $\mathbf{P}$ where c is a winner. Similarly, c is a *necessary unique winner* if c is a unique winner in every completion $\mathbf{T}$ of $\mathbf{P}$, and c is a *possible unique winner* if there exists a completion $\mathbf{T}$ of $\mathbf{P}$ where c is a unique winner.

The decision problems associated to a positional scoring rule r are those of determining, given a partial profile $\mathbf{P}$ and a candidate c, whether c is a necessary winner, a necessary unique winner, a possible winner, and a possible unique winner. We denote these problems by NW, NU, PW and PU, respectively. A known classification of the complexity of these problems is the following.

Table 1: Overview of the results for positional scoring rules. $\overline{k}$ stands for $m-k+1$ where m is the number of candidates. Results on parameterized complexity take k as the parameter.

Problem	plurality, veto	pure – {pl., veto}	non-pure	comment
Min{<}	NP-c W[2]-hard for pl.	NP-c	NP-c	NP-c: [Thm. 3.1] W[2]: [Thm. 3.5]
Max{>}	NP-c W[1]-hard	NP-c W[1]-hard	NP-c W[1]-hard	[Thm. 3.2] [Thm. 3.10]
Min{< k}	P	NP-c for strongly pure w/ poly. scores	?	P: [Thm. 3.4] NP-c: [Thm. 3.6]
Max{> k}	P	P for poly. scores	P for poly. scores	[Thm. 3.7]
Min{< $\overline{k}$}	P	P for poly. scores	P for poly. scores	[Thm. 3.12]
Max{> $\overline{k}$}	P	NP-c for strongly pure bounded	?	P: [Cor. 3.13] NP-c: [Thm. 3.14]

Theorem 2.1 (Classification Theorem [3, 4, 28]). *Each of* NW *and* NU *can be solved in polynomial time for every positional scoring rule. Each of* PW *and* PU *is solvable in polynomial time for plurality and veto; for all other pure scoring rules,* PW *and* PU *are NP-complete.*

We aim at generalizing the Classification Theorem to determine the *minimal and maximal ranks*, as we formalize next.

Minimal and Maximal Ranks. The *rank* of a candidate is its position in the list of candidates, sorted by the sum of scores from the voters. However, for a precise definition, we need to resolve potential ties. Formally, let r be a positional scoring rule, C be a set of candidates, $\mathbf{T}$ a voting profile, and τ a *tiebreaker*, which is simply a linear order over C. Let $R_{\mathbf{T}}$ be the linear order on C that sorts the candidates by their scores and then by τ; that is,

$$R_{\mathbf{T}} := \left\{c \succ c' : s(\mathbf{T}, c) > s(\mathbf{T}, c') \vee (s(\mathbf{T}, c) = s(\mathbf{T}, c') \wedge c\,\tau\,c')\right\}.$$

The rank of c is the position of c in $R_{\mathbf{T}}$, and we denote it by $\text{rank}(c \mid \mathbf{T}, \tau)$. If $\mathbf{T}$ is replaced with a partial voting profile $\mathbf{P}$, then we define $\text{ranks}(c \mid \mathbf{P}, \tau)$ as the set of ranks that c gets in the different completions of $\mathbf{P}$:

$$\text{ranks}(c \mid \mathbf{P}, \tau) := \{\text{rank}(c \mid \mathbf{T}, \tau) \mid \mathbf{T} \text{ extends } \mathbf{P}\}$$

The minimal and maximal positions in $\text{ranks}(c \mid \mathbf{P}, \tau)$ are denoted by $\min(c \mid \mathbf{P}, \tau)$ and $\max(c \mid \mathbf{P}, \tau)$, respectively.

Observe the following for a partial profile $\mathbf{P}$ and a candidate c:

- c is a possible winner if and only if $\min(c \mid \mathbf{P}, \tau) = 1$ (or $\min(c \mid \mathbf{P}, \tau) < 2$) for any tiebreaker τ that positions c first.
- c is a possible unique winner if and only if $\min(c \mid \mathbf{P}, \tau) = 1$ for any tiebreaker τ that positions c last.
- c is a necessary winner if and only if $\max(c \mid \mathbf{P}, \tau) = 1$ (or $\max(c \mid \mathbf{P}, \tau) < 2$) for any tiebreaker τ that positions c first.
- c is a necessary unique winner if and only if $\max(c \mid \mathbf{P}, \tau) = 1$ for any tiebreaker τ that positions c last.

To investigate the computational complexity of calculating the minimal and maximal ranks for a scoring rule r, we will consider the decision problems of determining, given $\mathbf{P}$, c, τ and a position k, whether $X(c \mid \mathbf{P}, \tau)\ \theta\ k$ where X is one of min and max and θ is one of $<$ and $>$. We denote these problems by $\text{Min}_r\{\theta\}$ and $\text{Max}_r\{\theta\}$. Moreover, we will omit the rule r when it is clear from the context. For example, $\text{Min}_r\{<\}$ (or just $\text{Min}\{<\}$) is the decision problem of determining whether $\min(c \mid \mathbf{P}, \tau) < k$, and $\text{Max}_r\{>\}$ (or just $\text{Max}\{>\}$) decides whether $\max(c \mid \mathbf{P}, \tau) > k$.

Observe that for every scoring rule r, if we can compute the scores of the candidates within a complete profile in polynomial time, then Min{<} and Max{>} are in NP. Also observe that if Min{<} is solvable in polynomial time, then so is Min{>}. Conversely, if Min{<} is NP-complete then Min{>} is coNP-complete. The same holds for the complexity of Max{>} in comparison to Max{<}. Hence, in the remainder of the paper we will restrict the discussion to Min{<} and Max{>}.

Additional Notation. For a set A and a partition $A_1, \dots, A_t$ of A, $P(A_1, \dots, A_t)$ denotes the partitioned partial order

$$\{a_1 \succ \cdots \succ a_t : \forall i \in [t], a_i \in A_i\}$$

and $O(A_1, \dots, A_t)$ denotes an arbitrary linear order on A that completes $P(A_1, \dots, A_t)$. A linear order $a_1 \succ \cdots \succ a_t$ is also denoted as a vector $(a_1, \dots, a_t)$. The *concatenation* $(a_1, \dots, a_t) \circ (b_1, \dots, b_\ell)$ is $(a_1, \dots, a_t, b_1, \dots, b_\ell)$.

3 POSITIONAL SCORING RULES

In this section, we show that the problems we study are computationally hard for *all* positional scoring rules. We start with the hardness of computing the minimal rank.

Theorem 3.1. *For every positional scoring rule r,* $\text{Min}_r\{<\}$ *is NP-complete.*

Proof. Let $r = \{\vec{s}_m\}_{m>1}$ be a positional scoring rule. We assume, without loss of generality, that $\vec{s}_m(m) = 0$ for every $m > 1$. (Otherwise, we can subtract $\vec{s}_m(m)$ from all the entries in the vector without affecting the ranks in any profile.) The membership of $\text{Min}_r\{<\}$ in NP is straightforward. We show hardness by a reduction from the vertex-cover problem: given an undirected graph G and an integer k, is there a set B of k or fewer vertices such that every edge is incident to at least one vertex in B? This problem is known to be NP-complete even on regular graphs [15], and we will assume that G is indeed regular.

Let $G = (U, E)$ be a regular graph with $U = \{u_1, \dots, u_n\}$. In the reduction, the vertices will correspond to candidates, and the edges

Voter	1	2	$\cdots$	$\ell-1$	$\ell, \ell+1$	$\ell+2$	$\cdots$	$n+2$
$P_e^1(i)$	c_1	c_2	$\cdots$	$c_{\ell-1}$	$\{u, w\}$	c_ℓ	$\cdots$	c_n
$P_e^1(2)$	c_2	c_3	$\cdots$	c_ℓ	$\{u, w\}$	$c_{\ell+1}$	$\cdots$	c_1
$\vdots$								
$P_e^1(n)$	c_n	c_1	$\cdots$	$c_{\ell-2}$	$\{u, w\}$	$c_{\ell-1}$	$\cdots$	c_{n-1}

Figure 1: The voters of the profile $\mathrm{P}_e^1 = (P_e^1(1), \ldots, P_e^1(n))$ for the edge $e = \{u, w\}$ used in the proof of Theorem 3.1. The other candidates are denoted as $C \setminus U = \{c_1, \ldots, c_n\}$.

will be voters that will need to select one of their incident vertices. Hence, the edges jointly select a vertex cover. The question will be whether this vertex cover is small enough. The details follow.

We construct an instance $(C, \mathbf{P}, \tau)$ under r. The candidate set is $C = U \cup \{c^*, d\}$ and the tiebreaker is $\tau = O(\{c^*, d\}, U)$. The voting profile $\mathbf{P} = \mathbf{P}^1 \circ \mathbf{T}^2$ is the concatenation of two parts $\mathbf{P}^1$ and $\mathbf{T}^2$ that we describe next.

Note that $|C| = n+2$. Let $\ell < n+2$ be an index where $\vec{s}_{n+2}(\ell) > \vec{s}_{n+2}(\ell+1) = 0$. We know that such ℓ exists due to the definition of a scoring rule and our assumption that $\vec{s}_m(m) = 0$ for every $m > 1$.

The first part of the profile contains a profile for every edge $\mathbf{P}^1 = \{\mathbf{P}_e^1\}_{e \in E}$. For every edge $e = \{u, w\} \in E$, the profile $\mathbf{P}_e^1 = (P_e^1(1), \ldots, P_e^1(n))$ consists of n voters, as illustrated in Figure 1. For every $i \in [n]$, denote $M_i(C \setminus e) = (c_{i_1}, \ldots, c_{i_n})$ where M_i is the i*th circular vote* as defined by Baumeister, Roos and Jörg [2]:

$$M_i(a_1, \ldots, a_t) := (a_i, a_{i+1}, \ldots, a_t, a_1, a_2, \ldots, a_{i-1}).$$

Then $P_e^1(i) := \left(c_{i_1}, c_{i_2}, \ldots, c_{i_{\ell-1}}, \{u, w\}, c_{i_\ell}, c_{i_{\ell+1}}, \ldots, c_{i_n}\right)$ is the ith voter in $\mathbf{P}_e^1$. This means that in $\mathbf{P}_e^1$, the candidates u and w can only be at positions ℓ and $\ell+1$, and the other candidates are circulating at all other positions. The decision whether to rank u or w at the ℓth position represents the selection of e between its vertices, to construct a vertex cover.

The second part of the profile, $\mathbf{T}^2$, is constructed such that for every completion $\mathbf{T}$ of $\mathbf{P}$ and vertex $u \in U$, the candidate c^* defeats u if and only if for every edge e incident to u, all voters of $\mathbf{P}_e^1$ rank u at the $(\ell+1)$st position. (This means that none of the edges have selected u, and so u is not in the constructed cover.) Formally, recall that G is regular, and let Δ be the common degree of all the vertices of G. The profile $\mathbf{T}^2$ consists of Δ copies of $(T_1^2, \ldots, T_n^2)$, as illustrated in Figure 2. For every $i \in [n]$, denote $M_i(U) = (c_{i_1}, \ldots, c_{i_n})$ and define $T_i^2 = (c_{i_1}, c_{i_2}, \ldots, c_{i_{\ell-1}}, d, c^*, c_{i_\ell}, c_{i_{\ell+1}}, \ldots, c_{i_n})$. This means that d and c^* are always at positions ℓ and $\ell+1$, respectively, and the candidates of U are circulating at all other positions. This completes the construction of $(C, \mathbf{P}, \tau)$.

Voter	1	2	$\cdots$	$\ell-1$	ℓ	$\ell+1$	$\ell+2$	$\cdots$	$n+2$
T_1^2	u_1	u_2	$\cdots$	$u_{\ell-1}$	d	c^*	u_ℓ	$\cdots$	u_n
T_2^2	u_2	u_3	$\cdots$	u_ℓ	d	c^*	$u_{\ell+1}$	$\cdots$	u_1
$\vdots$									
T_n^2	u_n	u_1	$\cdots$	$u_{\ell-2}$	d	c^*	$u_{\ell-1}$	$\cdots$	u_{n-1}

Figure 2: The voters of the profile $(T_1^2, \ldots, T_n^2)$ used in the proof of Theorem 3.1.

We now discuss the correctness of the reduction. Denote by $\alpha(G)$ the minimal size of a vertex cover in G. In the complete proof, by analyzing the scores of the candidates in $\mathbf{P}^1$ and $\mathbf{T}^2$, we show that for every completion $\mathbf{T} = \{\mathbf{T}_e^1\}_{e \in E} \circ \mathbf{T}^2$ of $\mathbf{P}$ it holds that d defeats c^*. For every $u \in U$, the candidate c^* defeats u if and only if $\sum_{e \in E(u)} s(\mathbf{T}_e^1, u) = 0$, where $E(u)$ is the set of edges incident to u. Then, we show that for any k it is the case that $\min(c^* | \mathbf{P}, \tau) \leq k+2$ if and only if $\alpha(G) \leq k$. We conclude the correctness of the reduction and, hence, the NP-completeness of $\mathsf{Min}_r\{<\}$. □

Theorem 3.1 stated the hardness of $\mathsf{Min}_r\{<\}$ for every positional scoring rule r. The next theorem states the hardness of $\mathsf{Max}_r\{>\}$ for every such r.

Theorem 3.2. *For every positional scoring rule r, $\mathsf{Max}_r\{>\}$ is NP-complete.*

Proof. This proof uses parts of the proof of Theorem 3.1. Let $r = \{\vec{s}_m\}_{m>1}$ be a positional scoring rule. We again assume (w.l.o.g.) that $\vec{s}_m(m) = 0$ for every $m > 1$. Membership of $\mathsf{Max}_r\{>\}$ in NP is straightforward. We show hardness by a reduction from the independent-set problem: given an undirected graph G and an integer k, is there any set $B \subseteq U$ of k or more vertices such that no two vertices in B are connected by an edge? Again, we use the NP-complete variant of the problem where G is regular [15].

Let $G = (U, E)$ be a regular graph with $U = \{u_1, \ldots, u_n\}$, and let Δ be the degree of all vertices. As in the proof of Theorem 3.1, we will make every edge (voter) select an incident vertex (candidate). Let $B \subseteq U$ be the vertices who receive Δ votes. Observe that B is necessarily an independent set. The question is whether we can construct a big enough such B. Details follow.

We construct an instance $(C, \mathbf{P}, \tau)$ under r, as follows. The candidates set is $C = U \cup \{c^*, d\}$ and the tiebreaker is $\tau = O(U, \{c^*, d\})$. Note that $|C| = n+2$. The voting profile is the concatenation $\mathbf{P} = \mathbf{P}^1 \circ \mathbf{T}^2 \circ \mathbf{T}^3$ of three parts described next.

Let $\ell < n+2$ be an index such that $\vec{s}_{n+2}(\ell) > \vec{s}_{n+2}(\ell+1) = 0$. The first two parts $\mathbf{P}^1 = \{\mathbf{P}_e^1\}_{e \in E}$ and $\mathbf{T}^2$ are the same as in the proof of Theorem 3.1. Recall that for every edge $e = \{u, w\}$, only the positions of u and w are not determined in the voters of $\mathbf{P}_e^1$. The edge e "selects" the vertex that is put in the ℓth position.

The third part, $\mathbf{T}^3$, is constructed such that for every completion $\mathbf{T}$ of $\mathbf{P}$ and vertex $u \in U$ it holds that u defeats c^* if and only if all voters of $\mathbf{P}_e^1$ rank u at the ℓth position for every edge e incident to u. (This means that all edges incident to u select u.) Formally, $\mathbf{T}^3$ consists of Δn copies of the profile $(T_1^3, \ldots, T_{n+2}^3)$, as illustrated in Figure 3. We start with $T_i^3 = M_i(u_1, \ldots, u_n, d, c^*)$ for the circular votes as defined in the proof of Theorem 3.1, and then perform the following change. There exists some $i \in [n+2]$ such that d and c^* are placed at positions ℓ and $\ell+1$, respectively, in T_i^3. In this voter, switch the positions of d and c^*. This means that in $(T_1^3, \ldots, T_{n+2}^3)$, the candidate c^* is placed at the ℓth position twice, and d is placed at the $(\ell+1)$st position twice.

For the correctness of the reduction, let $\beta(G)$ be the maximal size of an independent set of G. In the complete proof, by inspecting the scores of the candidates in $\mathbf{P}^1, \mathbf{T}^2$ and $\mathbf{T}^3$ we show that in each completion $\mathbf{T} = \{\mathbf{T}_e^1\}_{e \in E} \circ \mathbf{T}^2 \circ \mathbf{T}^3$ of $\mathbf{P}$, the candidate d is defeated by all other candidates. For all $u \in U$, the candidate u defeats c^* if and only if $\sum_{e \in E(u)} s(\mathbf{T}_e^1, u) = \Delta n \cdot \vec{s}_{n+2}(\ell)$ where $E(u)$ is the set of

Voter	1	2	$\cdots$	$\ell-1$	ℓ	$\ell+1$	$\ell+2$	$\cdots$	n	$n+1$	$n+2$
T_1^3	u_1	u_2	$\cdots$	$u_{\ell-1}$	u_ℓ	$u_{\ell+1}$	$u_{\ell+2}$	$\cdots$	u_n	d	c^*
T_2^3	u_2	u_3	$\cdots$	u_ℓ	$u_{\ell+1}$	$u_{\ell+2}$	$u_{\ell+3}$	$\cdots$	d	c^*	u_1
$\vdots$											
$T_{n-\ell+1}^3$	$u_{n-\ell+1}$	$u_{n-\ell+2}$	$\cdots$	u_{n-1}	u_n	d	c^*	$\cdots$	$u_{n-\ell-2}$	$u_{n-\ell-1}$	$u_{n-\ell}$
$T_{n-\ell+2}^3$	$u_{n-\ell+2}$	$u_{n-\ell+3}$	$\cdots$	u_n	c^*	d	u_1	$\cdots$	$u_{n-\ell-1}$	$u_{n-\ell}$	$u_{n-\ell+1}$
$T_{n-\ell+3}^3$	$u_{n-\ell+3}$	$u_{n-\ell+4}$	$\cdots$	d	c^*	u_1	u_2	$\cdots$	$u_{n-\ell}$	$u_{n-\ell+1}$	$u_{n-\ell+2}$
$\vdots$											
T_{n+2}^3	c^*	u_1	$\cdots$	$u_{\ell-2}$	$u_{\ell-1}$	u_ℓ	$u_{\ell+1}$	$\cdots$	u_{n-1}	u_n	d

Figure 3: The voters of the profile $(T_1^3, \ldots, T_{n+2}^3)$ used in the proof of Theorem 3.2.

edges incident to u. Then, we show that for every k it holds that $\max(c^* \mid \mathbf{P}, \tau) \geq k+1$ if and only if $\beta(G) \geq k$. Hence the correctness of the reduction and the NP-completeness of $\mathsf{Max}_r\{>\}$. □

3.1 Comparison to a Bounded Rank

In the previous section, we established that the problems of computing the minimal and maximal ranks are very often intractable. We now investigate the complexity of comparing the minimal and maximal ranks to some fixed rank k. Hence, the input consists of only $\mathbf{P}$, c and τ, but not k. We denote these problems by $\mathsf{Min}_r\{>k\}$ and $\mathsf{Max}_r\{<k\}$. Again, we will omit the rule r when it is clear from the context. For example, $\mathsf{Min}\{<k\}$ is the decision problem of determining whether $\min(c \mid \mathbf{P}, \tau) < k$.

We will show that the complexity picture for $\mathsf{Min}\{<k\}$ and $\mathsf{Max}\{>k\}$ is way more positive, as we generalize the tractability of *almost* all tractable scoring rules for NW and PW. We will also generalize hardness results from PW to $\mathsf{Min}\{<k\}$; interestingly, this generalization turns out to be nontrivial.

In addition to comparing to the fixed k, we will consider the problem of comparing to $\overline{k} := m - k + 1$ where m is, as usual, the number of candidates. Note that the position $\overline{k}$ is the kth rank from the end (bottom). For instance, $\mathsf{Max}\{>\overline{4}\}$ decides whether the candidate can end up in one of the bottom 3 positions.

3.1.1 Complexity of $\mathsf{Min}\{<k\}$. We first show that the positional scoring rules that are tractable for PW, namely plurality and veto, are also tractable for $\mathsf{Min}\{<k\}$. This is proved via a reduction to the problem of *polygamous matching* [17]: given a bipartite graph $G = (U \cup W, E)$ and natural numbers $\alpha_w \leq \beta_w$ for all $w \in W$, determine whether there is a subset of E where each $u \in U$ is incident to exactly one edge and every $w \in W$ is incident to at least α_w edges and at most β_w edges. This problem is known to be solvable in polynomial time.

Lemma 3.3. *The following decision problem can be solved in polynomial time for the plurality and veto rules: given a partial profile* $\mathbf{P}$ *over a set* C *of candidates and numbers* $\gamma_c \leq \delta_c$ *for every candidate* c*, is there a completion* $\mathbf{T}$ *such that* $\gamma_c \leq s(\mathbf{T}, c) \leq \delta_c$ *for every* $c \in C$*?*

To solve $\mathsf{Min}\{<k\}$ given $C, \mathbf{P}, \tau$ and c, we search for a completion where c defeats more than $m - k$ candidates. For this goal we consider every set $D \subseteq C \setminus \{c\}$ of size $m - k + 1$ and search for a completion where c defeats all candidates of D. For that, we iterate over every integer score $0 \leq s \leq n$ and use Lemma 3.3 to test whether there exists a completion $\mathbf{T}$ such that $s(\mathbf{T}, c) \geq s$, and for every $d \in D$ we have $s(\mathbf{T}, d) \leq s$ if $c \,\tau\, d$ or $s(\mathbf{T}, d) < s$ otherwise. We conclude that:

Theorem 3.4. *For every fixed* $k \geq 1$*,* $\mathsf{Min}\{<k\}$ *is solvable in polynomial time under the plurality and veto rules.*

The polynomial degree in Theorem 3.4 depends on k. The following result shows that this is unavoidable, at least for the plurality rule, under conventional assumptions in parameterized complexity.

Theorem 3.5. *Under the plurality rule,* $\mathsf{Min}\{<\}$ *is* W[2]*-hard for the parameter* k.

Proof. We show an FPT reduction from the *dominating set* problem, which is the following: Given an undirected graph $G = (U, E)$ and an integer k, is there a set $D \subseteq U$ of size k such that every vertex is either in D or adjacent to some vertex in D? This problem is known to be W[2]-hard for the parameter k [11].

Given $G = (U, E)$, we construct an instance of $\mathsf{Min}\{<\}$ under plurality where the candidate set is $C = U \cup \{c^*\}$, the tiebreaker is $\tau = O(\{c^*\}, U)$, and the voting profile is $\mathbf{P} = \{P_u\}_{u \in U}$ where P_u is defined as follows. Let $N(u)$ be the set of neighbours of $u \in U$ and $N[u] = N(u) \cup \{u\}$. We define $P_u := P(N[u], U \setminus N[u], \{c^*\})$. Hence, the voter with preferences P_u can vote only for vertices that dominate u. In the complete proof, we show that the graph has a dominating set of size k if and only if $\min(c^* \mid \mathbf{P}, \tau) < k + 2$. □

Beyond Plurality and Veto. The Classification Theorem (Theorem 2.1) states that PW is intractable for every pure scoring rule r other than plurality or veto. While this hardness easily generalizes to $\mathsf{Min}_r\{<k\}$ for $k = 2$, it is not at all clear how to generalize it to any $k > 2$. In particular, we cannot see how to reduce PW to $\mathsf{Min}_r\{<k\}$ while assuming only the purity of the rule. We can, however, show such a reduction under a stronger notion of purity, as long as the scores are bounded by a polynomial in the number m of candidates. In this case, we say that the rule has *polynomial scores*. Note that all of the specific rules mentioned so far (i.e., t-approval, t-veto, Borda and so on) have polynomial scores; an example of a rule that does *not* have polynomial scores is $r(m, j) = 2^{m-j}$. Also note that this assumption is made in addition to our usual assumption that the scores can be computed in polynomial time.

A rule r is *strongly pure* if the score sequence for $m+1$ candidates is obtained from the score sequence for m candidates by inserting a new score to *either the beginning or the end of the sequence.* More

Voter	1	2	$\cdots$	$k-1$	k	$k+1$	$\cdots$	$k+m-1$
$M_{1,1}$	d_1	d_2	$\dots$	d_{k-1}	c_1	c_2	$\cdots$	c_m
$M_{1,2}$	d_1	d_2	$\dots$	d_{k-1}	c_2	c_3	$\cdots$	c_1
$\vdots$								
$M_{1,m}$	d_1	d_2	$\dots$	d_{k-1}	c_m	c_1	$\cdots$	c_{m-1}
$M_{2,1}$	d_2	d_3	$\dots$	d_1	c_1	c_2	$\cdots$	c_m
$M_{2,2}$	d_2	d_3	$\dots$	d_1	c_2	c_3	$\cdots$	c_1
$\vdots$								
$M_{2,m}$	d_2	d_3	$\dots$	d_1	c_m	c_1	$\cdots$	c_{m-1}
$\vdots$								
$M_{k-1,1}$	d_{k-1}	d_1	$\dots$	d_{k-2}	c_1	c_2	$\cdots$	c_m
$M_{k-1,2}$	d_{k-1}	d_1	$\dots$	d_{k-2}	c_2	c_3	$\cdots$	c_1
$\vdots$								
$M_{k-1,m}$	d_{k-1}	d_1	$\dots$	d_{k-2}	c_m	c_1	$\cdots$	c_{m-1}

Figure 4: The voters $M_{i,j}$ used in the proof of Theorem 3.6.

formally, $r = \{\vec{s}_m\}_{m\in\mathbb{N}^+}$ is strongly pure if for all $m \geq 1$, either $\vec{s}_{m+1} = \vec{s}_{m+1}(1)\circ\vec{s}_m$ or $\vec{s}_{m+1} = \vec{s}_m\circ\vec{s}_{m+1}(m+1)$. Note that t-approval, t-veto and Borda are all strongly pure.

Theorem 3.6. *Suppose that a positional scoring rule is strongly pure, has polynomial scores, and is neither plurality nor veto. Then* $\mathsf{Min}\{<k\}$ *is NP-complete for all fixed* $k \geq 2$.

Proof. Let $r = \{\vec{s}_m\}_{m>1}$ be a positional scoring rule that satisfies the conditions of the theorem, and let $k \geq 1$. We show a reduction from PW under r to $\mathsf{Min}_r\{<k+1\}$. The idea is to add $k-1$ new candidates and modify the voters so that the new candidates are always the top $k-1$ candidates, and the score of each of the original candidates is increased by the same amount.

Consider the input $\mathbf{P} = (P_1, \dots, P_n)$ and c for PW over a set C of m candidates. Let $m' = m + k - 1$. Since r is strongly pure, there is an index $t \leq k-1$ such that

$$\vec{s}_{m'} = (\vec{s}_{m'}(1), \dots, \vec{s}_{m'}(t)) \circ \vec{s}_m \circ (\vec{s}_{m'}(t+m+1), \dots, \vec{s}_{m'}(m'))\,.$$

That is, $\vec{s}_{m'}$ is obtained from $\vec{s}_m$ by inserting t values at the top coordinates and $k-1-t$ values at the bottom coordinates. We define C', $\mathbf{P}'$ and τ' as follows.

The candidate set is $C' = C \cup D_1 \cup D_2$ where $D_1 = \{d_1, \dots, d_t\}$ and $D_2 = \{d_{t+1}, \dots, d_{k-1}\}$. Denote $D = D_1 \cup D_2$. The tiebreaker is $\tau' = O(D, \{c\}, C \setminus \{c\})$. The profile $\mathbf{P}'$ is the concatenation $\mathbf{Q} \circ \mathbf{M}$ of two voting profiles. The first is $\mathbf{Q} = (Q_1, \dots, Q_n)$, where Q_i is the same as P_i, except that the candidates of D_1 are placed at the top positions and the candidates of D_2 are placed at the bottom positions. Formally, $Q_i := P_i \cup P(D_1, C, D_2)$. The second, $\mathbf{M}$, consists of $n \cdot \vec{s}_{m'}(1)$ copies of the profile $\left\{M_{i,j}\right\}_{i=1,\dots,k-1,\, j=1,\dots,m}$ where $M_{i,j}$ is $M_i(D) \circ M_j(C)$ for the circular votes $M_i(D)$ and $M_j(C)$ as defined in the proof of Theorem 3.1. (See Figure 4.) Note that by the conditions of the theorem, $n \cdot \vec{s}_{m'}(1)$ is polynomial in n, m.

In the complete proof, we show that the candidates of D always defeat all other candidates, and that c is a possible winner for $\mathbf{P}$ if and only if $\min(c \mid \mathbf{P}', \tau') < k+1$. □

3.1.2 *Complexity of* $\mathsf{Max}\{>k\}$.

The following theorem states that $\mathsf{Max}\{>k\}$ is tractable for every fixed k and every positional scoring rule (pure or not) with polynomial scores.

Theorem 3.7. *For all fixed* $k \geq 1$ *and positional scoring rules* r *with polynomial scores,* $\mathsf{Max}_r\{>k\}$ *is solvable in polynomial time.*

Next, we prove Theorem 3.7. To determine whether $\max(c \mid \mathbf{P}, \tau) > k$, we search for k candidates that defeat c in some completion $\mathbf{T}$, since $\mathrm{rank}(c \mid \mathbf{T}, \tau) > k$ if and only if at least k candidates defeat c in $\mathbf{T}$. For that, we consider each subset $\{c_1, \dots, c_k\} \subseteq C \setminus \{c\}$ and determine whether these $k+1$ candidates can get a combination of scores where $c_1, \dots, c_k$ all defeat c.

More formally, let C be a set of candidates and r a positional scoring rule. For a partial profile $\mathbf{P} = (P_1, \dots, P_n)$ and a sequence $S = (c_1, \dots, c_q)$ of candidates from C, we denote by $\pi(\mathbf{P}, S)$ the set of all possible scores that the candidates in S can obtain jointly in a completion: $\pi(\mathbf{P}, S) := \left\{(s(\mathbf{T}, c_1), \dots, s(\mathbf{T}, c_q)) \mid \mathbf{T} \text{ completes } \mathbf{P}\right\}$. Note that $\pi(\mathbf{P}, S) \subseteq \{0, \dots, n \cdot \vec{s}_m(1)\}^q$. When $\mathbf{P}$ consists of a single voter P, we write $\pi(P, S)$ instead of $\pi(\mathbf{P}, S)$. To show that $\max(c \mid \mathbf{P}, \tau) > k$ we need to find a sequence $S = (c_1, \dots, c_q)$ of distinct candidates where $q = k+1$ and $c_q = c$, and a sequence $(s_1, \dots, s_q) \in \pi(\mathbf{P}, S)$ such that the following holds: for $i = 1, \dots k$ we have $s_i \geq s_q$ if $c_i \,\tau\, c_q$ and $s_i > s_q$ if $c_q \,\tau\, c_i$. The following two lemmas show that if such a sequence exists, then we can find it in polynomial time.

Lemma 3.8. *Let* q *be a fixed natural number and* r *a positional scoring rule. Whether* $(s_1, \dots, s_q) \in \pi(P, S)$ *can be determined in polynomial time, given a partial order* P *over a set of candidates, a sequence* S *of* q *candidates, and scores* $s_1, \dots, s_q$.

Proof. We use a reduction to a scheduling problem where tasks have *execution times*, *release times*, *deadlines*, and *precedence constraints* (i.e., task x should be completed before starting task y). This scheduling problem can be solved in polynomial time [14]. In the reduction, each candidate c is a task with a unit execution time. For every c_i in S, the release time is $\min\{j \in [n] : r(m, j) = s_i\}$, and the deadline is $1 + \max\{j \in [n] : r(m, j) = s_i\}$. For the rest of the candidates, the release time is 1 and the deadline is $m+1$. The precedence constraints are P. It holds that $(s_1, \dots, s_q) \in \pi(P, S)$ if and only if the tasks can be scheduled according to all the requirements. □

From Lemma 3.8 we can conclude that when q is fixed and r has polynomial scores, we can construct $\pi(\mathbf{P}, S)$ in polynomial time, via simple dynamic programming.

Lemma 3.9. *Let* q *be a fixed natural number and* r *a positional scoring rule with polynomial scores. The set* $\pi(\mathbf{P}, S)$ *can be computed in polynomial time, given a partial profile* $\mathbf{P}$ *and a sequence* S *of* q *candidates.*

From Lemma 3.9 we conclude Theorem 3.7. Note that the polynomial degree depends on k. This is unavoidable under conventional assumptions in parameterized complexity—we can modify the proof of Theorem 3.2 to get an FPT reduction from the regular clique problem, which is W[1]-hard [20], to $\mathsf{Max}\{>\}$. Therefore:

Theorem 3.10. *For every positional scoring rule,* $\mathsf{Max}\{>\}$ *is* W[1]*-hard for the parameter* k.

3.1.3 Complexity of $\mathrm{Min}\{<\bar{k}\}$. Recall that $\bar{k} := m - k + 1$. We now show that the problem of $\mathrm{Min}\{<\bar{k}\}$ is tractable for every positional scoring rule with polynomial scores. We find it surprising because $\mathrm{Min}\{<2\}$ is NP-complete for every pure positional scoring rule other than plurality and veto, by a reduction from PW.

Given a positional scoring rule r and functions $a, b : \mathbb{N}_+ \to \mathbb{N}_+$, we define the (a, b)*-reversed* scoring rule, denoted $r^{a,b}$, to be the one given by $r^{a,b}(m, i) = a(m) - b(m) \cdot r(m, m+1-i)$. For example, the $(1, 1)$-reversed rule of plurality is veto, and more generally, the $(1, 1)$-reversed rule of t-approval is t-veto. Also, the $(m, 1)$-reversed rule of Borda is Borda itself. In the following lemma, we use a generalized notation for our decision problems where, instead of fixed k or $\bar{k}$, we use a fixed function f that is is applied to the number m of candidates to produce a number $f(m)$.

Lemma 3.11. *Let r be a positional scoring rule, let $f, a, b : \mathbb{N}_+ \to \mathbb{N}_+$, and let $\bar{f}(m) = m + 1 - f(m)$. There exists a reduction from $\mathrm{Min}_r\{<f\}$ to $\mathrm{Max}_{r^{a,b}}\{>\bar{f}\}$, and from $\mathrm{Max}_r\{>f\}$ to $\mathrm{Min}_{r^{a,b}}\{<\bar{f}\}$.*

Using Lemma 3.11 and Theorem 3.7, we can show that:

Theorem 3.12. *$\mathrm{Min}\{<\bar{k}\}$ is solvable in polynomial time for every fixed $k \geq 1$ and positional scoring rule r with polynomial scores.*

3.1.4 Complexity of $\mathrm{Max}\{>\bar{k}\}$. First, for plurality and veto, by Theorem 3.4 and Lemma 3.11, we can deduce the following:

Corollary 3.13. *For every fixed $k \geq 1$, $\mathrm{Max}\{>\bar{k}\}$ is solvable in polynomial time under the plurality and veto rules.*

A positional scoring rule r is *p-valued*, where p is a positive integer greater than 1, if there exists a positive integer m_0 such that for all $m \geq m_0$, the scoring vector $\vec{s}_m$ of r contains exactly p distinct values. A rule is *bounded* if it is p-valued for some $p > 1$. Note that for a pure bounded rule there exists some constant t such that for every m, the values in $\vec{s}_m$ are at most t, since for all $m > m_0$ the vector $\vec{s}_m$ cannot contain values that do not appear in $\vec{s}_{m_0}$. Combining Theorem 3.6 and Lemma 3.11, we get the following:

Theorem 3.14. *Suppose that a positional scoring rule r is bounded, strongly pure, and is neither plurality nor veto. Then $\mathrm{Max}_r\{>\bar{k}\}$ is NP-complete for all fixed $k \geq 2$.*

4 ADDITIONAL VOTING RULES

In this section, we consider other, non-positional voting rules. In each rule, we recall the definition of the score of a candidate that is used for winner determination (i.e., top-score candidates). Once we have the score, we automatically get the rank of a candidate, namely $\mathrm{rank}(c \mid \mathbf{T}, \tau)$, and the minimal and maximal ranks, namely $\min(c \mid \mathbf{P}, \tau), \max(c \mid \mathbf{P}, \tau)$, respectively, in the same way as the positional scoring rules. Our results are summarized in Table 2.

4.1 Copeland

We say that a candidate c *defeats* c' in a pairwise election if the majority of the votes rank c ahead of c'. In the Copeland rule, the score of c is the number of candidates $c' \neq c$ that c defeats in a pairwise election. A winner is a candidate with a maximal score. It is known that PW is NP-complete and NW is coNP-complete with respect to Copeland [28]. We use reductions from PW and NW under Copeland to obtain hardness of computing the minimal and maximal ranks, respectively.

Table 2: Results for non-positional voting rules.

Problem	Copeland		Bucklin		Maximin	
PW	NP-c	[28]	NP-c	[28]	NP-c	[28]
NW	coNP-c	[28]	P	[28]	P	[28]
Min{<}	NP-c	[Thm. 4.1]	NP-c	[Thm. 4.2]	NP-c	[Thm. 4.6]
Max{>}	NP-c	[Thm. 4.1]	NP-c	[Thm. 4.2]	NP-c	[Thm. 4.7]
Min{< k}	NP-c	[Thm. 4.1]	NP-c	[Thm. 4.3]	NP-c	[Thm. 4.6]
Max{> k}	NP-c	[Thm. 4.1]	P	[Thm. 4.4]	NP-c	[Thm. 4.7]

Theorem 4.1. *For the Copeland rule, $\mathrm{Min}\{<k\}$ is NP-complete for all fixed $k \geq 2$, and $\mathrm{Max}\{>k\}$ is NP-complete for all fixed $k \geq 1$.*

4.2 Bucklin

Under the Bucklin rule, the score of a candidate c is the smallest number t such that more than half of the voters rank c among the top t candidates. A winner is a candidate with a *minimal* Bucklin score. Since we prefer the minimal score rather than the maximal score, we need to modify the definition of the rank: Let $R'_{\mathbf{T}}$ be the linear order on C that sorts the candidates by their scores in *increasing* order and then by τ. The rank of c is the position of c in $R'_{\mathbf{T}}$, which we denote again by $\mathrm{rank}(c \mid \mathbf{T}, \tau)$. It is known that PW is NP-complete and NW is in polynomial time with respect to Bucklin [28]. We show that computing the minimal and maximal ranks is hard for the Bucklin rule.

Theorem 4.2. *For the Bucklin rule, both $\mathrm{Min}\{<\}$ and $\mathrm{Max}\{>\}$ are NP-complete.*

Proof. First, $\mathrm{Min}\{<\}$ is NP-complete for Bucklin by a straightforward reduction from PW (as PW is the special case of $\mathrm{Min}\{<2\}$). For $\mathrm{Max}\{>\}$. We show a reduction from the independent-set problem in 3-regular graphs, as defined in the proof of Theorem 3.2. Let $G = (U, E)$ be a 3-regular graph with $U = \{u_1, \ldots, u_n\}$. We construct an instance $(C, \mathbf{P}, \tau)$ under Bucklin. The candidates set is $C = U \cup \{c^*, d\} \cup F$ where $F = \{f_1, \ldots, f_{n-1}\}$ and the tiebreaker is $\tau = O(F, U, \{c^*\}, \{d\})$. The voting profile $\mathbf{P} = \mathbf{P}^1 \circ \mathbf{T}^2$ is the concatenation of two parts described next.

The first part, $\mathbf{P}^1 = \{P^1_e\}_{e \in E}$, contains a voter for every edge e. For each edge $e = \{u, w\} \in E$, define $P^1_e = P(F, \{c^*\}, e, U \setminus e, \{d\})$. Then, in three arbitrary voters in $\mathbf{P}^1$, switch between c^* and d (i.e., the profile becomes $P^1_e = P(F, \{d\}, e, U \setminus e, \{c^*\})$). The second part is $\mathbf{T}^2 = (T^2_1, \ldots, T^2_{|E|-4})$ where every voter is $T^2_i = O(U, \{c^*\}, F, \{d\})$. Overall, there are $2n + 1$ candidates and $2|E| - 4$ voters.

In the complete proof, we show that for all k we have $\beta(G) \geq k$ if and only if $\max(c^* \mid \mathbf{P}, \tau) \geq k + |F| + 1$, where $\beta(G)$ is the maximal size of an independent set in G. Thus, $\mathrm{Max}\{>\}$ is NP-hard. □

For comparing the minimal and maximal ranks to some fixed rank k, we show that the complexity is the same as of PW and NW under Bucklin. Namely, $\mathrm{Min}\{<k\}$ is NP-complete and $\mathrm{Max}\{>k\}$ can be solved in polynomial time.

The following theorem states the hardness of $\mathrm{Min}\{<k\}$ under Maximin. The proof is by the same reduction as that of Theorem 4.1.

Theorem 4.3. *For the Bucklin rule,* $\mathsf{Min}\{<k\}$ *is NP-complete for all fixed* $k \geq 2$.

In contrast, the following theorem states that $\mathsf{Max}\{>k\}$ is solvable in polynomial time for all fixed $k \geq 1$ under the Bucklin rule. The proof follows a strategy similar to the proof of Theorem 3.7.

Theorem 4.4. *For all fixed* $k \geq 1$, $\mathsf{Max}\{>k\}$ *is solvable in polynomial time under the Bucklin rule.*

4.3 Maximin

Let $N_{\mathbf{T}}(c, c')$ be the number of votes that rank c ahead of c' in the profile $\mathbf{T}$. The score of c is $s(\mathbf{T}, c) = \min\{N_{\mathbf{T}}(c, c') : c' \in C \setminus \{c\}\}$. A winner is a candidate with a maximal score. Xia and Conitzer [28] established that under Maximin, PW is NP-complete; we generalize it and show that $\mathsf{Min}\{<k\}$ is NP-complete for every $k > 1$. They also show that NW is tractable, and their polynomial-time algorithm can be easily adjusted to solve $\mathsf{Max}\{>1\}$ by accommodating tie-breaking. In contrast, we show that $\mathsf{Max}\{>k\}$ is NP-complete for every $k > 1$.

For the hardness results, we use the following technique. For a profile $\mathbf{T}$ and a pair of candidates c, c' define the *pairwise score difference* $D_{\mathbf{T}}(c, c') = N_{\mathbf{T}}(c, c') - N_{\mathbf{T}}(c', c)$. Note that $D_{\mathbf{T}}(c, c') = -D_{\mathbf{T}}(c', c)$ and $D_{\mathbf{T}}(c, c') = 2N_{\mathbf{T}}(c, c') - n$, so we can define the score under Maximin to be $s(\mathbf{T}, c) = \min\{D_{\mathbf{T}}(c, c') : c' \in C \setminus \{c\}\}$. The following lemma states that we can change the values of $D_{\mathbf{T}}$ to any other values, as long as the parity of the values is unchanged.

Lemma 4.5 (Main theorem in [21]). *Let* $\mathbf{T}$ *be a profile and* $F : C \times C \to \mathbb{Z}$ *be a skew-symmetric function (i.e.,* $F(c_1, c_2) = -F(c_2, c_1)$*) such that for all pairs* $c, c' \in C$ *of candidates,* $F(c, c') - D_{\mathbf{T}}(c, c')$ *is even. There exists a profile* $\mathbf{T}'$ *such that* $D_{\mathbf{T} \circ \mathbf{T}'} = F$ *and* $|\mathbf{T}'| \leq \frac{1}{2}\sum_{c,c'}(|F(c, c') - D_{\mathbf{T}}(c, c')| + 1)$.

If the values $|F(c, c') - D_{\mathbf{T}}(c, c')|$ are polynomial in n and m, then $\mathbf{T}'$ of Lemma 4.5 can be constructed in polynomial time. This is used for establishing the following results.

Theorem 4.6. *Under Maximin,* $\mathsf{Min}\{<k\}$ *is NP-complete for all fixed* $k \geq 2$.

Proof. Let $k \geq 1$. We show a reduction from PW to $\mathsf{Min}\{<k+1\}$ under Maximin. Let $\mathbf{P} = (P_1, \dots, P_n)$ and c^* be an input for PW over a set C of m candidates. By the proof of Xia and Conitzer [28] that PW is hard for Maximin, we can assume that for every completion $\mathbf{T}$ of $\mathbf{P}$ the score of c^* satisfies $s(\mathbf{T}, c^*) \leq -2$. As in the proof of Theorem 3.6, the idea is to add $k-1$ new candidates and modify the voters so that the new candidates are always the top $k-1$ candidates, and the score of every original candidate is increased by the same amount.

We define $C', \mathbf{P}'$ and τ as follows. The candidate set is $C' = C \cup D$ where $D = \{d_1, \dots, d_{k-1}\}$ and the tiebreaker is $\tau = O(D, \{c^*\}, C \setminus \{c^*\})$. The profile $\mathbf{P}' = \mathbf{P}_1 \circ \mathbf{T}_2$ is the concatenation of two parts. The first part is $\mathbf{P}_1 = (P'_1, \dots, P'_n)$ where P'_i is the same as P_i, except that the candidates of D are placed at the bottom positions. Formally, $P'_i := P_i \cup P(C, d_1, \dots, d_{k-1})$. Observe that for every $c \in C$ and $d \in D$, the pairwise score difference $D_{\mathbf{T}_1}(c, d)$ is the same in every completions $\mathbf{T}_1$ of $\mathbf{P}_1$. The same holds for $D_{\mathbf{T}_1}(d, d')$ on all $d, d' \in D$.

The second part, $\mathbf{T}_2$, is the complete profile that exists due to Lemma 4.5 such that for every completion $\mathbf{T}' = \mathbf{T}_1 \circ \mathbf{T}_2$ of $\mathbf{P}'$, the pairwise scores differences satisfy:

- $D_{\mathbf{T}'}(d, c) \in \{-1, 0, 1\}$ for all $d \in D$ and $c \in C' \setminus \{d\}$;
- $D_{\mathbf{T}'}(c, c') = D_{\mathbf{T}_1}(c, c')$ for all $c, c' \in C$.

In the complete proof, we show that c^* is a possible winner of $\mathbf{P}$ if and only if $\min(c \mid \mathbf{P}', \tau) < k + 1$. ◻

Theorem 4.7. *Under Maximin,* $\mathsf{Max}\{>k\}$ *is solvable in polynomial time for* $k = 1$ *and is NP-complete for all* $k > 1$.

Proof. As said earlier, tractability for $k = 1$ is obtained by adjusting the NW algorithm of Xia and Conitzer [28]. For $k > 1$, we show a reduction from *exact cover by-3sets* (X3C): given a vertex set $U = \{u_1, \dots, u_{3q}\}$ and a collection $E = \{e_1, \dots, e_m\}$ of 3-element subsets of U, can we cover all the elements of U using q pairwise-disjoint sets from E? This problem is known to be NP-complete [13].

Given U and E, we construct an instance $(C, \mathbf{P}, \tau)$ under Maximin. The candidate set is $C = U \cup \{c^*, w\} \cup D$ where $D = \{d_1, \dots, d_k\}$, and the tiebreaker is $\tau = O(D, \{c^*\}, U \cup \{w\})$. The voting profile $\mathbf{P} = \mathbf{P}_1 \circ \mathbf{T}_2$ is the concatenation of two parts that we describe next.

The first part $\mathbf{P}_1 = \{P_e\}_{e \in E}$ contains a voter for every set in E. For every $e \in E$, define a complete order $T_e = O(w, c^*, U \setminus e, e, d_1, \dots, d_k)$. The partial order P_e is obtained from T_e by removing the relations in $(e \cup \{d_1, \dots, d_{k-1}\}) \times \{d_k\}$. Denote $\mathbf{T}_1 = \{T_e\}_{e \in E}$. The idea is that ranking d_k higher than the candidates of e indicates that e is in the cover, and ranking d_k in the last position indicates that e is not in the cover. The second part $\mathbf{T}_2$ is the profile that exists due to Lemma 4.5 such that the pairwise scores differences of $\mathbf{T} = \mathbf{T}_1 \circ \mathbf{T}_2$ satisfy:

- $D_{\mathbf{T}}(w, c^*) = m$, $D_{\mathbf{T}}(w, d_1) = -m - 2$, and $D_{\mathbf{T}}(w, u) = m + 2$ for all $u \in U$.
- $D_{\mathbf{T}}(d_k, d_i) = 2q - m$ for $i < k$, and $D_{\mathbf{T}}(d_k, u) = -m - 2$ for all $u \in U$.
- $D_{\mathbf{T}}(c_1, c_2) \in \{-1, 0, 1\}$ for every other pair $c_1, c_2 \in C$.

In the complete proof, we show that c^* always defeats the candidates of $U \cup \{w\}$, and that there is an exact cover if and only if c^* is defeated by all candidates of D in some completion. Hence, there is an exact cover if and only if $\max(c^* \mid \mathbf{P}, \tau) > k$. ◻

5 CONCLUDING REMARKS

We studied the problems of determining the minimal and maximal ranks of a candidate in a partial voting profile, for positional scoring rules and for several other voting rules that are based on scores (namely Bucklin, Copeland and Maximin). We showed that these problems are fundamentally harder than the necessary and possible winners that reason about being top ranked. For example, comparing the maximal/minimal rank to a given number is NP-hard for *every* positional scoring rule, pure or not, including plurality and veto. For the problems of comparison to a fixed k, we have generally recovered the tractable cases of the necessary winners (for maximum rank) and possible winners (for minimum rank). An exception is the Maximin rule, where the problem is tractable for $k = 1$ but intractable for every $k > 1$. Many problems are left for investigation in future research, including: *(a)* establishing useful tractability conditions for an input k; *(b)* completing a full classification of the class of (pure) positional scoring rules for fixed k; *and (c)* determining the parameterized complexity of the problems when k is the parameter.

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