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ON MEASURABLE SOLUTIONS OF A FUNCTIONAL EQUATION AND ITS APPLICATION TO INFORMATION THEORY

GUR DIAL

In this paper, the measurable solutions of a functional equation with two unknown functions are obtained. As an application of the measurable solutions, characterization of three measures of information is given.

1. INTRODUCTION

Let $\Delta_n = \{P = (p_1, \ldots, p_n); p_i \geqq 0,\ i = 1, \ldots, n, \sum_{i=1}^{n} p_i = 1\}$ for $n \geqq 1$ be the set of n-complete probability distributions.

Let $\mathbb{R}$ be the set of all real numbers and let $I = [0, 1]$.

Let us consider measurable functions $h, g : I \to \mathbb{R}$ satisfying the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} h(x_i y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} g(x_i)\, h(y_j) + \sum_{i=1}^{n} \sum_{j=1}^{m} g(y_j)\, h(x_i) \tag{1.1}$$

where $X = (x_1, \ldots, x_n) \in \Delta_n$, $Y = (y_1, \ldots, y_m) \in \Delta_m$ for $n, m = 2, 3$.

The continuous solutions of (1.1) were given by Sharma and Taneja [3].

The objective of this paper is to find the measurable solutions of the functional equation (1.1) and given its application to information theory.

2. MEASURABLE SOLUTIONS OF (1.1)

In the following theorem, we will give the measurable solutions of system (1.1) of functional equations.

Theorem 1. If h and g are Lebesgue measurable solutions of system (1.1) of functional equations for $X \in \Delta_n$, $Y \in \Delta_m$ where $n, m = 2, 3$, then they are given for

$x \in [0, 1]$, by one of the following solutions:

$$h(x) = Ax^{\alpha} \log x\,, \quad g(x) = x^{\alpha}\,, \quad \alpha > 0 \tag{2.2}$$

$$h(x) = 1/B(x^{\alpha} - x^{\beta})\,, \quad g(x) = 1/2(x^{\alpha} + x^{\beta})\,, \quad \alpha, \beta > 0 \tag{2.3}$$

$$h(x) = (x^{\alpha}/C) \sin(\beta \log x)\,, \quad g(x) = x^{\alpha} \cos(\beta \log x)\,, \quad \alpha > 0\,, \quad \beta \neq 0\,. \tag{2.4}$$

Proof. Putting $Y = (y, v, 1 - y - v) \in \Delta_3$ and $Y = (y + v, 1 - y - v) \in \Delta_2$ respectively in (1.1), we get

$$\sum_i (h(x_i y) + h(x_i v) + h(x_i(1 - y - v))) = \tag{2.5}$$
$$= \sum_i g(x_i)(h(y) + h(v) + h(1 - y - v)) + \sum_i h(x_i)(g(y) + g(v) + g(1 - y - v))$$

and

$$\sum_i (h(x_i(y + v) + h(x_i(1 - y - v))) = \tag{2.6}$$
$$= \sum_i g(x_i)(h(y + v) + h(1 - y - v)) + \sum_i h(x_i)(g(y + v) + g(1 - y - v))$$

Subtracting (2.6) from (2.5), we have

$$\sum_i h(x_i y) + h(x_i v) - h(x_i(y + v))) = \tag{2.7}$$
$$= \sum_i g(x_i)(h(y) + h(v) - h(y + v)) + \sum_i h(x_i)(g(y) + g(v) + g(1 - y - v))$$

For $X \in \Delta_n$, $n = 2, 3$, let

$$A_X(t) = \sum_i h(x_i t) - \sum_i g(x_i)\, h(t) - \sum_i h(x_i)\, g(t) \tag{2.8}$$

Using (2.8), (2.7) becomes

$$A_X(y + v) = A_X(y) + A_X(v) \tag{2.9}$$

It means that $A_X(.)$ is additive on I. We can conclude from the result of Daroczy and Losonczi [2] that the measurable solution of (2.9) is

$$A_X(t) = t\, A_X(1) \tag{2.10}$$

Thus, in order to see the expression of $A_X(t)$, we need to evaluate

$$A_X(1) = \sum_i h(x_i) - \sum_i g(x_i)\, h(1) - \sum_i h(x_i)\, g(1) \tag{2.11}$$

Substituting $Y = (1, 0)$ and $Y = (1, 0, 0)$ respectively in (1.1) we get

$$\sum_i h(x_i) + n\, h(0) = \sum_i g(x_i)(h(1) + h(0)) + \sum_i h(x_i)(g(1) + g(0)) \tag{2.12}$$

and

$$\sum_i h(x_i) + 2n\, h(0) = \sum_i g(x_i)(h(1) + 2h(0)) + \sum_i h(x_i)(g(1) + 2g(0)) \tag{2.13}$$

Subtracting (2.12) from (2.13), we have

$$n\,h(0) = \sum_i g(x_i)\,h(0) + \sum_i h(x_i)\,g(0) \tag{2.14}$$

Using (2.14), (2.12) becomes

$$\sum_i h(x_i) = \sum_i g(x_i)\,h(1) + \sum_i h(x_i)\,g(1) \tag{2.15}$$

so that $A_X(1) = 0$. Now by (2.10)

$$\sum_i h(x_i t) = \sum_i g(x_i)\,h(t) + \sum_i h(x_i)\,g(t) \tag{2.16}$$

for $X = (x_1, \ldots, x_n) \in \Delta_n$, $n = 2, 3$ and $t \in [0, 1]$.

Let $X = (x, u, 1 - x - u)$. Then (2.16) becomes

$$h(xt) + h(ut) + h((1 - x - u)\,t) = (g(x) + g(u) + g(1 - x - u))\,h(t) + \\ + (h(x) + h(u) + h(1 - x - u))\,g(t) \tag{2.17}$$

Again, if $X = (x + u, 1 - x - u)$ in (2.16), we have

$$h(x + u)\,t + h((1 - x - u)\,t) = (g(x + u) + g(1 - x - u))\,h(t) + \\ + (h(x + u) + h(1 - x - u))\,g(t) \tag{2.18}$$

Subtracting (2.18) from (2.17), we get

$$h(xt) + h(ut) - h((x + u)\,t) = (g(x) + g(u) - g(x + u))\,h(t) + \\ + (h(x) + h(u) - h(x + u))\,g(t) \tag{2.19}$$

For $t \in [0, 1]$, let us define

$$B_t(w) = h(wt) - g(w)\,h(t) - h(w)\,g(t), \quad w \in [0, 1] \tag{2.20}$$

Then, (2.19) can be written as

$$B_t(x + u) = B_t(x) + B_t(u) \quad \text{for} \quad x, u, x + u \in [0, 1] \tag{2.12}$$

Using again the result of Daroczy and Losonoczi [2], we have

$$B_t(w) = w\,B_t(1)\,, \quad w \in [0, 1] \tag{2.22}$$

$$B_t(1) = h(t) - g(1)\,h(t) - h(1)\,g(t)\,, \quad t \in [0, 1] \tag{2.23}$$

Putting $X = (1, 0)$ and $X = (1, 0, 0)$ respectively in (2.16), we get

$$h(t) + h(0) = (g(1) + g(0))\,h(t) + (h(t) + h(0))\,g(t) \tag{2.24}$$

and

$$h(t) + 2h(0) = (g(1) + 2g(0))\,h(t) + (h(1) + 2h(0))\,g(t) \tag{2.25}$$

Subtracting (2.24) from (2.25), we obtain

$$h(0) = g(0)\,h(t) + h(0)\,g(t) \tag{2.26}$$

Using (2.26), (2.24) becomes

$$h(t) = g(1)\, h(t) + h(1)\, g(t) \tag{2.27}$$

Hence we have

$$B_t(1) = 0 \tag{2.28}$$

Then (2.20) becomes

$$h(wt) = g(w)\, h(t) + h(w)\, g(t)\,, \quad w, t \in [0, 1] \tag{2.29}$$

But the most general complex solutions of (2.29) are given by (see [1])

$$h(w) = 0\,, \quad g(w) \text{ arbitrary}\,; \tag{2.30}$$

$$h(w) = e_0(w)\, a(w)\,, \quad g(w) = e_0(w)\,; \tag{2.31}$$

and

$$h(w) = (\tfrac{1}{2}k)\,(e_1(w) - e_2(w))\,, \quad g(w) = \tfrac{1}{2}(e_1(w) + e_2(w)) \tag{2.32}$$

where $k \neq 0$ is an arbitrary real or purely imaginary constant and $a(w)$,$e_t(w)$, $(t = 0, 1, 2)$ are arbitrary functions satisfying

$$a(wt) = a(w) + a(t)\,, \tag{2.33}$$

and

$$e_l(wt) = e_l(w)\, e_l(t)\,, \quad l = 0, 1, 2 \tag{2.34}$$

respectively.

From (2.30), (2.31), (2.32), (2.33) and (2.34) it is easy to see that the real measurable solutions h and g are given by (2.2), (2.3) and (2.4). This proves the theorem. □

3. APPLICATION TO INFORMATION THEORY

Let h be a real measurable function such that

$$H(P) = \sum_i h(p_i) \tag{3.1}$$

where $P \in \Delta_n$. Also suppose that h satisfies the normalizing condition $h(\tfrac{1}{2}) = 1$.

In the next theorem we give characterization of three measures of information satisfying (1.1), (3.1) and the normalizing condition.

Theorem 2. The entropies of a probability distribution $P \in \Delta_n$ corresponding to real measurable solution (2.2), (2.3) and (2.4) of the functional equation (1.1) under the normalization condition $h(\tfrac{1}{2}) = 1$ are given by

$$H_l(P) = -2^{\alpha-1} \sum_i p_i \log p_i\,, \quad \alpha > 0\,, \tag{3.2}$$

$$H_p^{(\alpha,\beta)}(P) = (2^{1-\alpha} - 2^{1-\beta})^{-1} \sum_i (p_i^\alpha - p_i^\beta)\,, \quad \alpha \neq \beta\,, \quad \alpha > 0\,, \quad \beta > 0 \tag{3.3}$$

$$H_s^{(\alpha,\beta)}(P) = (-2^{\alpha-1}/\sin \beta) \sum_i p_i^\alpha \sin(\beta \log p_i)\,, \quad \beta \neq 0\,, \quad \alpha > 0\,. \tag{3.4}$$

The proof is rather straighforward. □

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