# The Sophomore's Dream

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**Abstract**

This article provides a brief formalisation of the two equations known as the *Sophomore's Dream*, first discovered by Johann Bernoulli [1] in 1697:

$$\int_0^1 x^{-x}\,\mathrm{d}x = \sum_{n=1}^{\infty} n^{-n} \quad \text{and} \quad \int_0^1 x^{x}\,\mathrm{d}x = -\sum_{n=1}^{\infty} (-n)^{-n}$$


# Contents



# 1 The Sophomore's Dream

**theory** *Sophomores-Dream*
**imports** *HOL−Analysis.Analysis HOL−Real-Asymp.Real-Asymp*
**begin**

This formalisation mostly follows the very clear proof sketch from Wikipedia [3]. That article also provides an interesting historical perspective. A more detailed exploration of Bernoulli's historical proof can be found in the book by Dunham [2].

The name 'Sophomore's Dream' apparently comes from a book by Borwein et al., in analogy to the 'Freshman's Dream' equation $(x + y)^n = x^n + y^n$ (which is generally *not* true except in rings of characteristic $n$).

## 1.1 Continuity and bounds for $x \log x$

**lemma** *x-log-x-continuous*: *continuous-on* $\{0..1\}$ $(\lambda x{::}real.\ x * \mathit{ln}\ x)$
⟨*proof*⟩

**lemma** *x-log-x-within-01-le*:
  **assumes** $x \in \{0..(1::real)\}$
  **shows** $x * ln\ x \in \{-exp\ (-1)..0\}$
$\langle proof \rangle$

## 1.2 Convergence, Summability, Integrability

As a first result we can show that the two sums that occur in the two different versions of the Sophomore's Dream are absolutely summable. This is achieved by a simple comparison test with the series $\sum_{k=1}^{\infty} k^{-2}$, as $k^{-k} \in O(k^{-2})$.

**theorem** *abs-summable-sophomores-dream*: $summable\ (\lambda k.\ 1\ /\ real\ (k\ \hat{}\ k))$
$\langle proof \rangle$

The existence of the integral is also fairly easy to show since the integrand is continuous and the integration domain is compact. There is, however, one hiccup: The integrand is not actually continuous.

We have $\lim_{x \to 0} x^x = 1$, but in Isabelle $0^0$ is defined as $0$ (for real numbers). Thus, there is a discontinuity at $x = 0$

However, this is a removable discontinuity since for any $x > 0$ we have $x^x = e^{x \log x}$, and as we have just shown, $e^{x \log x}$ *is* continuous on $[0, 1]$. Since the two integrands differ only for $x = 0$ (which is negligible), the integral still exists.

**theorem** *integrable-sophomores-dream*: $(\lambda x::real.\ x\ powr\ x)$ *integrable-on* $\{0..1\}$
$\langle proof \rangle$

Next, we have to show the absolute convergence of the two auxiliary sums that will occur in our proofs so that we can exchange the order of integration and summation. This is done with a straightforward application of the Weierstraß $M$ test.

**lemma** *uniform-limit-sophomores-dream1*:
  *uniform-limit* $\{0..(1::real)\}$
    $(\lambda n\ x.\ \sum k{<}n.\ (x * ln\ x)\ \hat{}\ k\ /\ fact\ k)$
    $(\lambda x.\ \sum k.\ (x * ln\ x)\ \hat{}\ k\ /\ fact\ k)$
    *sequentially*
$\langle proof \rangle$

**lemma** *uniform-limit-sophomores-dream2*:
  *uniform-limit* $\{0..(1::real)\}$
    $(\lambda n\ x.\ \sum k{<}n.\ (-(x * ln\ x))\ \hat{}\ k\ /\ fact\ k)$
    $(\lambda x.\ \sum k.\ (-(x * ln\ x))\ \hat{}\ k\ /\ fact\ k)$
    *sequentially*
$\langle proof \rangle$

## 1.3 An auxiliary integral

Next we compute the integral

$$\int_0^1 (x \log x)^n \, \mathrm{d}x = \frac{(-1)^n \, n!}{(n+1)^{n+1}} \ ,$$

which is a key ingredient in our proof.

**lemma** *sophomores-dream-aux-integral*:
$((\lambda x.\ (x * \mathit{ln}\ x)\ \hat{}\ n)\ \mathit{has\text{-}integral}\ (-\ 1)\ \hat{}\ n * \mathit{fact}\ n\ /\ \mathit{real}\ ((n + 1)\ \hat{}\ (n + 1)))$
$\{0{<}..{<}1\}$
⟨*proof*⟩

## 1.4 Main proofs

We can now show the first formula: $\int_0^1 x^{-x} \, \mathrm{d}x = \sum_{n=1}^{\infty} n^{-n}$

**lemma** *sophomores-dream-aux1*:
$\mathit{summable}\ (\lambda k.\ 1\ /\ \mathit{real}\ ((k{+}1)\hat{}(k{+}1)))$
$\mathit{integral}\ \{0..1\}\ (\lambda x.\ x\ \mathit{powr}\ (-x)) = (\sum n.\ 1\ /\ (n{+}1)\hat{}(n{+}1))$
⟨*proof*⟩

**theorem** *sophomores-dream1*:
$(\lambda k{::}\mathit{nat}.\ \mathit{norm}\ (k\ \mathit{powi}\ (-k)))\ \mathit{summable\text{-}on}\ \{1..\}$
$\mathit{integral}\ \{0..1\}\ (\lambda x.\ x\ \mathit{powr}\ (-x)) = (\sum_\infty k{\in}\{(1{::}\mathit{nat})..\}.\ k\ \mathit{powi}\ (-k))$
⟨*proof*⟩

Next, we show the second formula: $\int_0^1 x^x \, \mathrm{d}x = -\sum_{n=1}^{\infty} (-n)^{-n}$

**lemma** *sophomores-dream-aux2*:
$\mathit{summable}\ (\lambda k.\ (-1)\ \hat{}\ k\ /\ \mathit{real}\ ((k{+}1)\hat{}(k{+}1)))$
$\mathit{integral}\ \{0..1\}\ (\lambda x.\ x\ \mathit{powr}\ x) = (\sum n.\ (-1)\hat{}n\ /\ (n{+}1)\hat{}(n{+}1))$
⟨*proof*⟩

**theorem** *sophomores-dream2*:
$(\lambda k{::}\mathit{nat}.\ \mathit{norm}\ ((-k)\ \mathit{powi}\ (-k)))\ \mathit{summable\text{-}on}\ \{1..\}$
$\mathit{integral}\ \{0..1\}\ (\lambda x.\ x\ \mathit{powr}\ x) = -(\sum_\infty k{\in}\{(1{::}\mathit{nat})..\}.\ (-k)\ \mathit{powi}\ (-k))$
⟨*proof*⟩

**end**

# References


[1] J. Bernoulli. *Opera omnia*, volume 3. 1697.

[2] W. Dunham. *The Calculus Gallery: Masterpieces from Newton to Lebesgue*. Princeton University Press, 2004.

[3] Wikipedia contributors. Sophomore's dream — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Sophomore%27s_dream&oldid=1053905038, 2021. [Online; accessed 10-April-2022].